Equilibria and field reversal limits in centrifugal mirror machines

B.S. Patel & the Gridfire team

Overview

- Impact of rotation on Grad-Shafranov equation
- Equilibrium calculations using FreeGSNKE
- Validity of a Maxwellian distribution
- Field reversal limit in centrifugal mirrors

Impact of rotation on Grad-Shafranov equation

Grad-Shafranov equation

- At sonic levels of rotation, inertial term becomes comparable to the pressure gradient

$$\mathbf{J} \times \mathbf{B} = \nabla p + \Sigma_s m_s n_s (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \qquad \boldsymbol{u} = \omega(\psi) R^2 \nabla\zeta$$

- GS equation combines radial force balance with Ampere's equation

$$\Delta^* \psi = -\mu_0 \frac{\partial p}{\partial \psi} R^2 - I \frac{\partial I}{\partial \psi} \longrightarrow \Delta^* \psi = -\mu_0 \left(\frac{\partial p}{\partial \psi} - \Sigma_s m_s n_s \omega^2 R \frac{\partial R}{\partial \psi} \right) R^2 - I \frac{\partial I}{\partial \psi}$$

Modification to pressure gradient

- With sonic rotation pressure no longer remains a flux function - see from poloidal component of force balance

$$\frac{\partial p}{\partial \theta} - \Sigma_s m_s n_s \omega^2 R \frac{\partial R}{\partial \theta} = 0$$

- Temperature remain a flux function but density picks up poloidal variation
  - $T(\psi) \rightarrow T(\psi)$: Fast parallel transport equilibrates temperature
  - $n(\psi) \rightarrow n(\psi, \theta)$: Centrifugal variation of density

- Note that tokamaks use flux surface coordinates (ψ, θ, ζ) but in mirrors cylindrical coordinates are much more natural (ψ, z, ζ) - similar argument applies

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Maxwellian plasma

- For a Maxwellian plasma it can be shown that the density takes the form [1] – rigorous derivation from Maxwell-Vlasov

$$n_s(\psi,\theta) = N_s(\psi)\mathrm{exp}\left[-\frac{\Xi_s}{T_s}\right]$$

$$\Xi_s = q_s\varphi_0 - \frac{m_s\omega^2R^2}{2}$$

- N_s is an arbitrary flux function
- ϕ_0 is an electrostatic potential
  - Deconfines ions and confines electrons

$$\Sigma_s q_s n_s(\psi,\theta) = 0$$

- To find ϕ_0 and N_s, one must solve quasineutrality

[1] I G Abel *et al* 2013 Rep. Prog. Phys. 76 116201

Rotating Grad-Shafranov equation

- Plugging this form of n and T into modified GS equation results in

$$\Delta^* \psi = -\mu_0 \left(\sum_s n_s \left\{ T_s \frac{\partial \ln N_s}{\partial \psi} + \left[q_s \varphi_0 - \frac{1}{2} m_s \omega^2 R^2 + T_s \right] \frac{\partial \ln T_s}{\partial \psi} \right\} - \sum_s m_s n_s R^2 \omega \frac{\partial \omega}{\partial \psi} \right) R^2 - I \frac{\partial I}{\partial \psi}$$

- Comprised of
  - Modified pressure gradient
  - Rotation term
  - Identical toroidal field term

- RHS is essentially the form for toroidal component of current

Quasineutrality solve

$$\varphi_0 = \frac{T_D T_e}{e\left(T_D + T_e\right)}\left[\frac{m_D}{2T_D} - \frac{m_e}{2T_e}\right]\omega^2 R^2$$

- Start with ψ(R,Z)
- Guess $T_s(\psi_n)$, $\omega(\psi_n)$ and $n_e(\psi_n, Z=0)$
  - Ideally these would really come from a transport solver

- We have a gauge freedom in setting ϕ_0
  - Set it such that $N_e(\psi_n=0.5) \equiv n_e(\psi_n=0.5, Z=0)$

- Use plasma composition at particular point to determine $N_{ion}(\psi_n)$
  - Can be outboard midplane (tokamaks) or Z=0 (mirrors)

- Adjust ϕ_0 until quasineutrality is satisfied - implemented a Newton method
  - For a two species plasma there is a relatively simple solution

Numerical solver - requirements for non-rotating equilibrium

- Magnet locations
- Plasma boundary conditions
- $p(\psi_n)$, $I(\psi_n)$

$$\psi_n = \frac{\psi - \psi_{\text{inner}}}{\psi_{\text{outer}} - \psi_{\text{inner}}}$$

- Guess $\psi(R,Z)$
- Find $J_\phi(R, Z)$ given $p(\psi_n)/I(\psi_n)$
- Use GS equation to find $\psi(R,Z)$ given $J_\phi(R,Z)$
- Iterate until converged - Picard or Newton-Krylov

Numerical solver - requirements for rotating equilibrium

- Magnet locations
- Plasma boundary conditions
- **$n_e(\psi_{n,}$, Z=0), $T_s(\psi_n)$, $\omega(\psi_n)$, $I(\psi_n)$, plasma composition at Z=0**

- Guess $\psi(R,Z)$
- **Solve quasineutrality to solve for $N_s(\psi_n)$ and ϕ_0**
- Find $J_\phi(R, Z)$ **given $N_s(\psi_n)$, $T_s(\psi_n)$, $\omega(\psi_n)$, $I(\psi_n)$ and $\phi_0(R,Z)$**
- Use modified GS equation to find $\psi(R,Z)$ given $J_\phi(R,Z)$
- Iterate until converged - Picard or Newton-Krylov

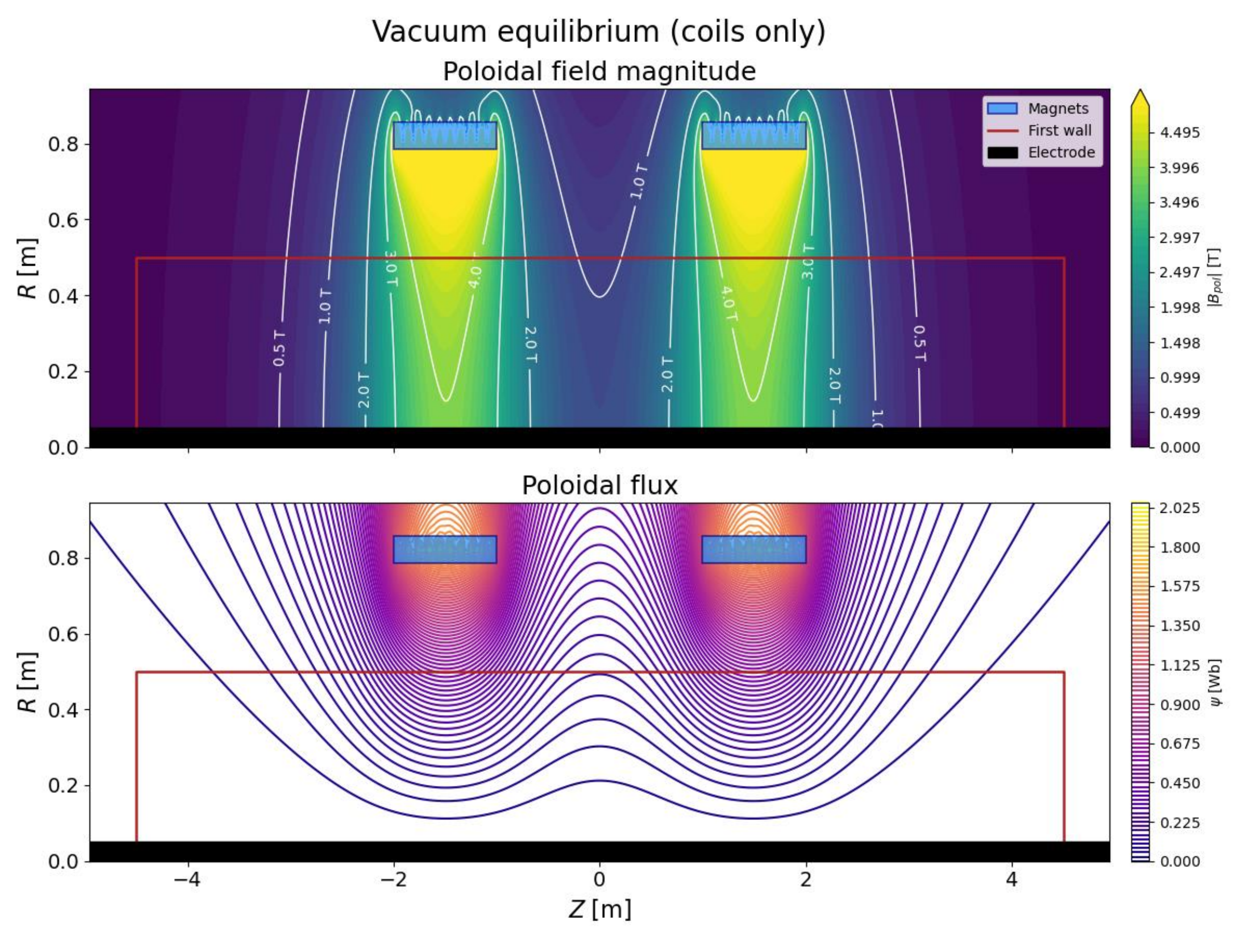
Equilibrium calculations using FreeGSNKE

FreeGSNKE - Vacuum solution

- FreeGSNKE is a free boundary equilibrium code used to solve Grad-Shafranov equation
  - Used extensively on MAST-U
  - Modified to calculate additional terms in J_ϕ

- Vacuum field trivial using Green's theorem

- Set up a 4T throat magnet configuration
  - Mirror ratio R=3.8
  - No toroidal field

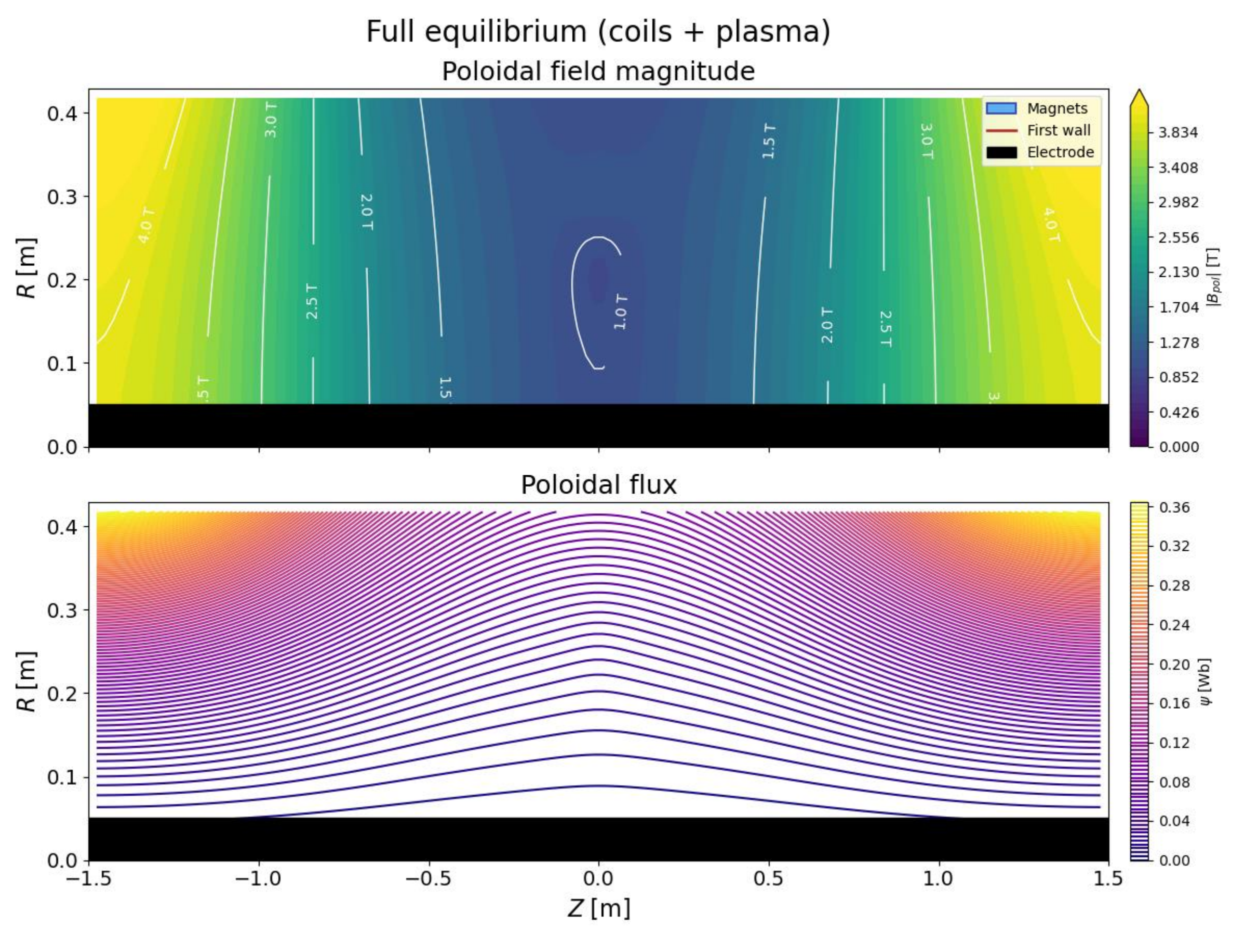


FreeGSNKE - Plasma solution

$$M = \frac{\omega R_0}{v_{th,D}}$$

- Assumed Gaussian profiles in T_s, n_e
  - Set ω such that M is constant
  - Can set M or voltage
  - Arbitrary profiles are possible

- $T_d(\psi_n=0.5)$ = 1.6keV
- $T_e(\psi_n=0.5)$ = 1.2keV
- $n_e(\psi_n=0.5) = 5e19m^{-3}$
- ψ_{inner}: R=0.1m, Z=0m
- ψ_{outer}: R=0.35m, Z=0m
- 3 species: electron, deuterium, carbon
- Z_{eff} = 1.5 at Z=0m



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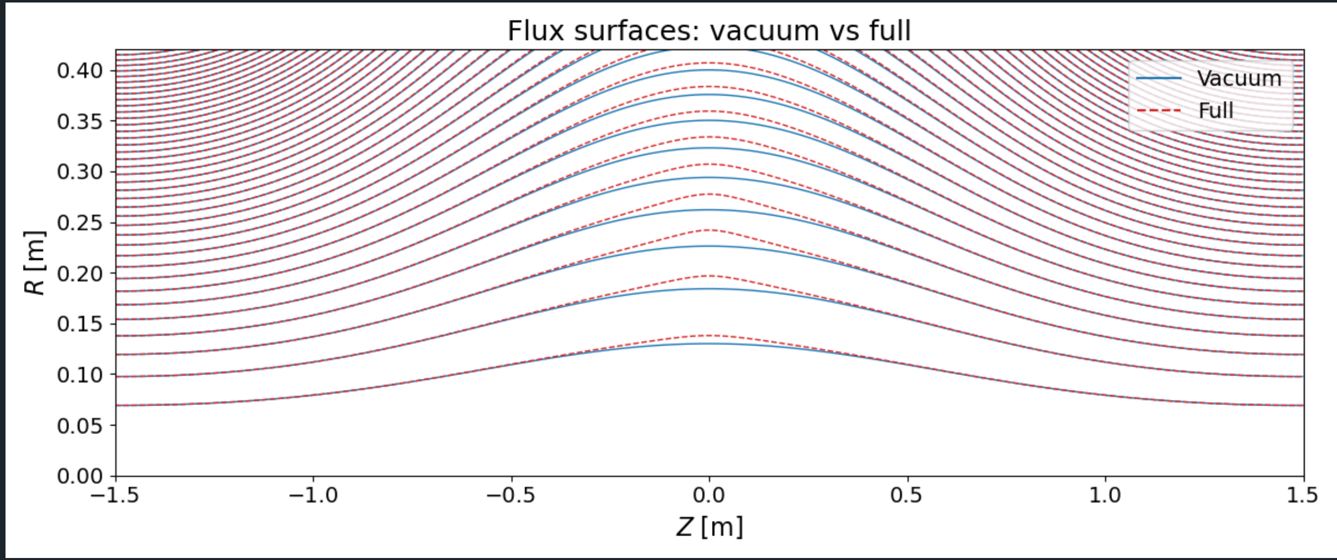


Flux functions

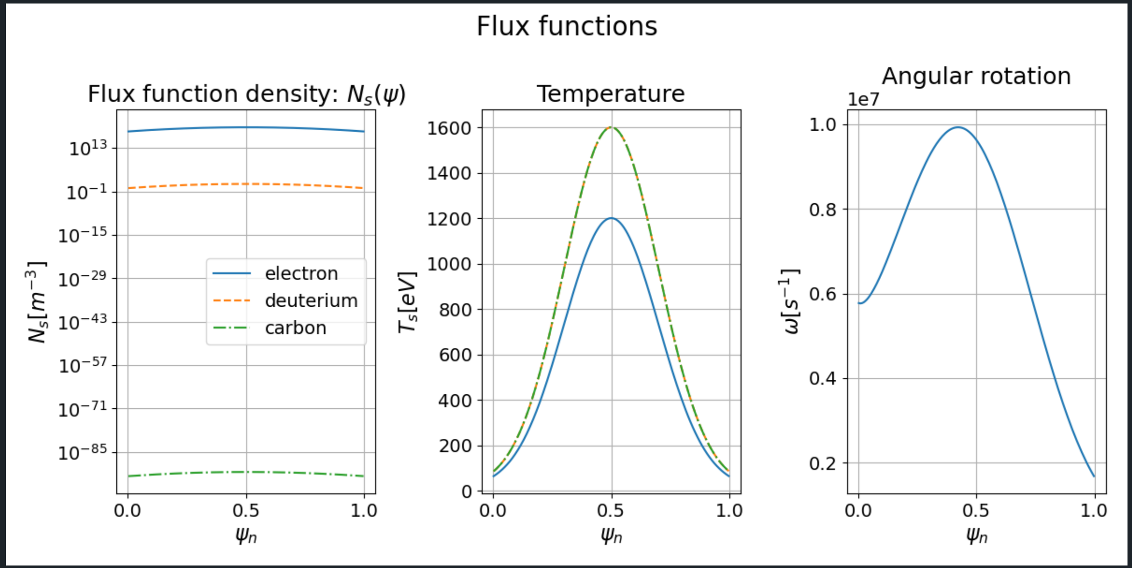
- Need to be careful when calculating quasineutrality

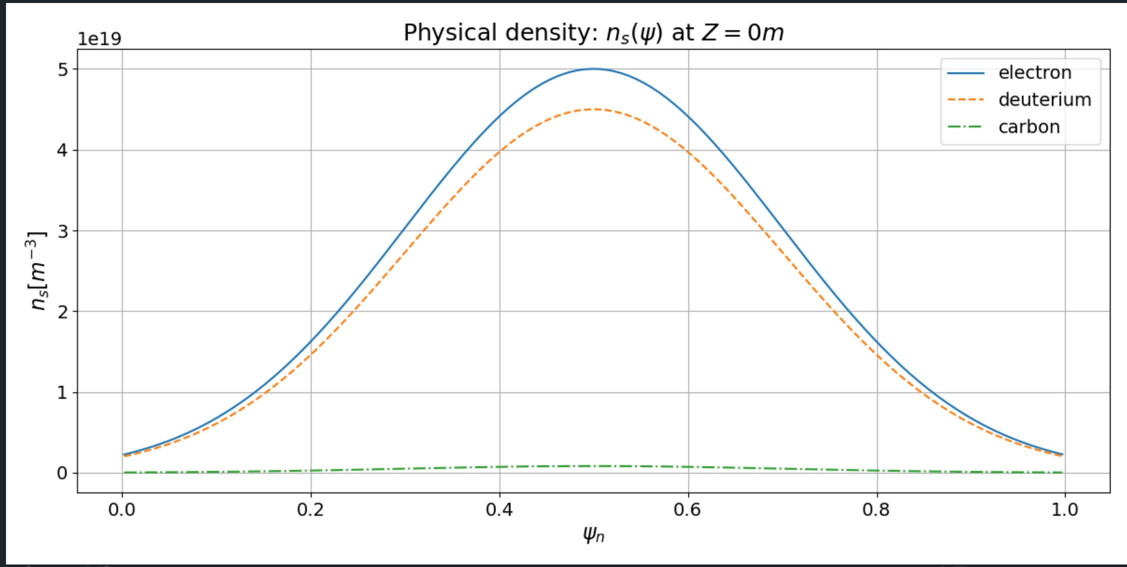
$$\frac{N_s(\psi)}{N_e(\psi)} \sim \frac{n_s(\psi,\theta=0)}{n_e(\psi,\theta=0)} \exp\left[-M^2 \frac{v_{th,d}}{v_{th,s}}\right]$$

- Assuming shear stabilisation you probably have the hottest plasma near peak in gradient of ω
  - Transport solver would be helpful…

- Result here at M~6
  - 400kV required to provide this rotation

$$\omega(\psi) = \frac{\partial \Phi(\psi)}{\partial \psi}$$





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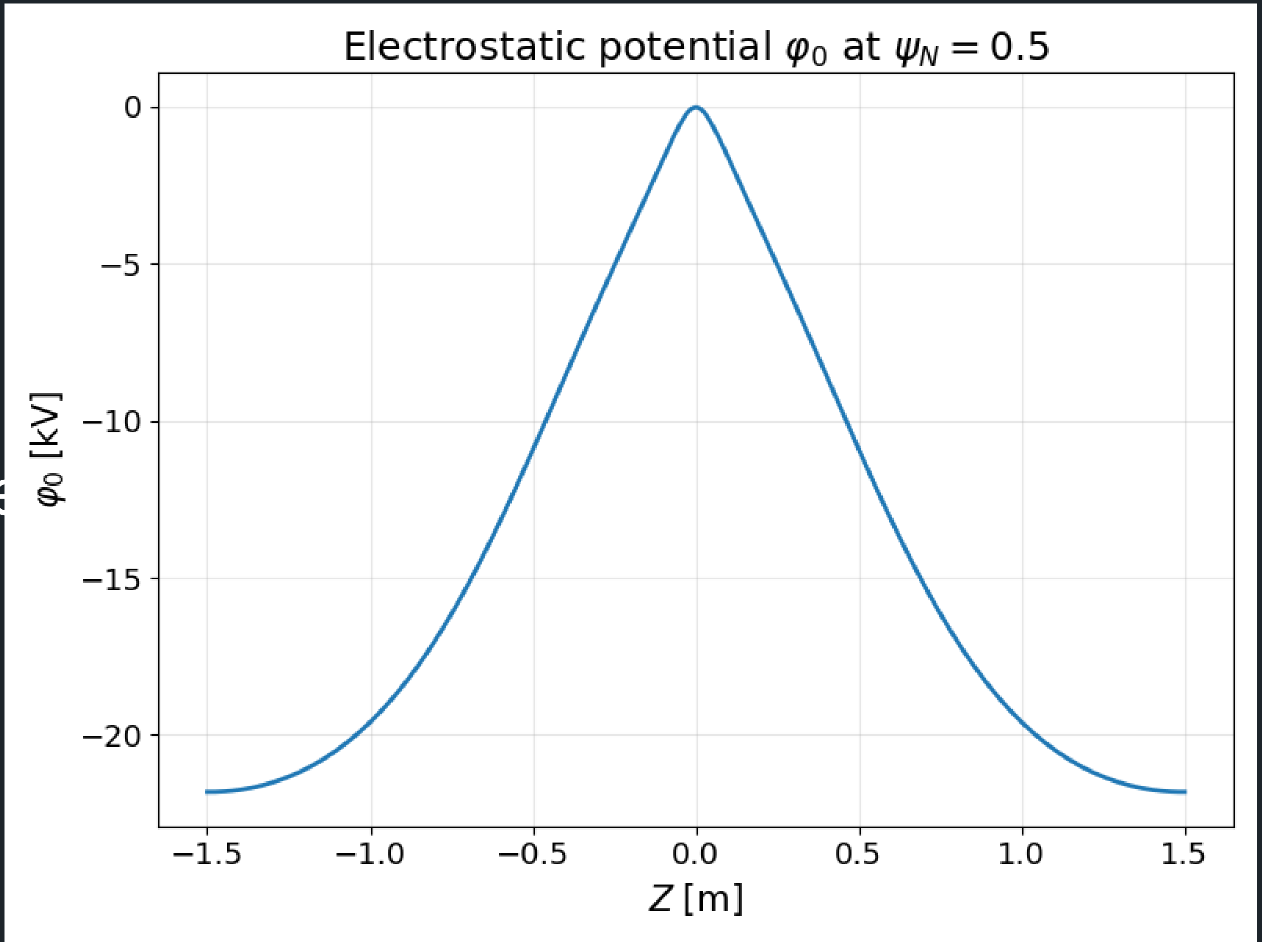
Confining potential ϕ_0

- Electrostatic potential hill forms
  - Confining for electrons
  - Deconfining for ions

- Small compared to applied voltage of 400kV



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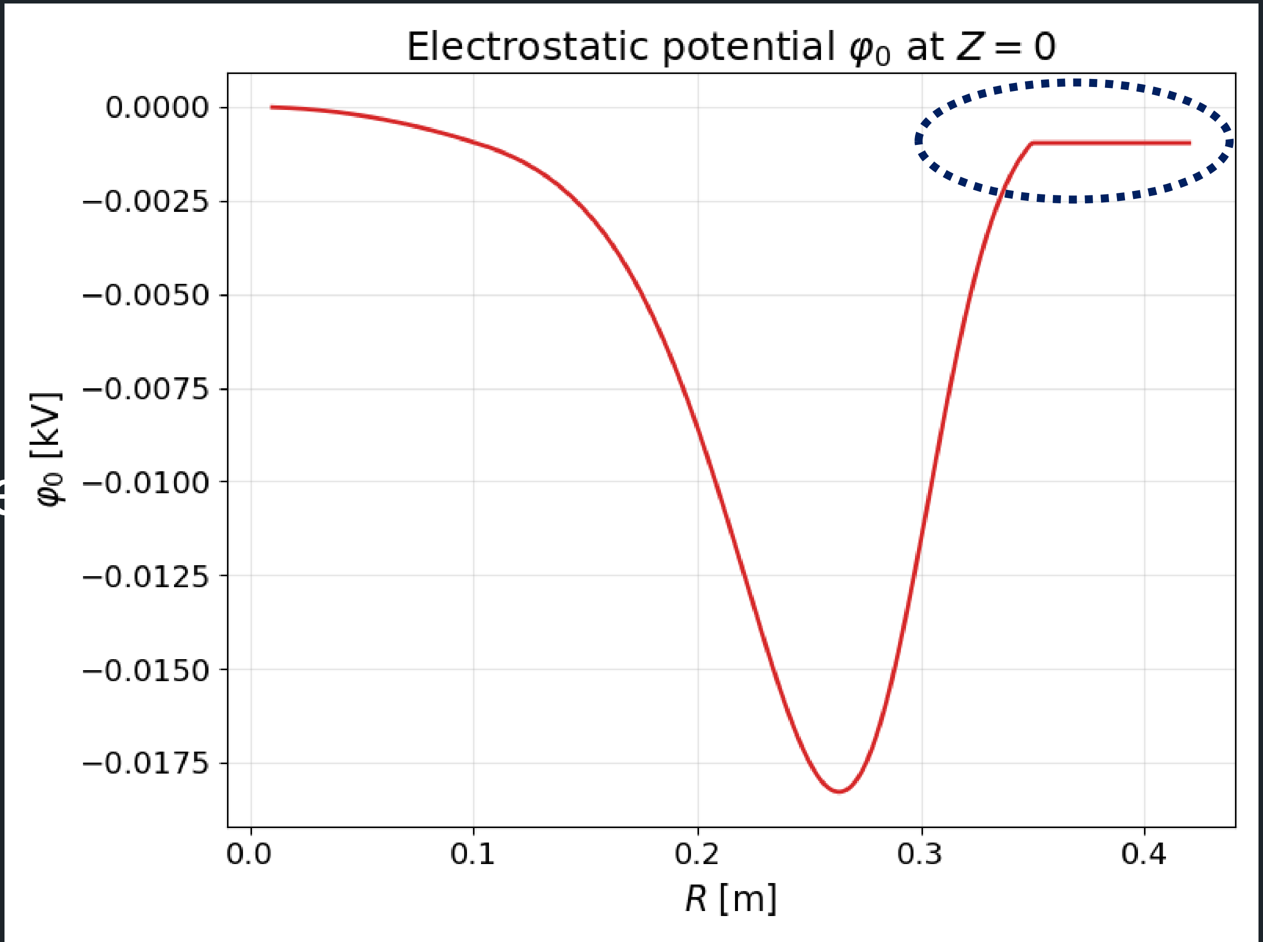
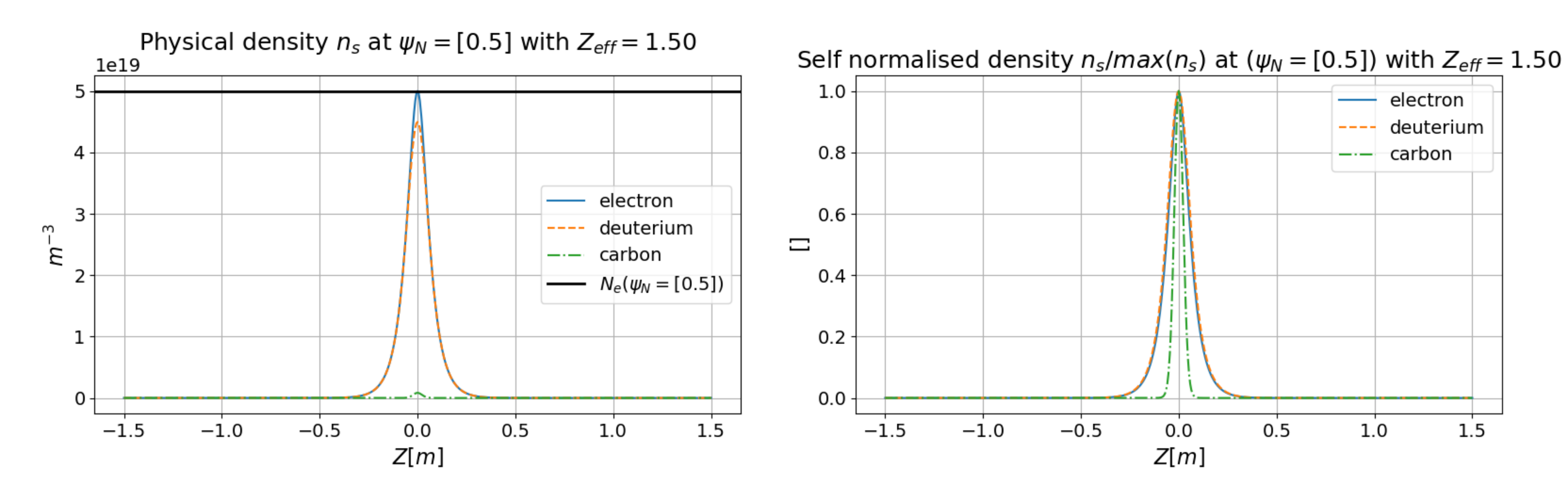


Density variation along field line

- Localisation around Z=0 (point of max R, min B)
- Quasineutrality maintained with strong impurity peaking
  - Peaking scales like exp(mass) so tungsten would be a lot worse



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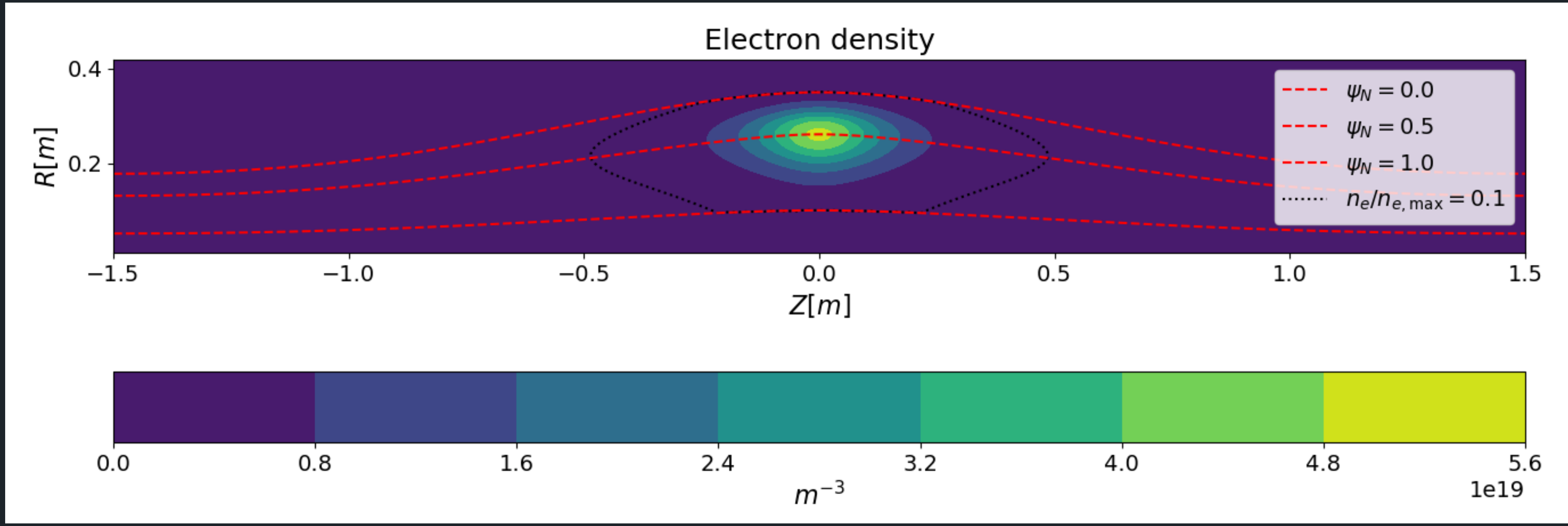

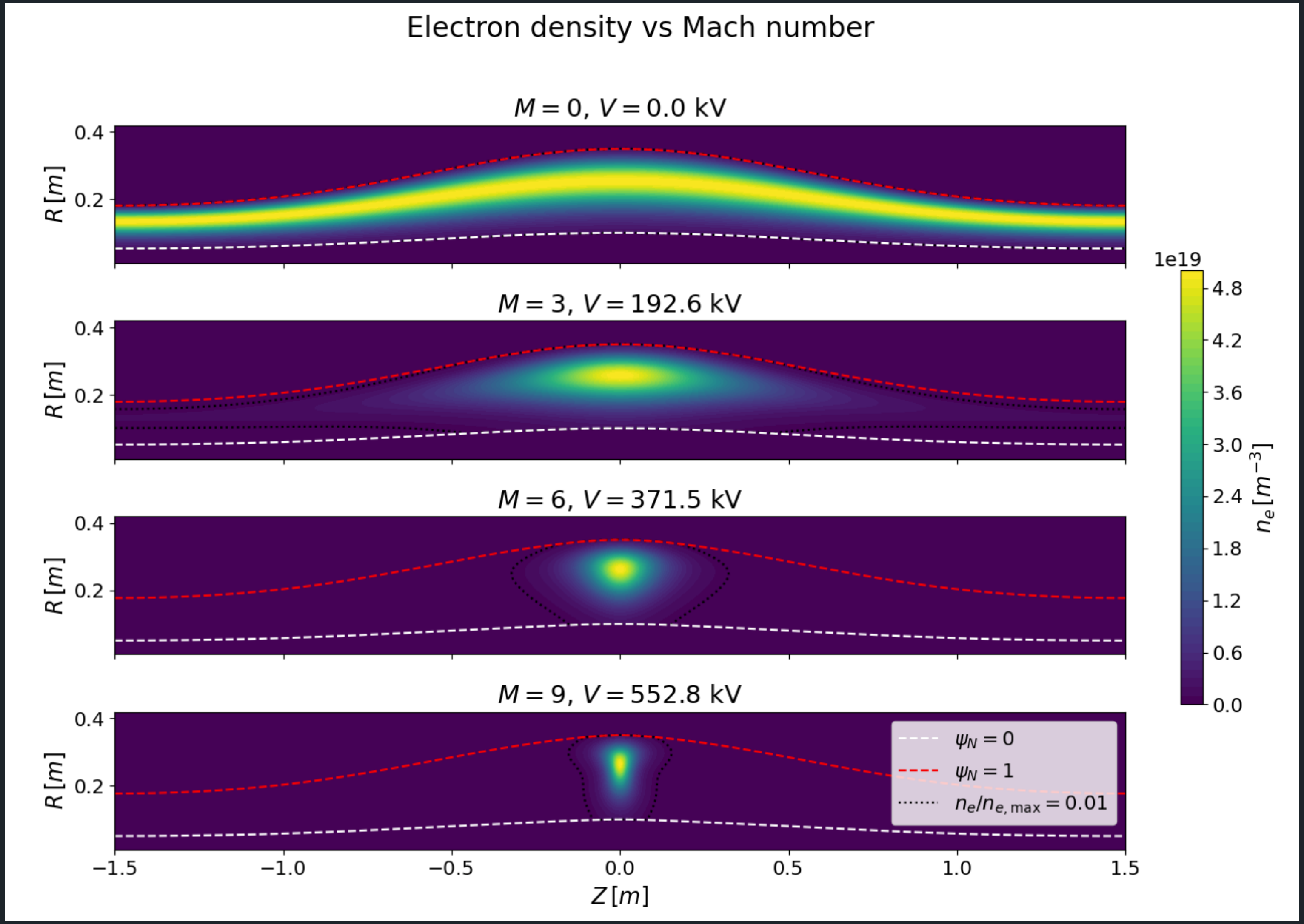

Electron density vs Mach number
$M = 0$, $V = 0.0$ kV
$M = 3$, $V = 192.6$ kV
$M = 6$, $V = 371.5$ kV
$M = 9$, $V = 552.8$ kV
$R\,[m]$
0.4
0.2
$Z\,[m]$
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
1e19
4.8
4.2
3.6
3.0
2.4
1.8
1.2
0.6
0.0
$n_e\,[m^{-3}]$
$\psi_N = 0$
$\psi_N = 1$
$n_e/n_{e,\max} = 0.01$

Validity of Maxwellian distribution

Maxwellian distribution

- Assumed a Maxwellian distribution when determining solution for quasineutrality
- Not strictly correct for slowing down distributions
  - NBI ions and fusion alphas
  - *Relevant for any power plant machine*
- Doesn't handle anisotropic pressures
  - Loss mechanism preferentially impacts high $v_{\parallel}$, low $v_{\perp}$
  - NBI ions and ICRH heated plasmas

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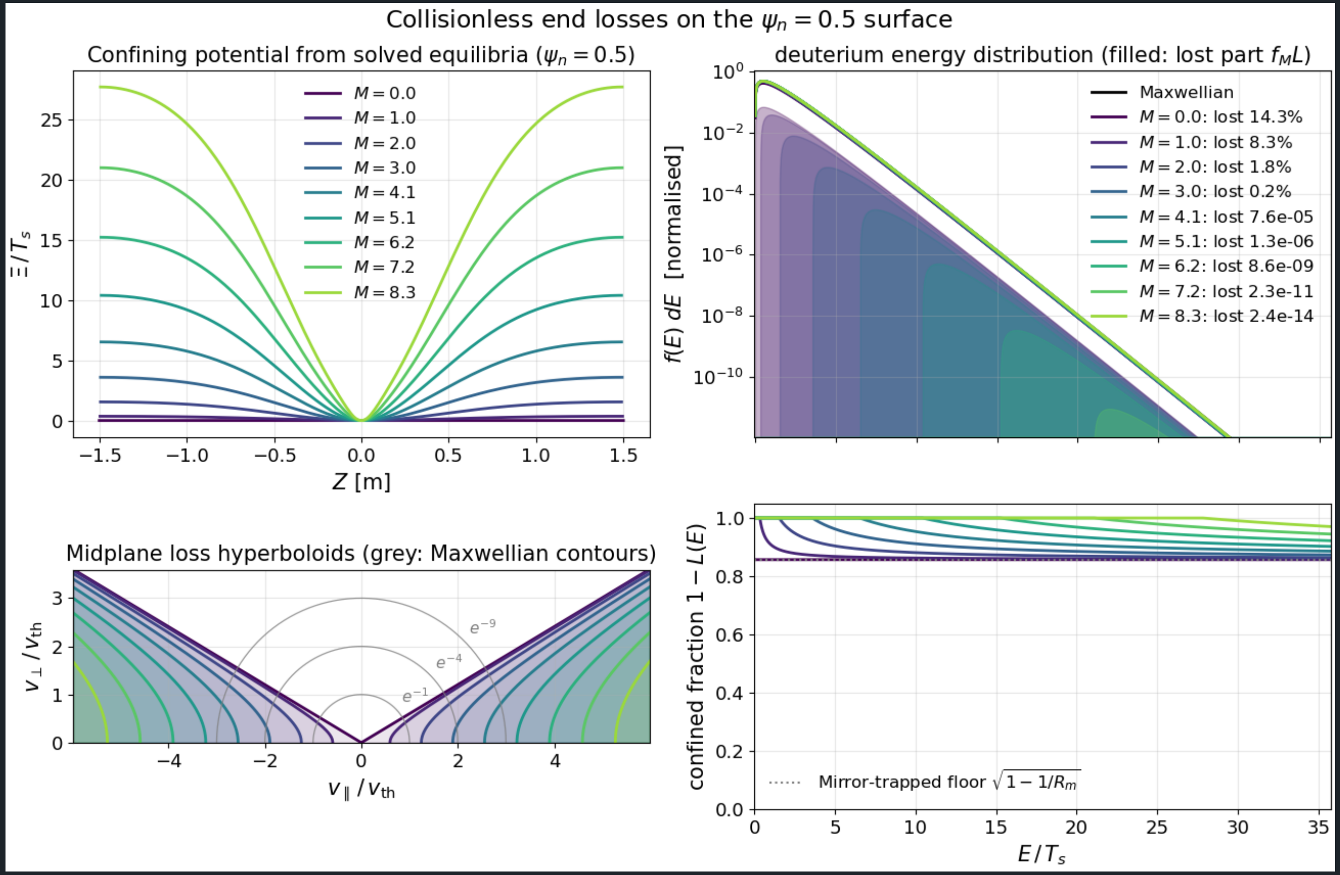
Loss rate - collisionless

$$f_{\text{lost}} = \sqrt{1 - 1/R_m}$$

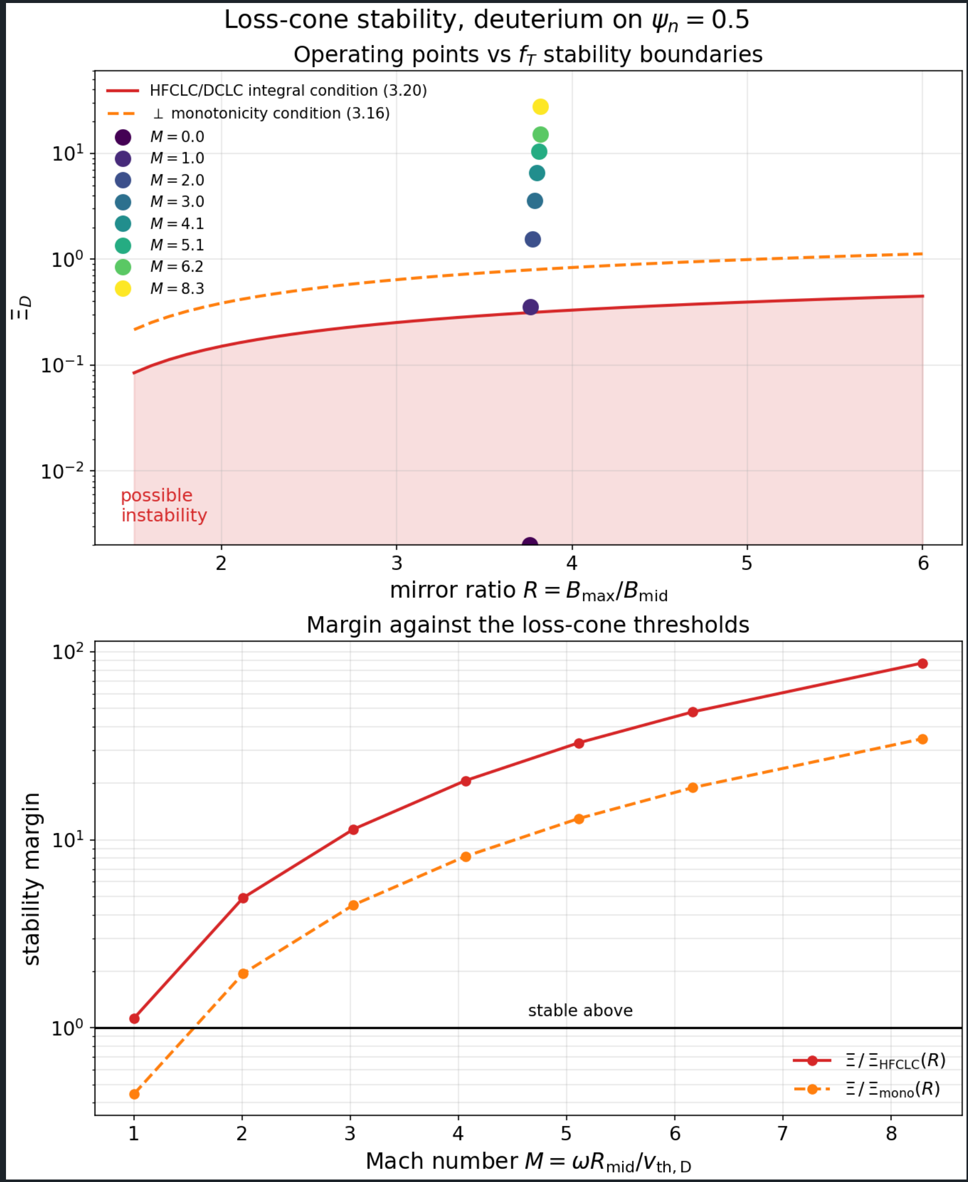
- Collisionless end losses on conventional mirrors mean poor confinement
  - Energy uniformly lost at all energies

- Additional potential closes loss cone at low velocity
  - Only very energetic particles can overcome hill



Loss cone instabilities

- Loss cone preferentially removes low $v_\perp$ particles creating a population inversion
  - Drives several instabilities like HFCLC, DCLC, DGH
  - HFCLC easiest to destabilise
- Kolmes et al uses a Maxwellian with Heaviside function to find a stability criteria for these modes [1]
  - With sufficient rotation, inversion can be avoided
- Generally satisfied for these conditions
  - What happens if $R = B_{max} / B_{min}$ increases significantly?



[1] Kolmes *et al* J. Plasma Phys. (2024), vol. 90, 905900203

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Loss rate - collisional

- Collisions can "bump" particles into the loss cone
  - Ideally need a FP/1D-2V code to model this loss mechanism

- Pastukhov result shows end losses in a mirror configuration [1,2]
  - Takes square well approximation - no variation of distribution along field line
    - *Could instead look at bounce average Fokker-Planck equation?*
  - Benchmarked against Fokker-Planck simulations [2]

$$\left(\frac{\partial n_s}{\partial t}\right)_{\|} = -\left(\frac{2 n_s \Sigma}{\sqrt{\pi}}\right)\nu_s \frac{1}{\ln\left(R_{\mathrm{m}}\Sigma\right)} \frac{\exp\left(-\Xi_s/T_s\right)}{\Xi_s/T_s} \qquad \Xi_s = q_s(\varphi_0+\varphi_1) - \frac{m_s \omega^2 R^2}{2}$$

- Can determine additional loss rate from collisions
  - Ambipolarity sets up an additional potential ϕ_1 as electrons can still leak out faster (not included in equilibrium calculation)

[1] V.P. Pastukhov *et al* Nucl. Fus. 14 1974
[2] R.H. Cohen *et al* Nucl. Fus. 18 9 1978

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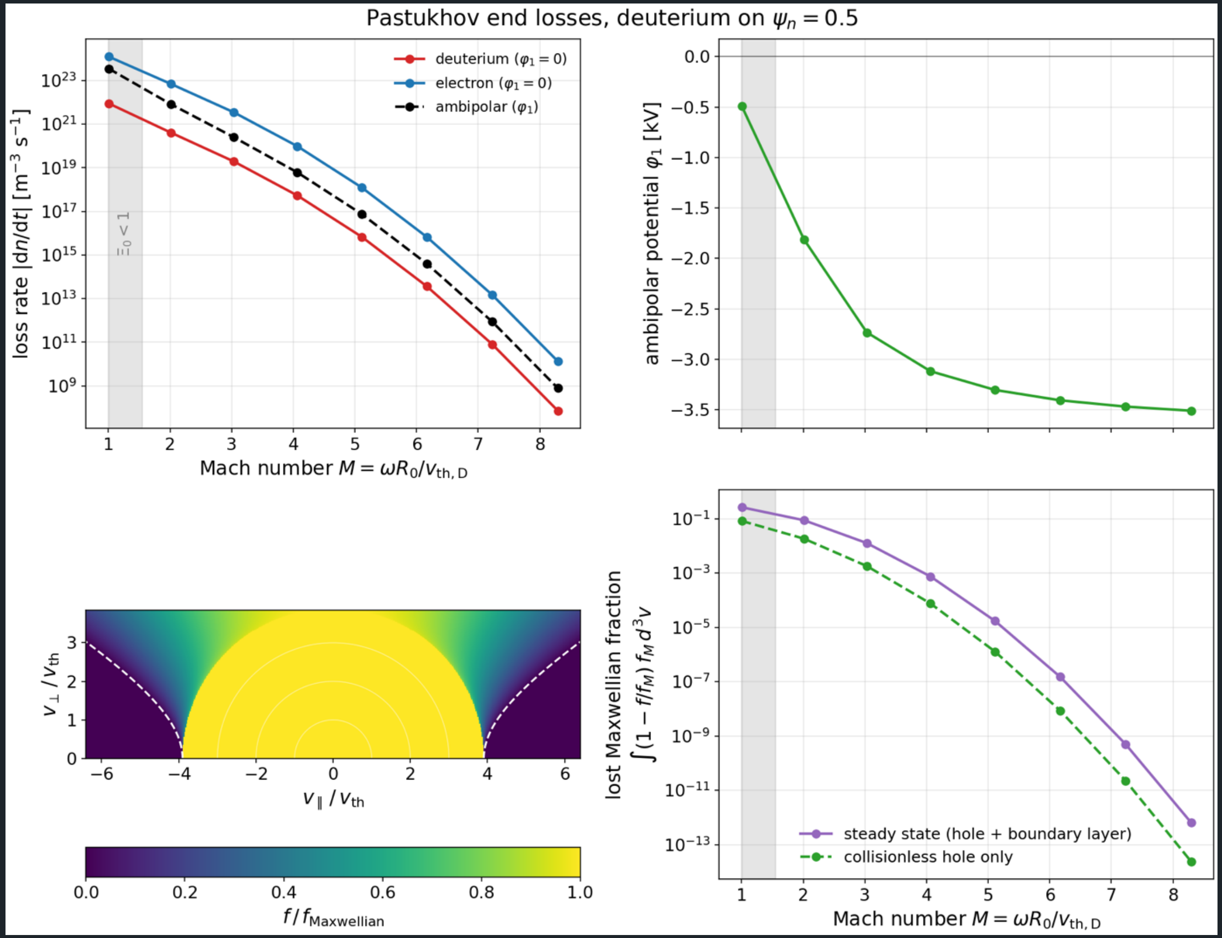
Pastukhov loss rate

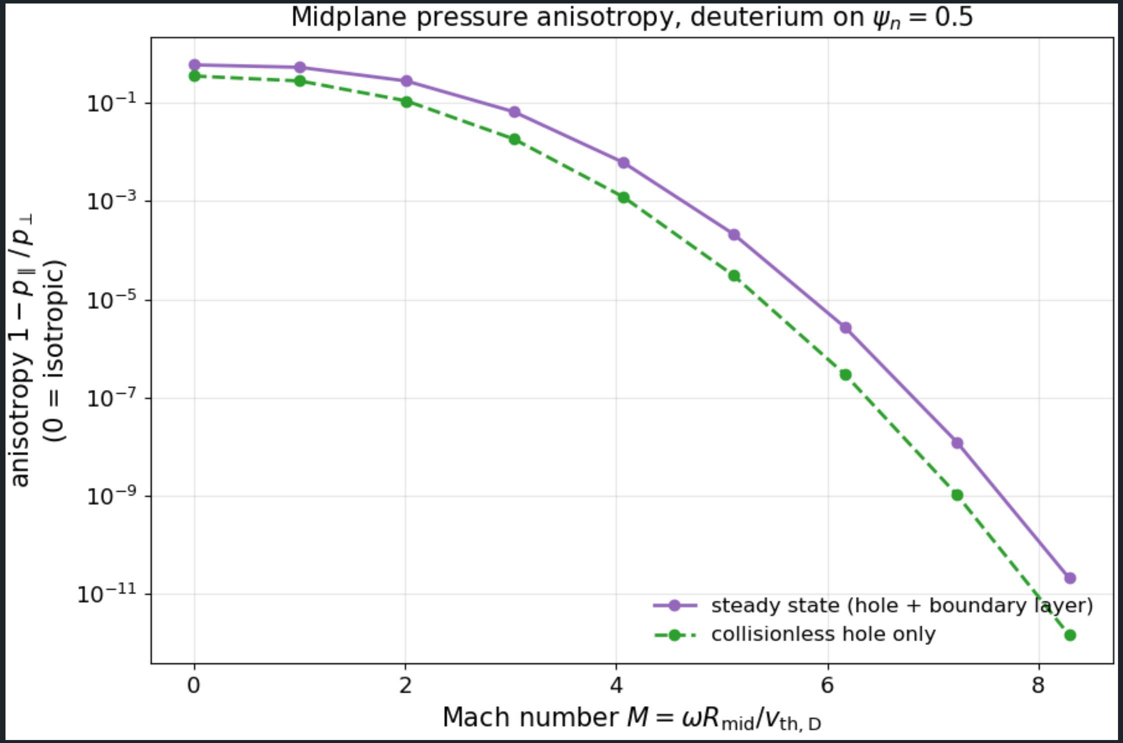
- Use ambipolarity to find new Ξ_s
  - *How to include φ_1 in equilibrium calculation?*
- Pitch angle scattering refills hyperboloid
  - No energy scattering
- Logarithmic distance from boundary empties into loss cone and escape out



Pressure anisotropy

$$f_{ani} = 1 - \frac{\int m v_\parallel^2 f d^3v}{\int \frac{m}{2} v_\perp^2 f d^3v}$$

- Given this Pastukhov style distribution function we can plot the level of pressure anisotropy from losses



Validity of Maxwellian

- Pushing to high Mach number significantly reduces the deviation from a Maxwellian
  - Collisionless loss cone shrinks significantly with losses ~1e-8
  - Pastukhov collisional transport increases losses to ~1e-6
- Most loss cone instabilities are stable when $M > 2$
- Minimal pressure anisotropy at high Mach number

Pushing to field reversal

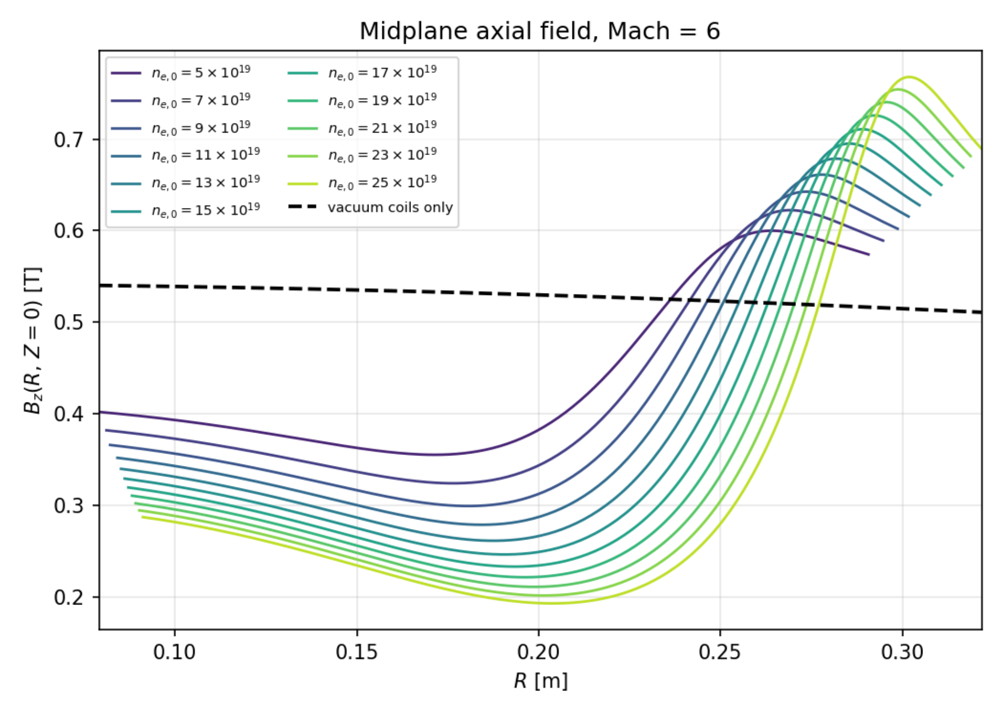
Plasma diamagnetism

- As the pressure/rotational gradients increase more toroidal current will be driven which will cancel out the vacuum B field

- What is the limit where this goes to 0?
  - ρ_* -> infinity so not best confinement regime
  - Large gradient in B_z forms…

- Happens when magnetic pressure completely balanced by thermal pressure?

- Define β_{dia} as metric for field reversal
  - As it approaches one we go to FRC state
  - FreeGSNKE methodology can't generate field reversed configuration
  - Essentially a measure of perpendicular pressure



$$\beta_{dia} = \left[1 - \left(\frac{B_z}{B_{vac}}\right)^2\right]_{min}$$

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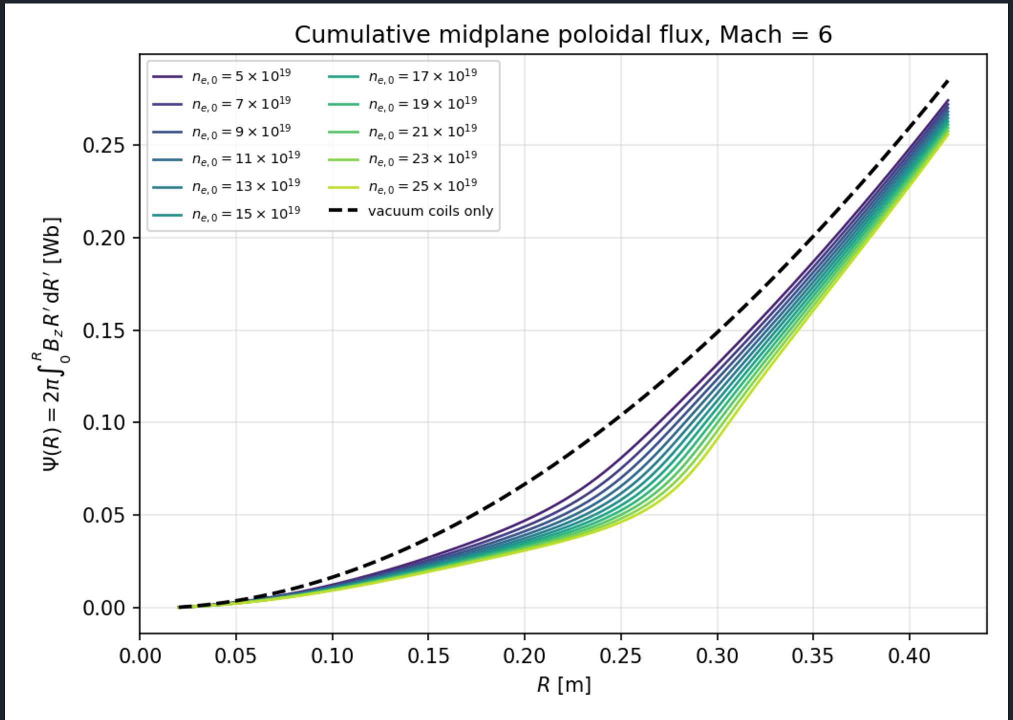

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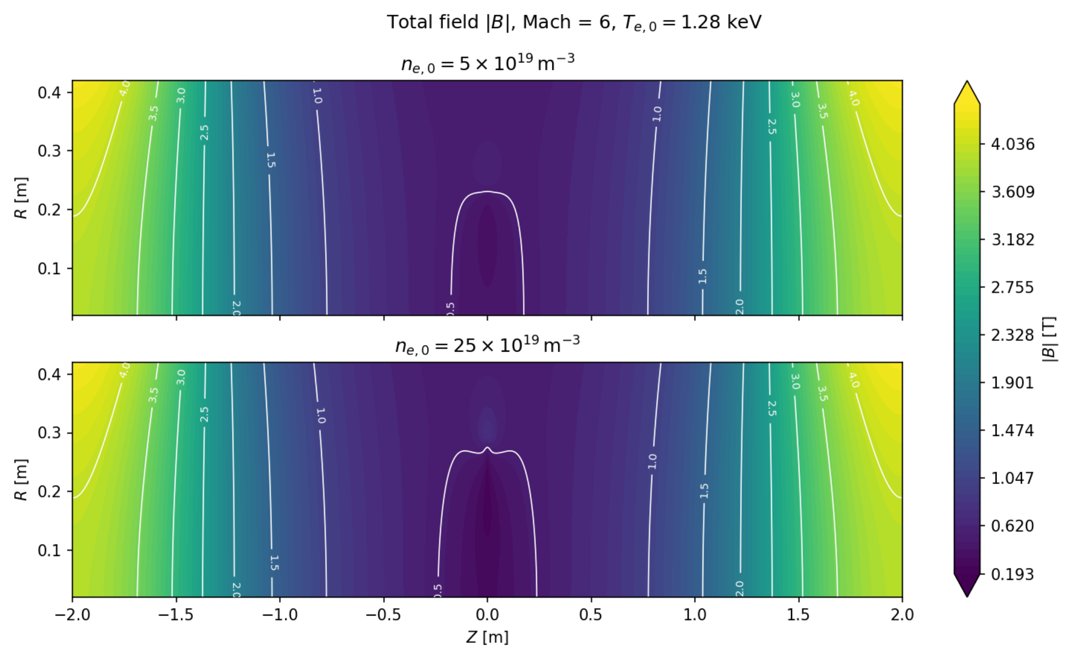

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Pushing to FRC

- Performed scans to determine limit
  - M: 4, 6, 8
  - Density: 1e19 -> 2.5e20
  - Temperature: 1keV -> 10keV
  - T_i/T_e: 1.2, 2.0
  - Mirror ratio: 3.8, 7.8
  - Magnet field strength: 4T, 6T
  - Boundary conditions at Z=2.0m
    - Radial shift outwards
    - Wider plasma

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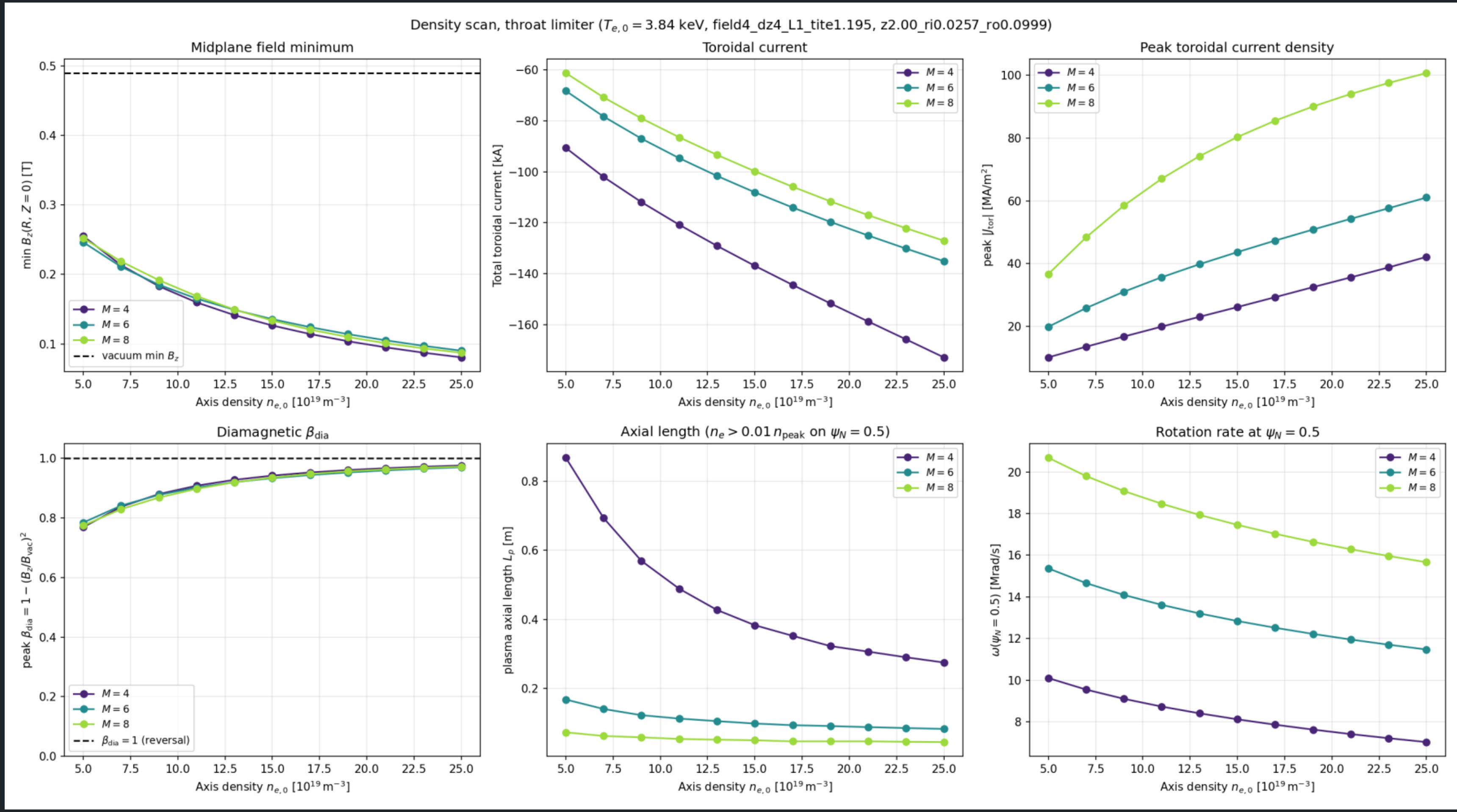
Mach number + density scan

- T_e = 3.8keV
- T_i = 4.8keV
- B_{min} = 0.5T
- B_{max} = 4.0T
- R_{inner} = 0.03m
- R_{outer} = 0.10m
- $Z_{limiter}$ = 2.0m



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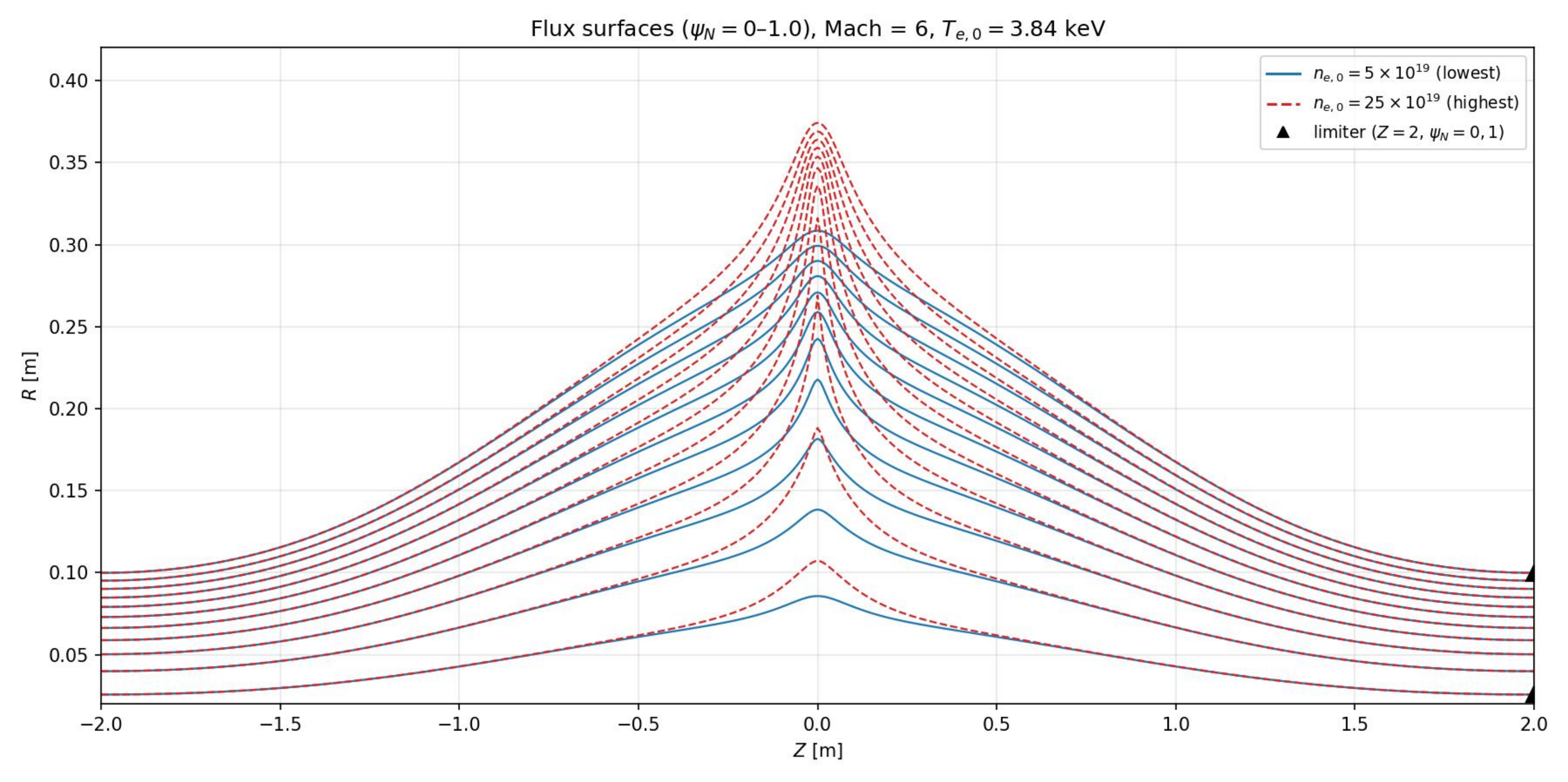

Dimensionless parameters

- $n_e = 1e20 m^{-3}$
- $B_{min} = 0.5T$
- $B_{max} = 4.0T$
- $R_{inner} = 0.03m$
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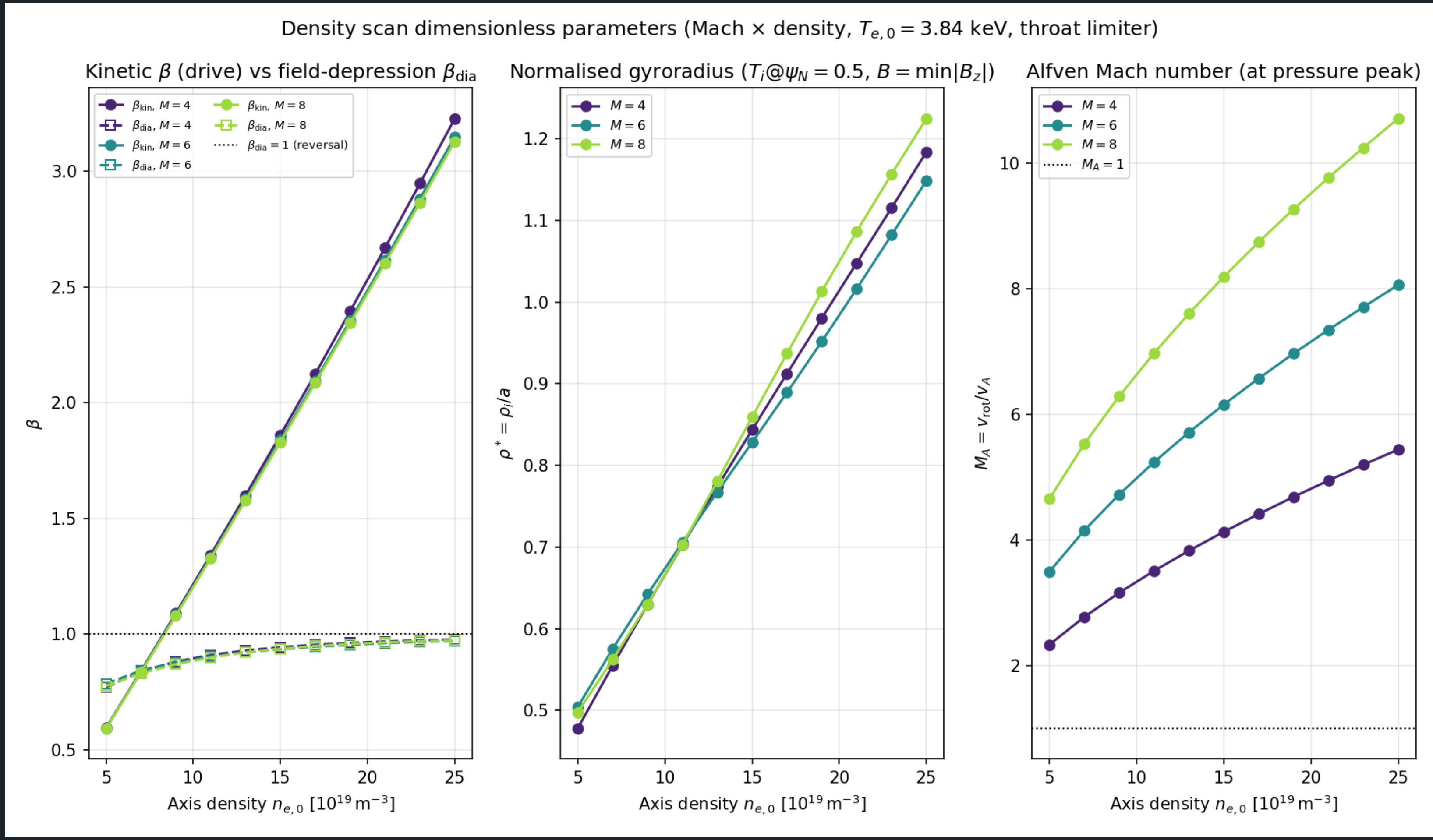
$$M_{alfven} = \frac{\omega R_0}{B/\sqrt{\mu_0 \rho}}$$



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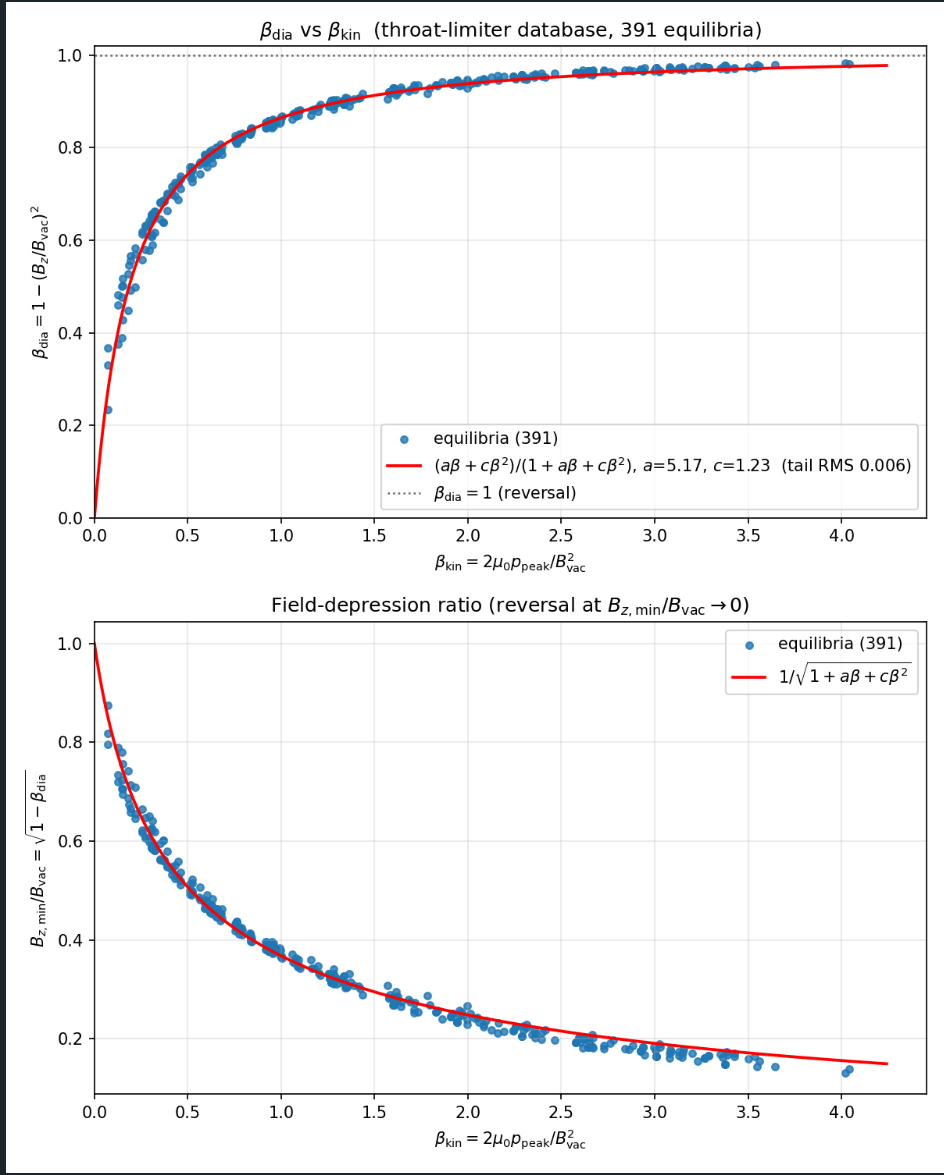
β limit

- Having done many scans in parameters overall find a soft β limit ~ 4-5
- Boundary conditions at Z=2.0m
- FreeGSNKE cannot model an actual FRC configuration

$$\frac{1}{1-\beta_{dia}} = \left(\frac{B_{\mathrm{vac}}}{B_{z,min}}\right)^2 \approx 1 + a\,\beta_{\mathrm{kin}} + c\,\beta_{\mathrm{kin}}^2$$



Radial force balance

- Start with radial force balance

$$\frac{d}{dR}\left(p+\frac{B_z^2}{2\mu_0}\right) = \underbrace{\rho\,\omega^2 R}_{\text{centrifugal}} + \underbrace{\frac{B_z}{\mu_0}\frac{\partial B_r}{\partial Z}}_{\text{tension}=(B_z^2/\mu_0)\kappa}$$

- Integrate from field minimum R_m to to edge R_e

$$\left(p+\frac{B_z^2}{2\mu_0}\right)_{R_e} - \left(p+\frac{B_z^2}{2\mu_0}\right)_{R_m} = \underbrace{\int_{R_m}^{R_e} \rho\,\omega^2 R\,dR}_{\equiv\, C} + \underbrace{\int_{R_m}^{R_e} \frac{B_z}{\mu_0}\frac{\partial B_r}{\partial Z}\,dR}_{\equiv\, T}$$

- Re-arrange for β_{dia}

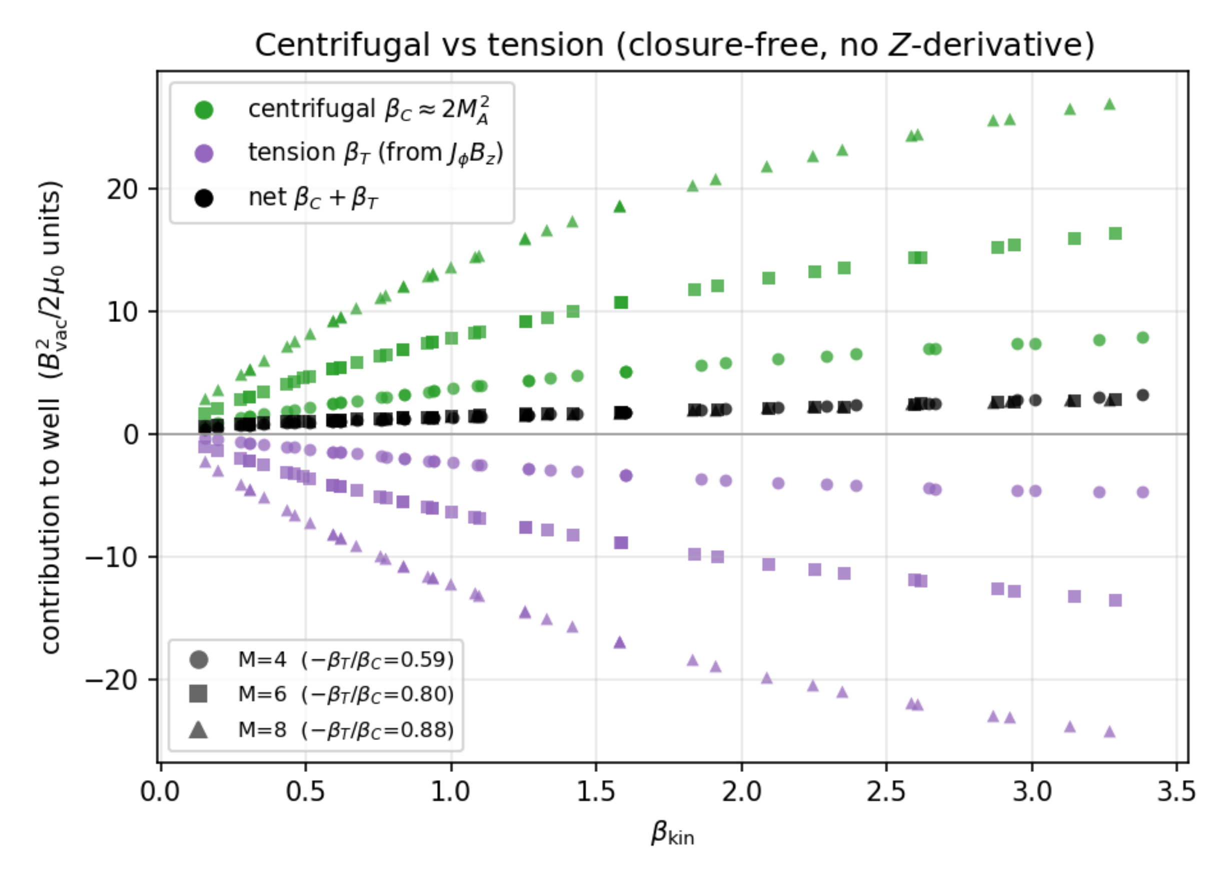
$$\underbrace{\frac{B_{\text{vac}}^2 - B_{z,min}^2}{B_{\text{vac}}^2}}_{=\ \beta_{dia}} = \underbrace{\frac{B_{\text{vac}}^2 - B_{z,e}^2}{B_{\text{vac}}^2}}_{\text{edge offset}} + \frac{2\mu_0}{B_{\text{vac}}^2}\Big[(p_m - p_e) + C + T\Big]$$

$$\beta_{dia} = \beta_{\text{ext}} + \beta_p + \left(\beta_C + \beta_T\right)$$

Tension vs rotation

$$\beta_{dia} = \beta_{\text{ext}} + \beta_p + (\beta_C + \beta_T)$$

- Bowing of field lines generates magnetic tension
- Curvature largely balances centrifugal pressure
- Residual kinetic pressure balances diamagnetic well



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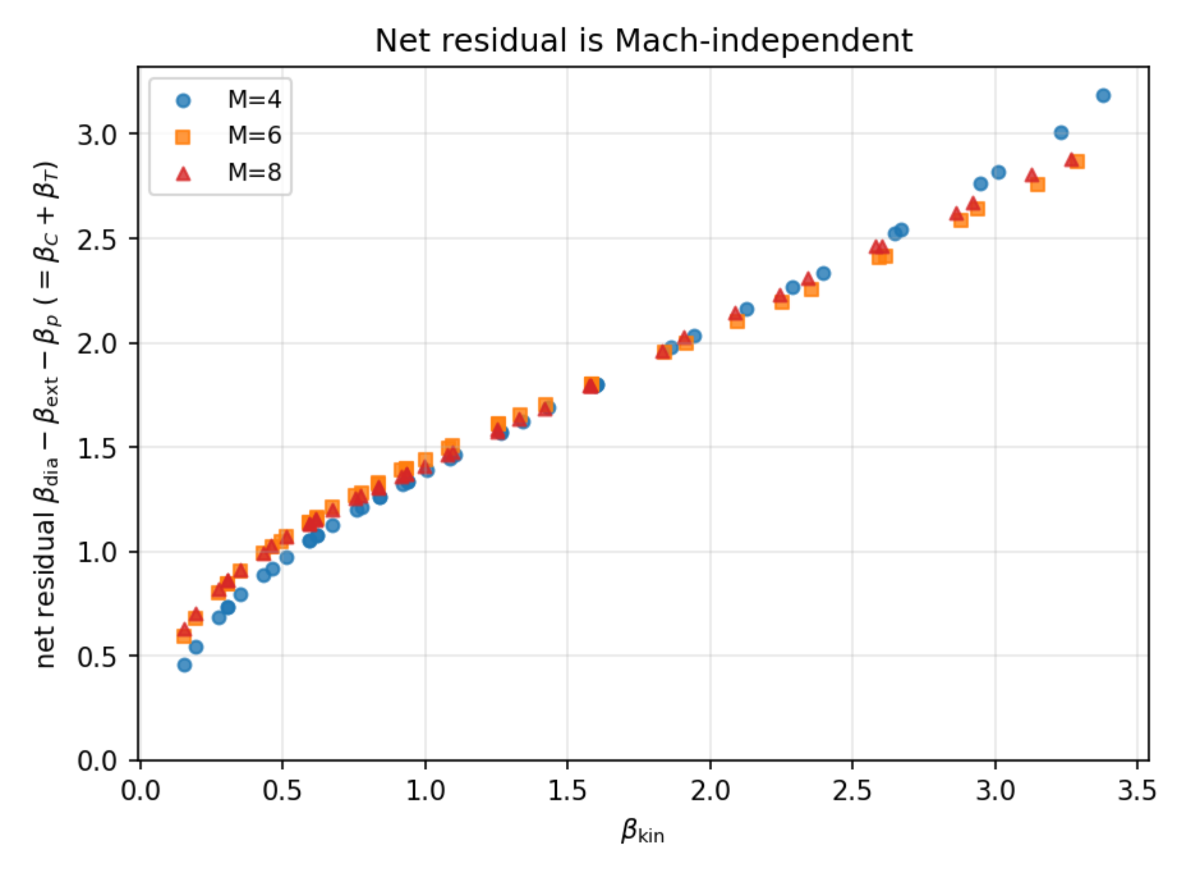
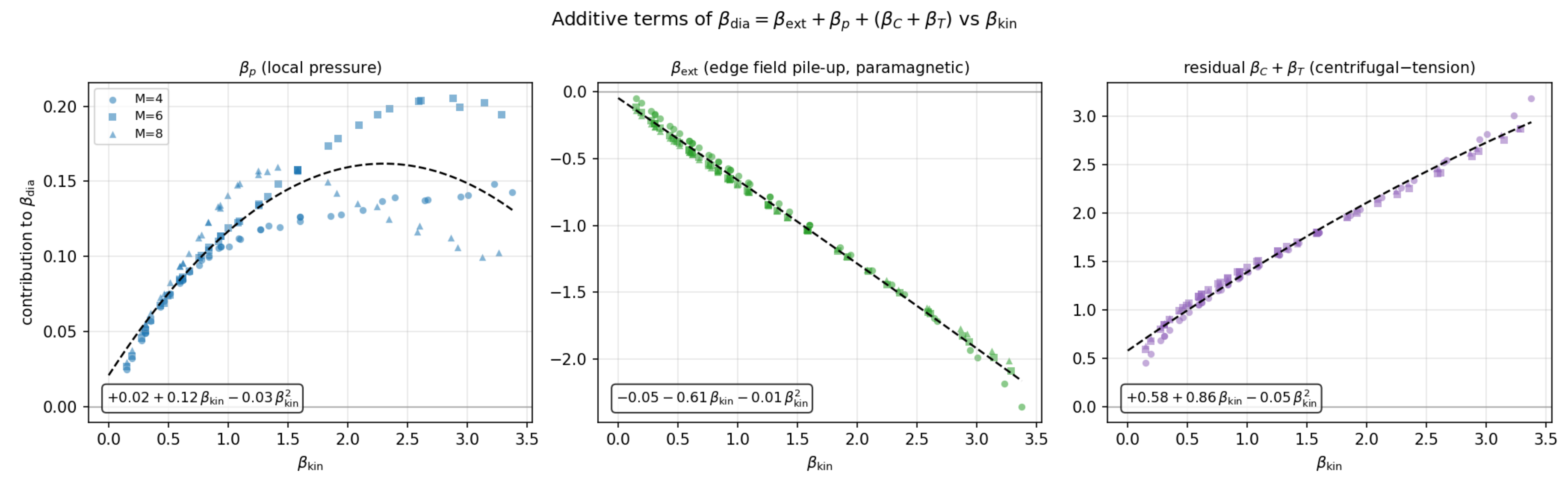
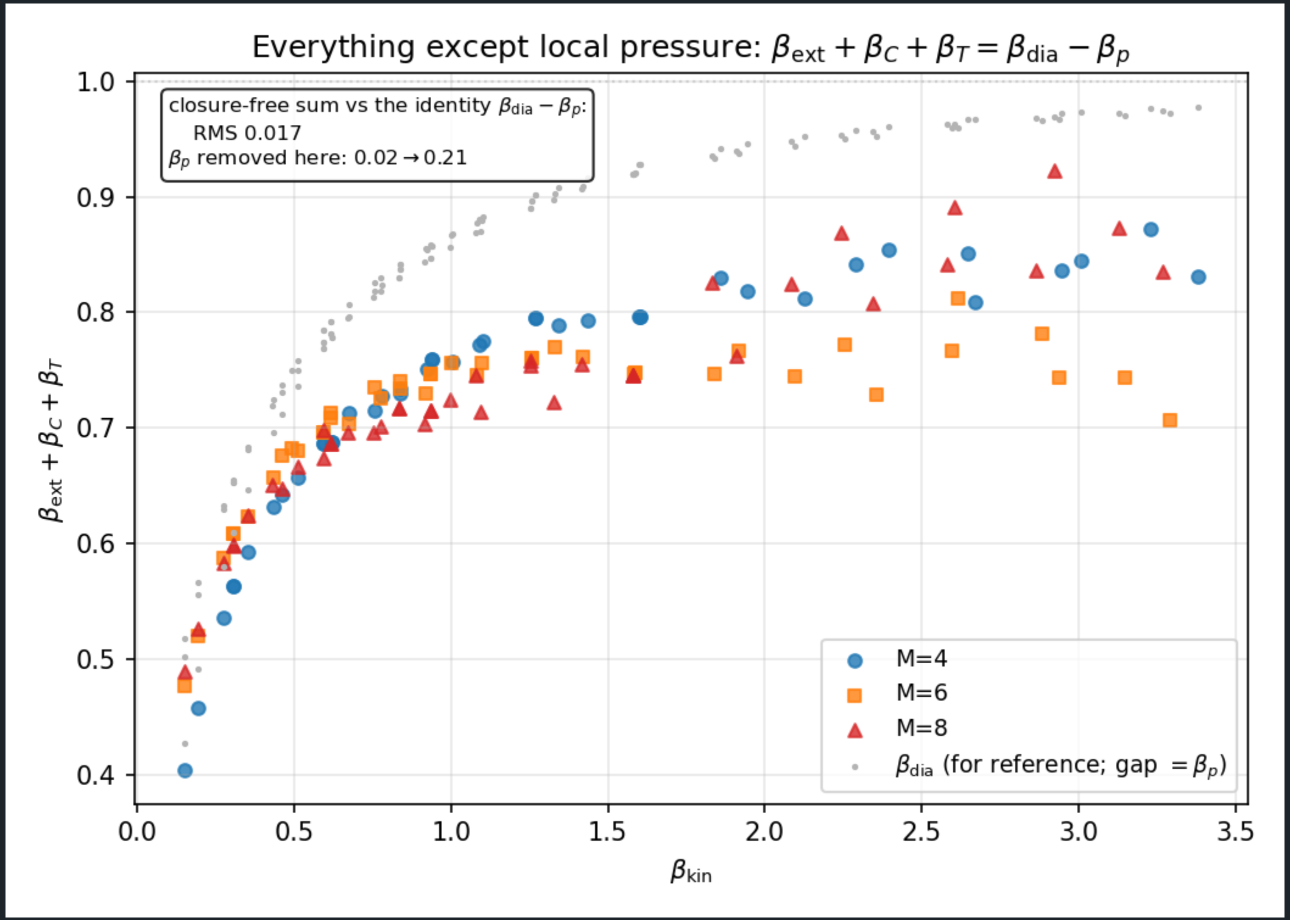


β breakdown


Additive terms of $\beta_{dia} = \beta_{ext} + \beta_p + (\beta_C + \beta_T)$ vs β_{kin}
β_p (local pressure)
β_{ext} (edge field pile-up, paramagnetic)
residual $\beta_C + \beta_T$ (centrifugal−tension)
contribution to β_{dia}
M=4
M=6
M=8
$+0.02 + 0.12\,\beta_{kin} - 0.03\,\beta_{kin}^2$
$-0.05 - 0.61\,\beta_{kin} - 0.01\,\beta_{kin}^2$
$+0.58 + 0.86\,\beta_{kin} - 0.05\,\beta_{kin}^2$
β_{kin}

β breakdown

- Residual from tension + centrifugal + external field shows saturation
- β_p pushes to limit to 1



Summary

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- Developed the capability to solve rotating Grad-Shafranov equation using FreeGSNKE
- Modelling shows at high Mach number, deviation from a Maxwellian is minimal and equilibria sit well above loss cone instabilities
- β limit found for field reversal in wide range of equilibria
  - Can reasonably predict where this happens but why