Kibble-Zurek mechanism

CMT Journal Club
Feb 21st, 2014 Thomas Veness

1 References

TWB Kibble 1976 J. Phys. A: Math. Gen. 9 1387
WH Zurek Nature 317, 505 (1985)

2 Introduction

The Kibble-Zurek Mechanism (KZM) describes the formation of defects in a system driven through a continuous phase transition at a finite rate. Originally formulated for thermal phase transitions, it also describes quantum phase transitions.

Across a continuous phase transition, the gap for excitations closes and the associated timescale diverges. This implies that *any* finite rate of change of the system will at some point fail to be adiabatic. The mechanism was originally proposed to explain the formation of defects in the early universe by Kibble and was later expanded upon for condensed matter systems by Zurek.

3 Heuristic example

If we consider a paramagnetic to ferromagnetic global quench, the ferromagnetic ground state has order whilst the paramagnetic state is disordered. For an infinitely slow quench, we would expect to end up in one of the two ferromagnetic ground states. For an infinitely fast i.e. instantaneous quench, we would expect to remain in a completely disordered state. We thus expect that in some intermediary regime, we introduce a finite density of defects.

It is important to note that in a physical ferromagnet the domains are caused by long-ranged dipolar forces. In the case under consideration, this is a purely non-equilibrium effect.

4 Crux

The KZM makes a simplification: considering a linear quench of some control parameter, the quench is divided into distinct regimes: a "freeze-out" time is identified at which the evolution changes from being *purely* adiabatic to "impulse-like". That is, we assume that very little of the physics is captured by behaviour that is not described by either of these regimes.

Implications:

- fluctuations present at freeze-out survive in the ordered phase
- the field configuration is independently determined in regions which never had a chance to interact

5 KZM argument

Consider a system with a tunable coupling $\varepsilon(t)$ such that a phase transition occurs at $\varepsilon = 0$. If we take the gap (which determines the relevant properties of the system) $\Delta \propto \varepsilon(t)$ then this determines a time-scale τ

$$\tau := \frac{\hbar}{\Delta} = \frac{\tau_0}{\varepsilon(t)} \tag{1}$$

We also have a healing length determined by the product of the speed of sound (assumed constant, true for TFIM) and the relaxation time.

$$\xi := c\tau \tag{2}$$

$$\xi = \frac{\xi_0}{\varepsilon(t)} \tag{3}$$

At the critical point, $\tau \to \infty$, this means that the system effectively can't respond to any changes. As mentioned before, we split the quench into two regimes, but we need to identify a sensible value for the freeze-out time. If we ramp ε as $\varepsilon(t) := \frac{t}{\tau_Q}$, then the relative couplings change on a time-scale

$$\frac{\varepsilon(t)}{\dot{\varepsilon}(t)} = t. \tag{4}$$

The switch between the two regimes therefore occurs at a time $\hat{t}$ defined by

$$\tau(\hat{t}) = \hat{t}, \tag{5}$$

i.e. the rate of change of the coupling is comparable to the response time of the system. Eq. (1) therefore allows us to write

$$\frac{\tau_0}{\frac{\hat{t}}{\tau_Q}} = \hat{t} \Rightarrow \boxed{\hat{t} = \sqrt{\tau_0 \tau_Q}}. \tag{6}$$

The system is therefore frozen for $t \in [-\hat{t}, \hat{t}]$.

$$\begin{aligned} \hat{\varepsilon} &:= \varepsilon(\hat{t}) &= \sqrt{\frac{\tau_0}{\tau_Q}} \\ \hat{\xi} &:= \xi(\hat{t}) &= \xi_0 \sqrt{\frac{\tau_Q}{\tau_0}} \end{aligned} \tag{7}$$

We expect $O(1)$ number of defects for every $\hat{\xi}$, which therefore gives us the main statement of the Kibble-Zurek mechanism

$$\nu_{KZM} \sim \frac{\alpha}{\hat{\xi}} \Rightarrow \boxed{\nu_{KZM} \propto \frac{1}{\sqrt{\tau_Q}}} \tag{8}$$

This is rather general, so it might be useful to explore a specific case. We will first consider a system which will hopefully be familiar, but not necessarily a convincing parallel to a phase transition.

6 Landau-Zener tunnelling

The Hamiltonian for this toy model is given by

$$H(t) = \frac{\hat{\Delta}}{2}\sigma_z + \frac{vt}{2}\sigma_x. \tag{9}$$

In the $(|\uparrow\rangle, |\downarrow\rangle)$ basis, this is equivalent to

$$H(t) = \frac{1}{2}\begin{pmatrix} \hat{\Delta} & vt \\ vt & -\hat{\Delta} \end{pmatrix} \Rightarrow E = \pm\frac{1}{2}\sqrt{(vt)^2 + \hat{\Delta}^2} \tag{10}$$

Sketch graph of states and energies. Intuitively, for adiabatic evolution $|\leftarrow\rangle \to |\rightarrow\rangle$ and for impulse-like evolution, the states stay the same: we call this case "tunnelling". If we take a state to have the general form

$$|\psi\rangle := A(t)|\leftarrow\rangle + B(t)|\rightarrow\rangle \tag{11}$$

Then the Schrödinger equation implies that

$$\boxed{\begin{aligned} i\dot{A} &= \frac{B\hat{\Delta}}{2} + \frac{vt}{2}A \\ i\dot{B} &= \frac{A\hat{\Delta}}{2} - \frac{vt}{2}B \end{aligned}} \tag{12}$$

This was originally solved by Zener: Proc R. Soc. 137,696-702 (1932). We find that

$$P_{\text{tun}} \approx \exp\left[-\frac{\pi\hat{\Delta}^2}{2\hbar|v|}\right] \tag{13}$$

When $t \gg \frac{\hat{\Delta}}{v}$, then the true gap Δ is given by $\Delta \approx vt \Rightarrow \dot{\Delta} \approx v$. We then have that

$$(1-f) = e\left[-\frac{\pi\hat{\Delta}^2}{2\hbar|v|}\right] \Rightarrow |v| = |\dot{\Delta}| \leq \frac{\pi\hat{\Delta}^2}{2\hbar\ln(1-f)} \tag{14}$$

For large N, $\hat{\Delta} \sim \frac{1}{N}$ and hence

$$\dot{\Delta} \propto \frac{1}{\tau_Q} \leq \frac{\pi\left(\frac{\alpha}{N}\right)^2}{2\hbar\ln(1-f)} \tag{15}$$

$$\frac{1}{N} \propto \nu \propto \frac{1}{\sqrt{\tau_Q}} \tag{16}$$

This is not particularly convincing as a phase transition, however.

7 Transverse field Ising model

$$H = -\sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + g\sigma_n^z\right) \tag{17}$$

Performing a Jordan-Wigner transformation and a Fourier decomposition implies that

$$H = \sum_k \Big(2(g - \cos k)c_k^\dagger c_k + \sin k(c_k^\dagger c_{-k}^\dagger + c_{-k}c_k) - g\Big) \tag{18}$$

If we now consider $g := g(t)$ and perform a Bogoliubov transformation

$$c_l = u_k(t)\gamma_k + v_{-k}^*(t)\gamma_{-k}^\dagger \tag{19}$$

$$c_l = u_k^*(t)\gamma_k^\dagger + v_{-k}(t)\gamma_{-k} \tag{20}$$

We require that, in the Heisenberg picture

$$i\frac{\mathrm{d}}{\mathrm{d}t}c_k = [c_k, H] \tag{21}$$

And this implies the equations

$$\boxed{\begin{aligned} i\dot{u}_k &= 2(g(t) - \cos k)u_k + 2\sin k v_k \\ i\dot{v}_k &= -2(g(t) - \cos k)v_k + 2\sin u_k \end{aligned}} \tag{22}$$

If we compare with eq (12), then we can see we simply need to make the identification of

$$\frac{\hat{\Delta}}{2} = 2\sin k; \frac{vt}{2} = 2(g(t) - \cos k) \tag{23}$$

As we have now mapped this to a free theory, each k-mode is independent. For a slow transition, we expect long-wavelength modes to be the most important and we thus consider small k.

$$p_k \approx \exp\left[-\frac{\pi \sin^2 k}{2\hbar (vt)'}\right] \tag{24}$$

We have

$$g(t) = 1 + \frac{t}{\tau_Q} \tag{25}$$

And hence

$$p_k \approx \exp\left[-\frac{\pi k^2 \tau_q}{2}\right] \tag{26}$$

The total number of kinks is given by the sum over all modes and, taking the thermodynamic limit, we can change this sum for an integral

$$\mathcal{N} \propto \int_{-A}^{A} \exp\left[-\pi k^2 \tau_Q/2\right] \mathrm{d}k \tag{27}$$

We can extend the ranges of this integral to infinity introducing only a small error and hence

$$\mathcal{N} \propto \frac{1}{\sqrt{\tau_Q}} \tag{28}$$

8 Conclusions and summary

- Breaks the quench into adiabatic and impulse regimes
- Estimate is typically order-of-magnitude
- Linear quench $\Rightarrow$ defect density scales at $\frac{1}{\sqrt{\tau_Q}}$
- Verified experimentally in liquid crystals, superfluids, superconductors and certain non-linear optics systems
- Provides a simple and powerful insight into a certain class of quenches
- Various numerical studies are in agreement with the principle