Phase-space turbulence of ion-acoustic fluctuations

Rishin Madan
Robert J. Ewart
Matthew W. Kunz

17th Plasma Kinetics Working Meeting

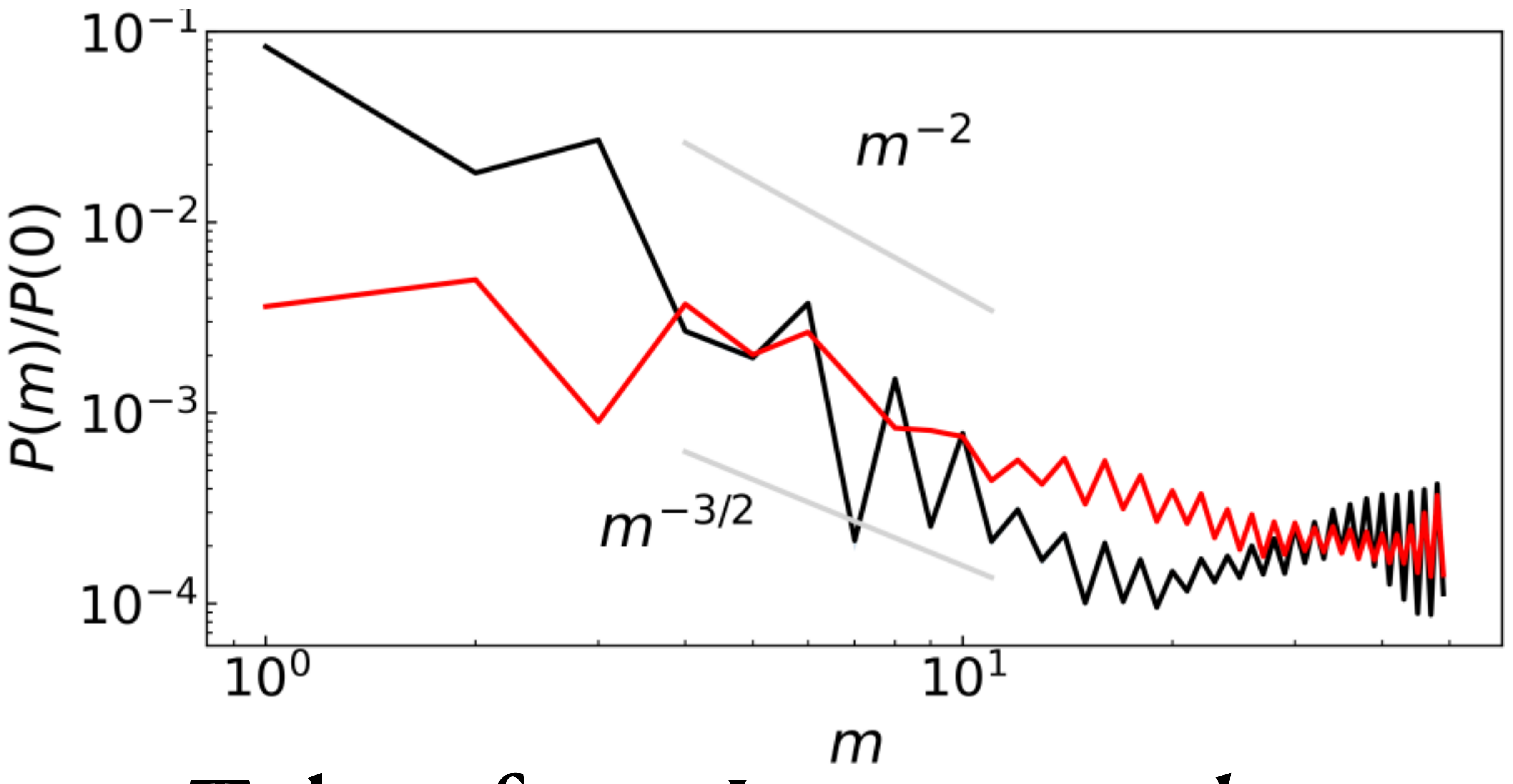
Setup

- A collisionless, 1D, electrostatic plasma is driven
  - on timescales $\gg \omega_{pe}^{-1}$
  - on lengthscales $\gg \lambda_{De}$
  - with $T_e \lesssim T_i$
  - creating strongly-damped (on ions) ion-acoustic fluctuations

$$\frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial x} - \frac{e}{m_i}\frac{\partial \phi}{\partial x}\frac{\partial f_i}{\partial v} = 0$$

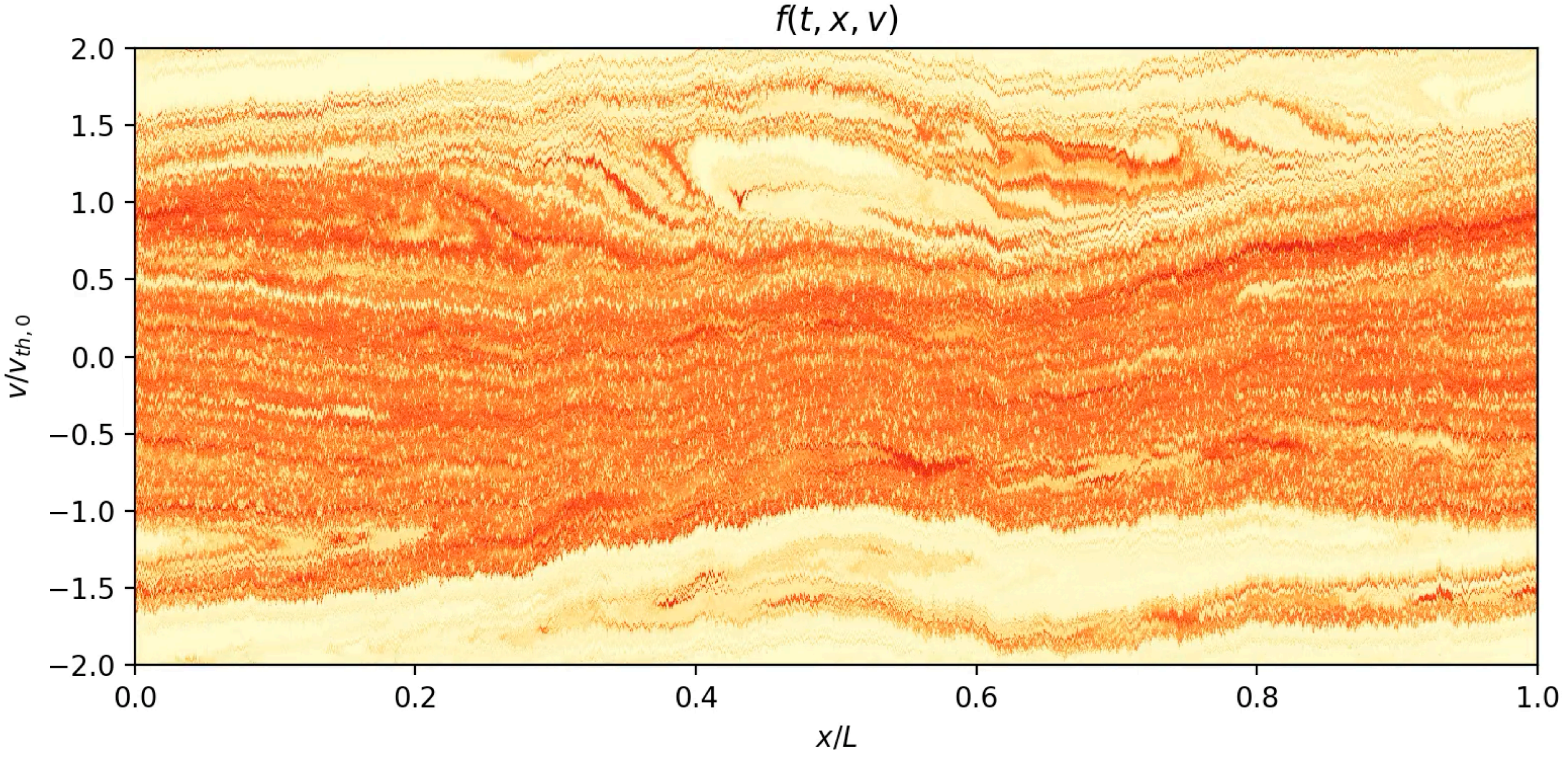
$$\frac{e\phi}{T_e} = \frac{\delta n_i}{n_0}$$

- **Why?**
  - 'Simple' example of disparate scales interacting in collisionless turbulence
  - Towards explaining (measurable) statistics in the phase-space density of ions in space plasmas



Taken from Larosa *et al.* 2025

What happens?

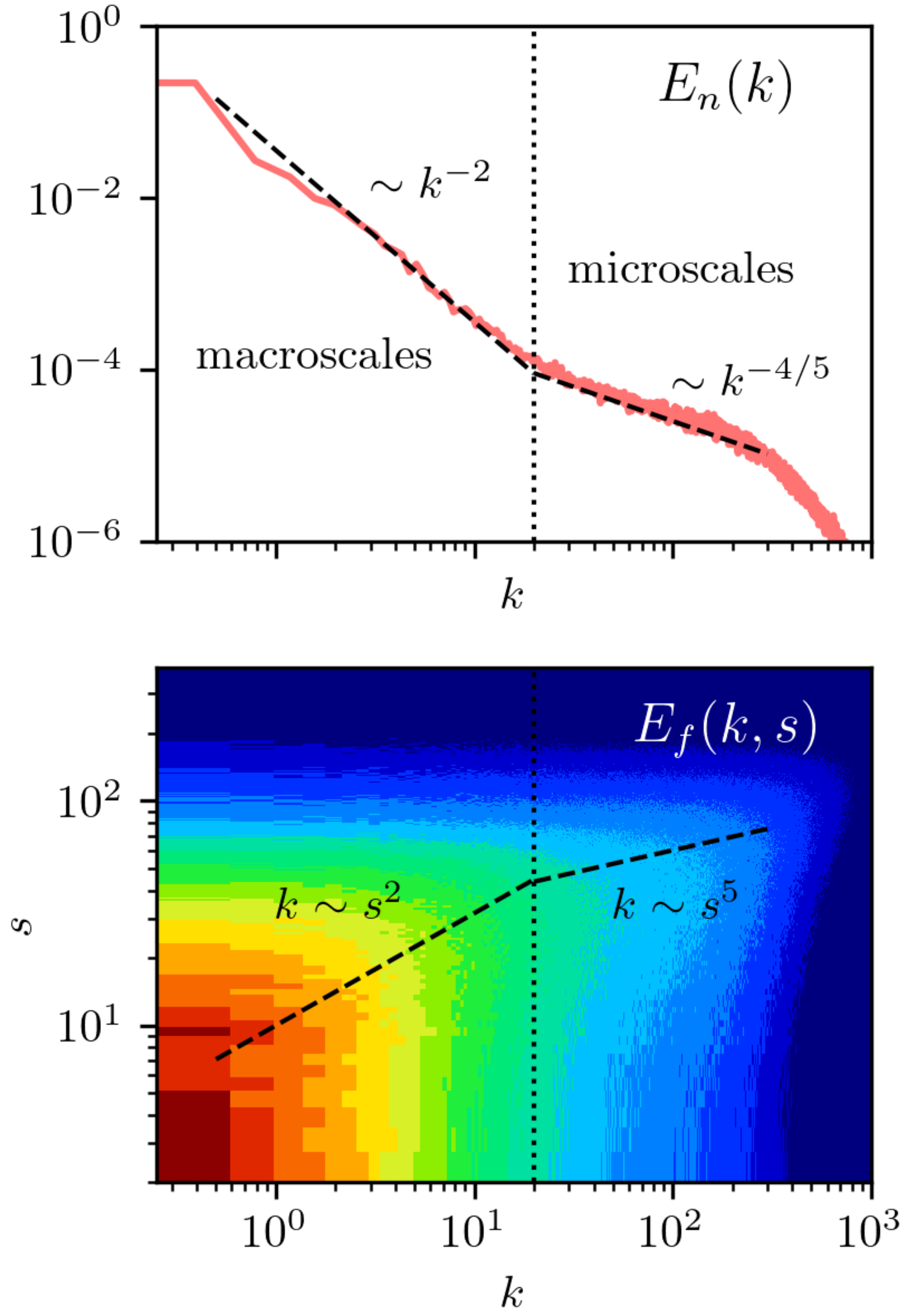


- $T_e/T_p = 0.05$
- Cell size takes place of Debye length
- 17280 cells
- 50,000 ppc
- Compressive time-correlated driving at box-scale

- Phase mixing → beam-like structures → two-stream microinstabilities

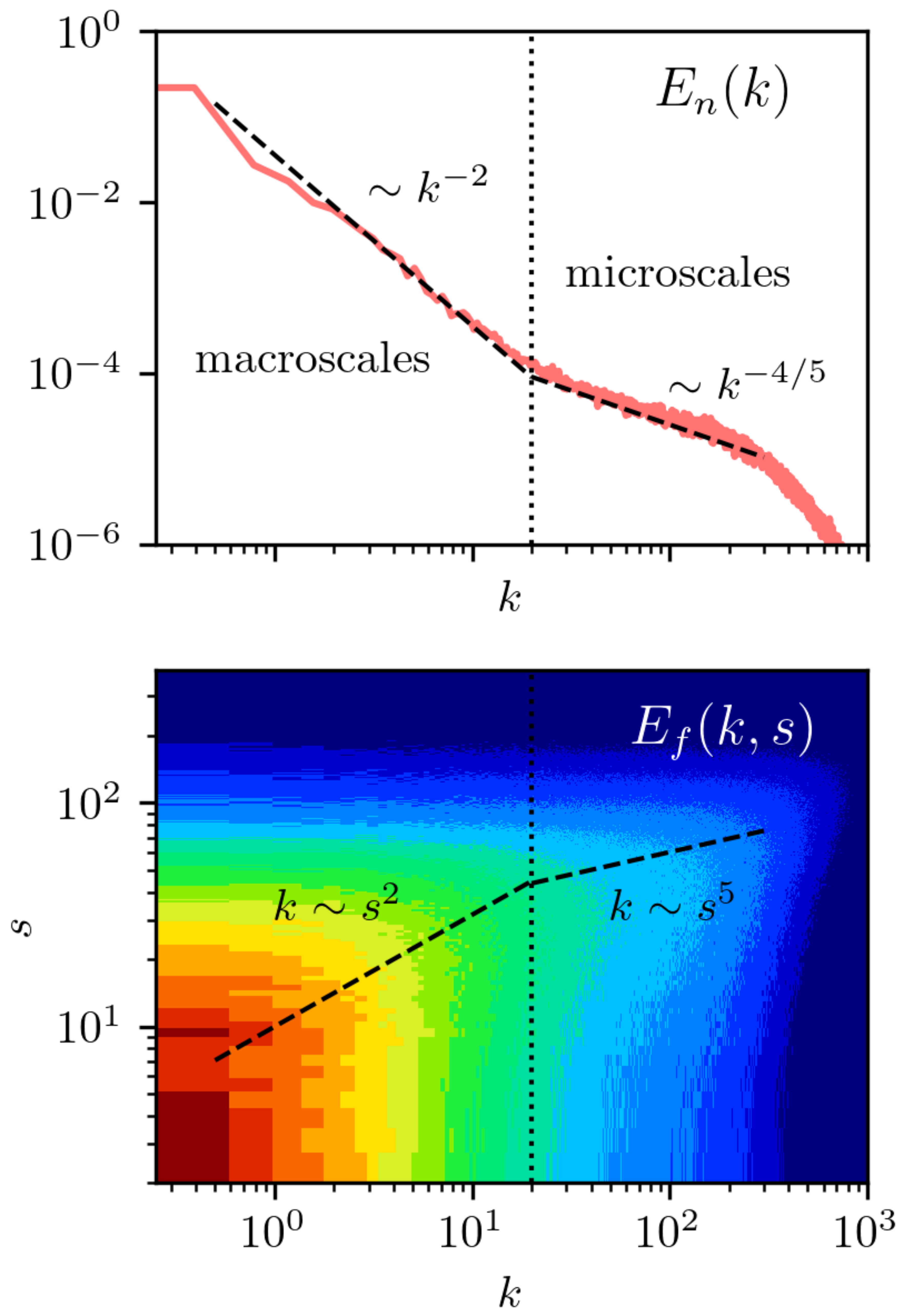
Two-scale system

- Microscales:
  - Fed by microinstabilities
  - Dominated by a bath of ion holes
- Macroscales:
  - Entropy cascade
  - Phase-space structures go unstable at Kolmogorov scale



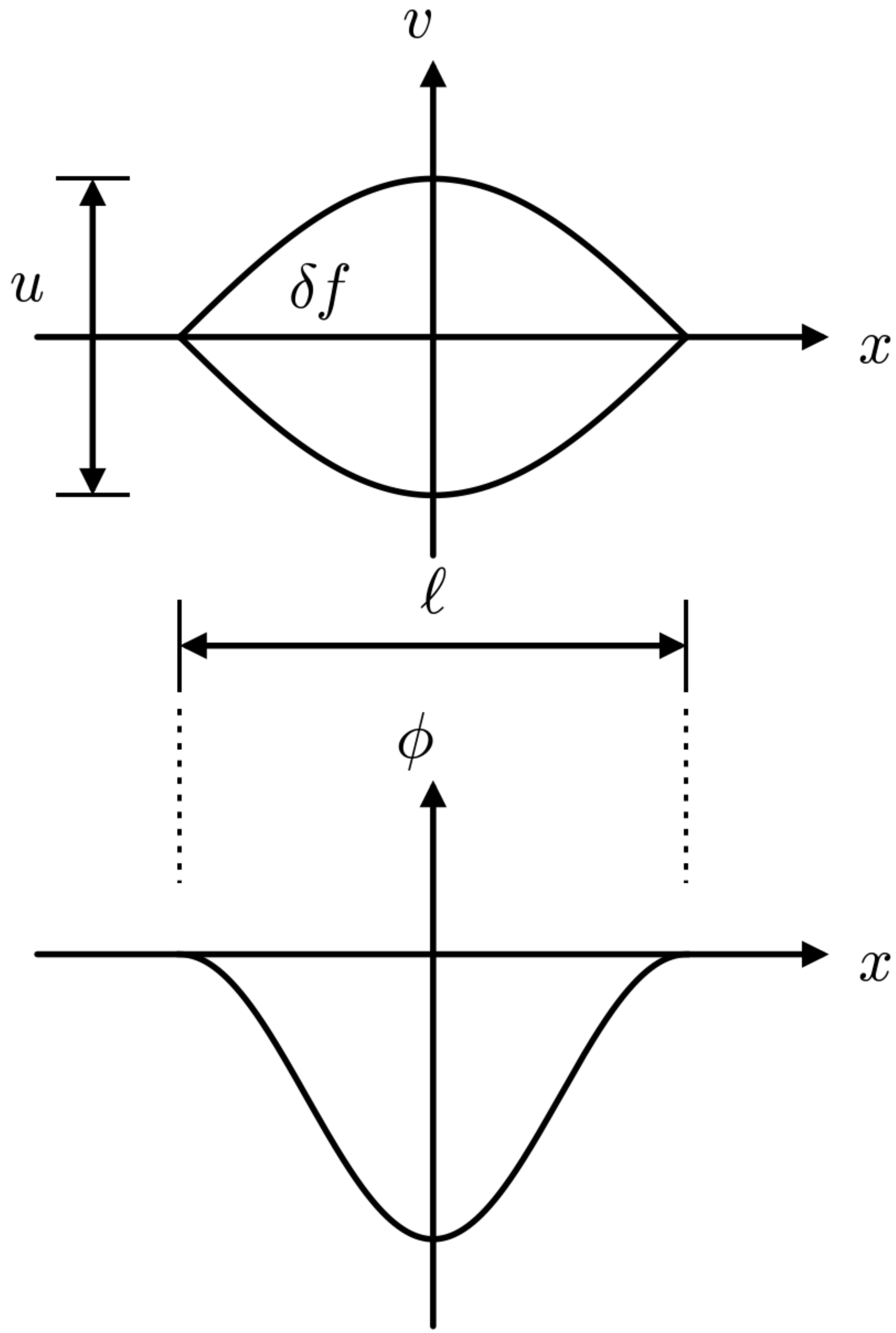
Two-scale system

- Microscales:
  - Fed by microinstabilities
  - Dominated by a bath of ion holes
- Macroscales:
  - Entropy cascade
  - Phase-space structures go unstable at Kolmogorov scale



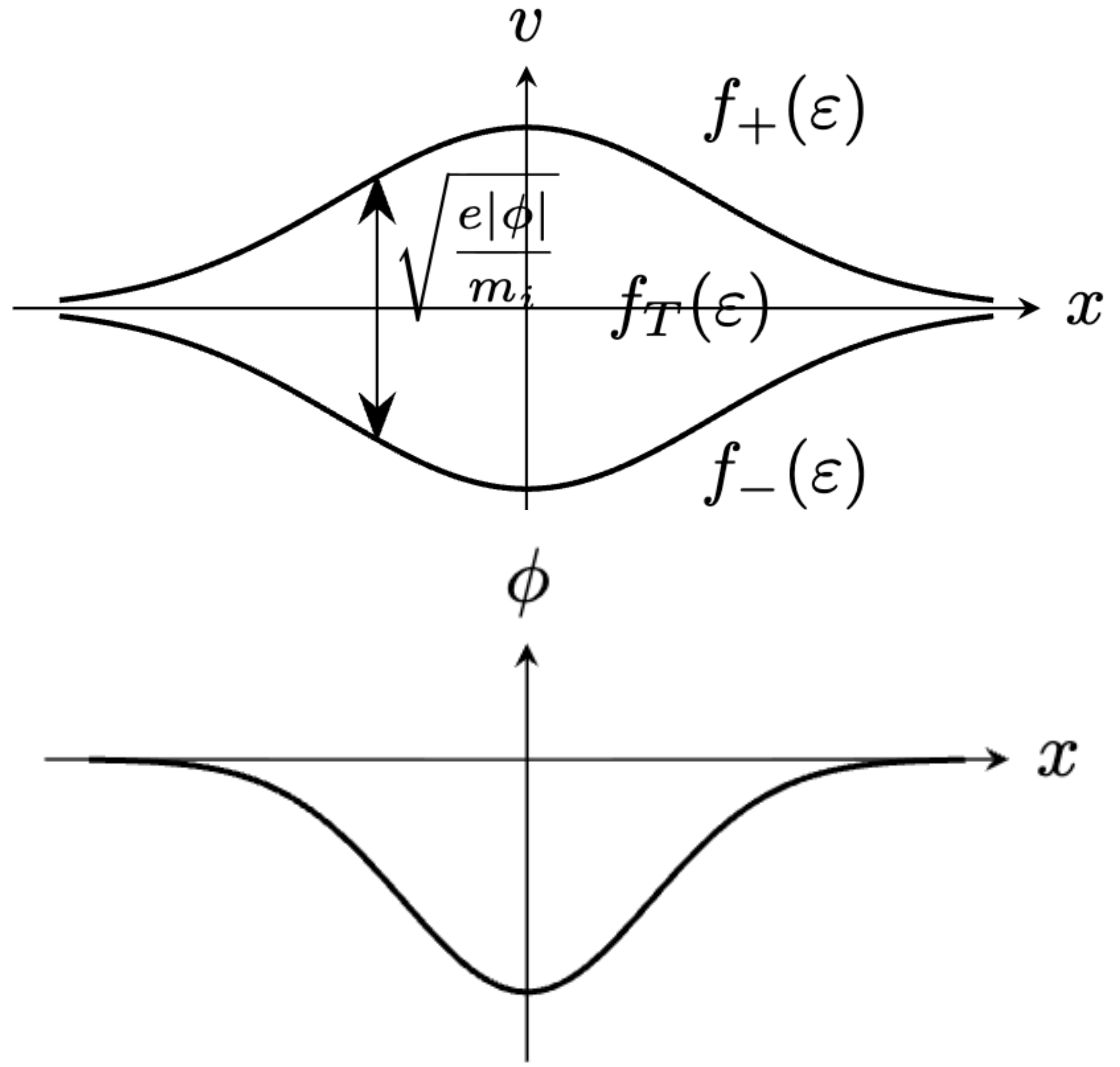
Ion holes

- Hole characterized by
  - Length $\ell \sim 1/k$
  - Height $u \sim 1/s \sim \sqrt{e\phi/m_i}$
  - Depth δf
  - Depth of potential well ϕ
- Holes merge to form larger holes
- What is the steady-state distribution of holes?



Ion holes as BGK modes

Step 1: derive relationship between δf and ϕ



- Exact, nonlinear, stationary (in hole frame) solutions
- BGK (1957) framework solves for trapped particle distribution:

$$f_T(\varepsilon) = \frac{\sqrt{2m_i}}{\pi}\int_\varepsilon^0 \mathrm{d}(e\phi)\,\frac{\frac{\mathrm{d}n(e\phi)}{\mathrm{d}(e\phi)}}{\sqrt{e\phi-\varepsilon}}, \qquad \varepsilon = \frac{m_i v^2}{2} + e\phi$$

where density of trapped-electrons is n_0 plus

$$n(e\phi) := \frac{n_0 e\phi}{T_e} - \sum_{\sigma=\pm}\int_0^\infty \frac{\mathrm{d}\varepsilon}{\sqrt{2m_i}}\frac{f_\sigma(\varepsilon)}{\sqrt{\varepsilon - e\phi}}$$

- Electron-Boltzmann term leads to $f_T(\varepsilon) \propto \sqrt{-\varepsilon}$: ‘mound’ not ‘hole’
- Hole behaviour arises with passing particle term in `cold' limit: $f_T(\varepsilon) \propto 1/\sqrt{-\varepsilon}$
- $\implies \delta f \propto \dfrac{1}{\sqrt{|\phi|}}$

Q: is there an obvious way to see this?

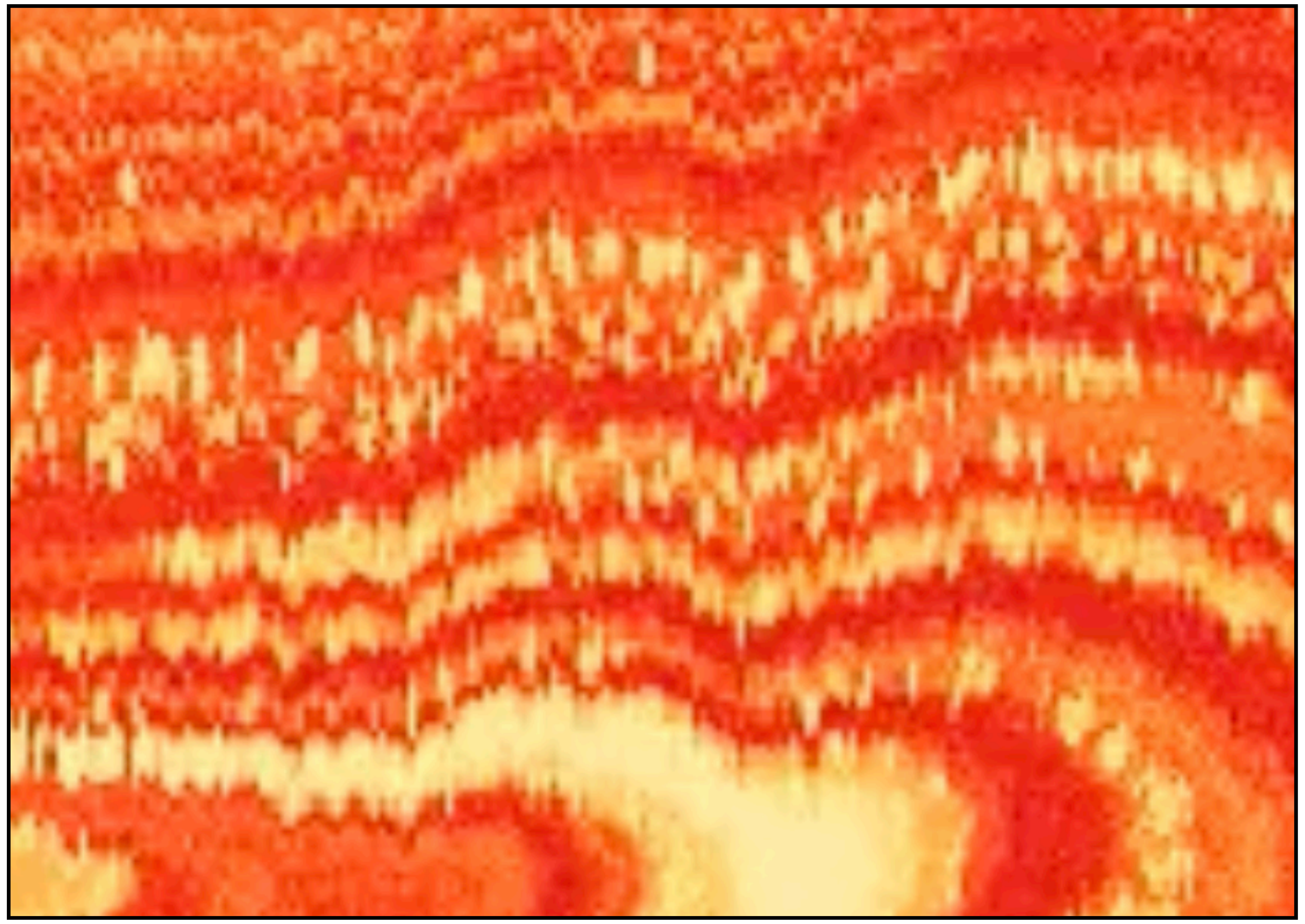
Bath of ion holes

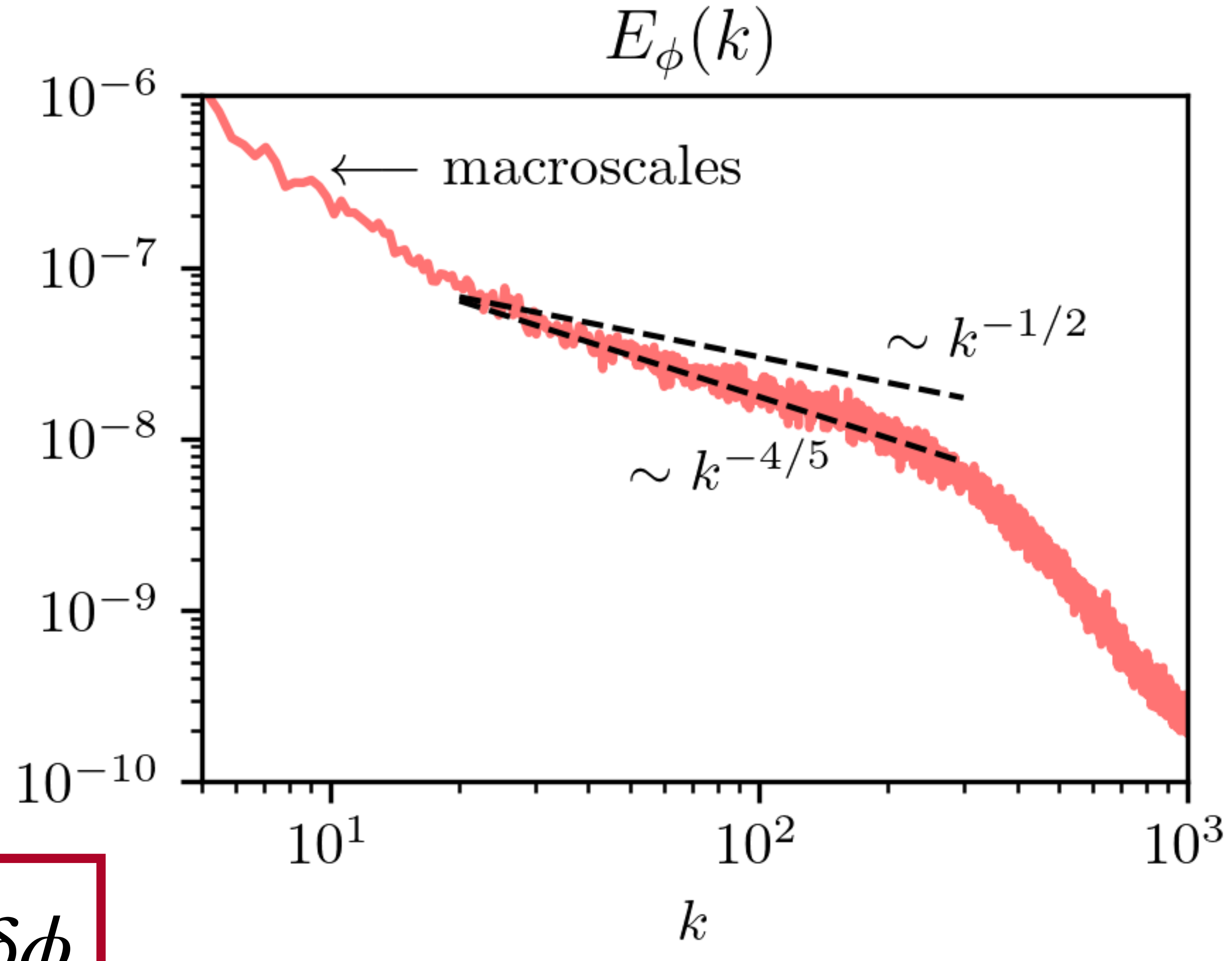
- Suppose $|\hat{\phi}_k|^2 \sim k^{-\alpha}$
- Velocity scale of holes set by energy conservation: $u \sim \sqrt{\phi_k} \implies s \sim k^{\alpha/4}$
- BGK: $\delta f \sim 1/\sqrt{\phi_k} \implies \delta f \sim k^{\alpha/4}$
- For $s \ll k^{4/\alpha}$, holes are uncorrelated, so $|\hat{f}_{k,s}|^2$ independent of s
  - Also, $|\hat{f}_{k,s=0}|^2 \sim |\hat{n}|^2 \sim |\hat{\phi}|^2 \sim k^{-\alpha}$
  - So,

$$\delta f^2 \sim k \int ds\, |\hat{f}_{k,s}|^2 \sim k \int^{k^{\alpha/4}} ds\, k^{-\alpha} \sim k^{1-3\alpha/4}$$

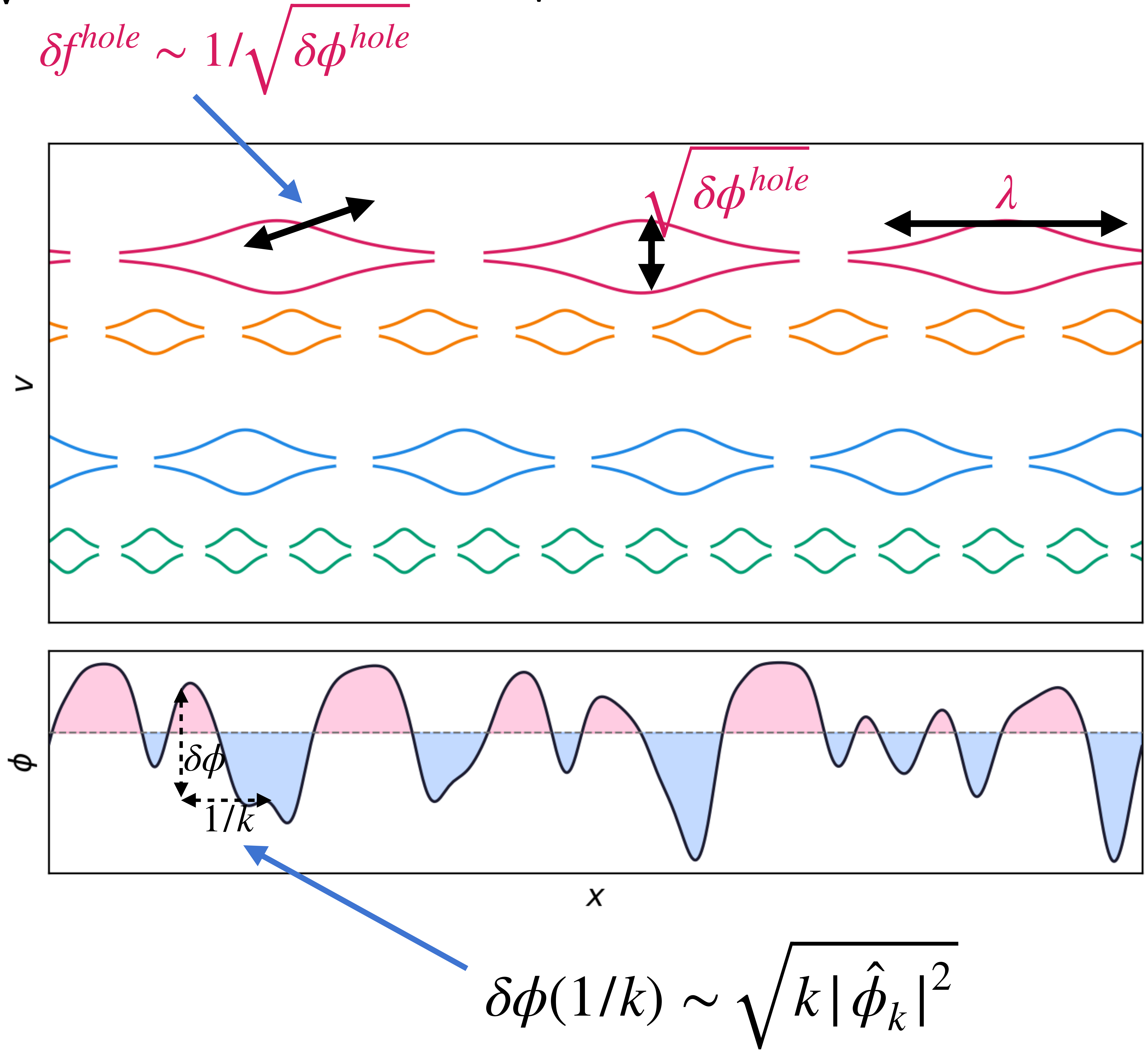
- $\implies \alpha = 4/5$

Disclaimer: will be justified in next slide why ϕ_k and not $\delta\phi$



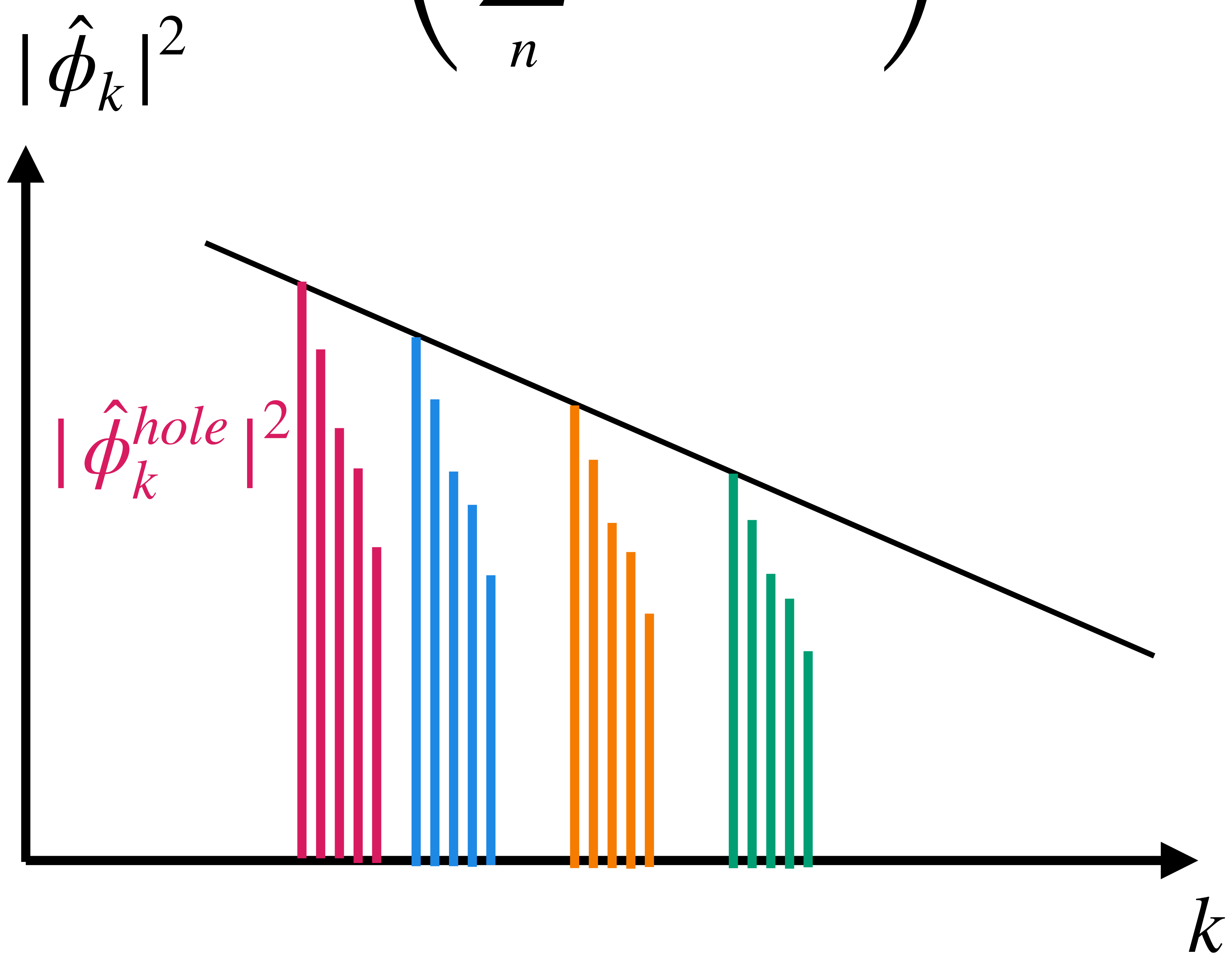


$\sqrt{\phi_k}$ instead of $\sqrt{\delta\phi}$



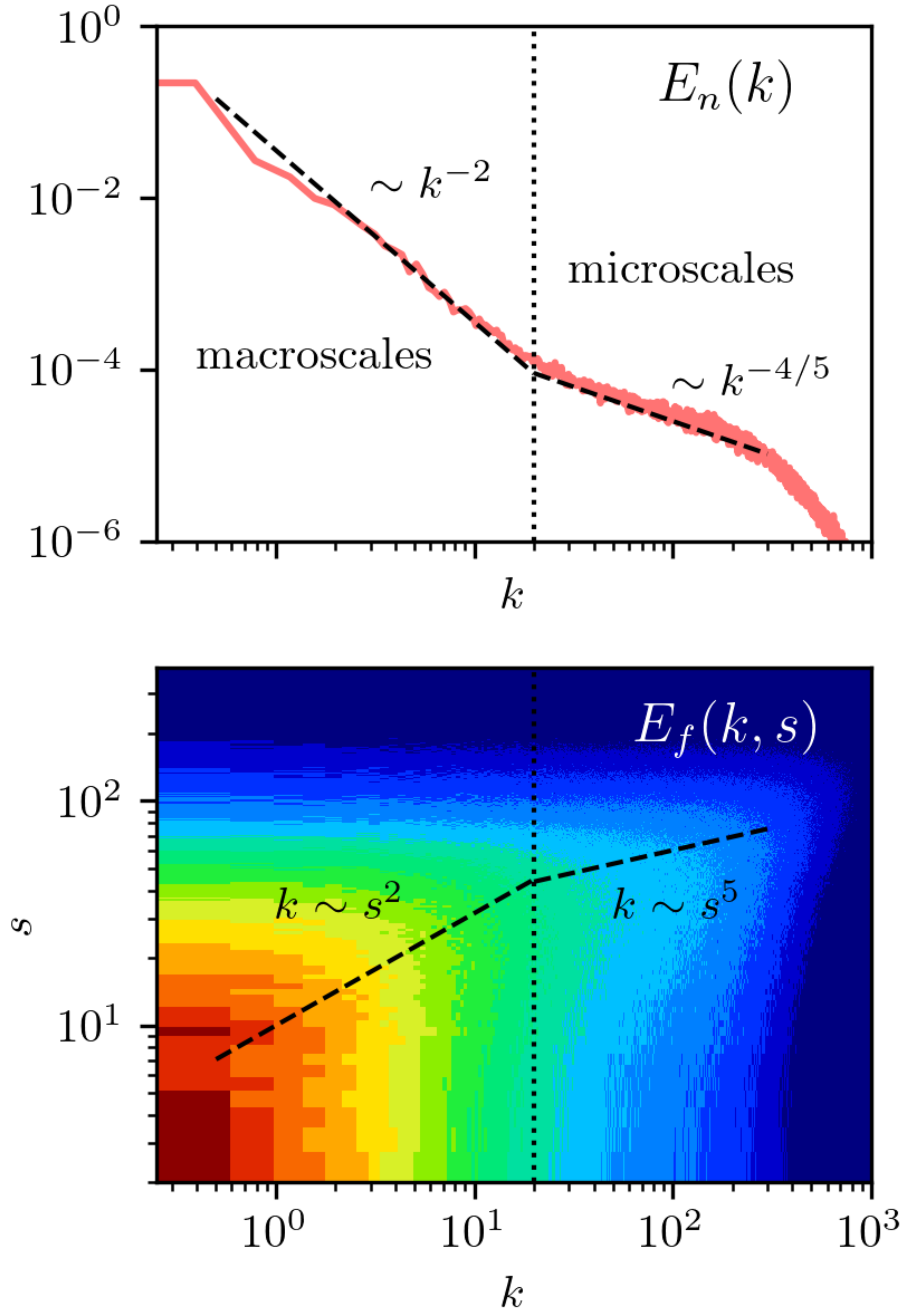
- Holes occur in periodic chains
- So, increasing λ for fixed amplitude just spaces $\hat{\phi}_k$ closer together, does not change amplitudes of individual modes
- So,

$$\delta\phi^{hole} \sim \left(\sum_n |\hat{\phi}^{hole}_{2\pi n/\lambda}|^2 \right)^{1/2} \propto |\hat{\phi}_{2\pi/\lambda}|$$



Two-scale system

- Microscales:
  - Fed by microinstabilities
  - Dominated by a bath of ion holes
- Macroscales:
  - Entropy cascade
  - Phase-space structures go unstable at Kolmogorov scale

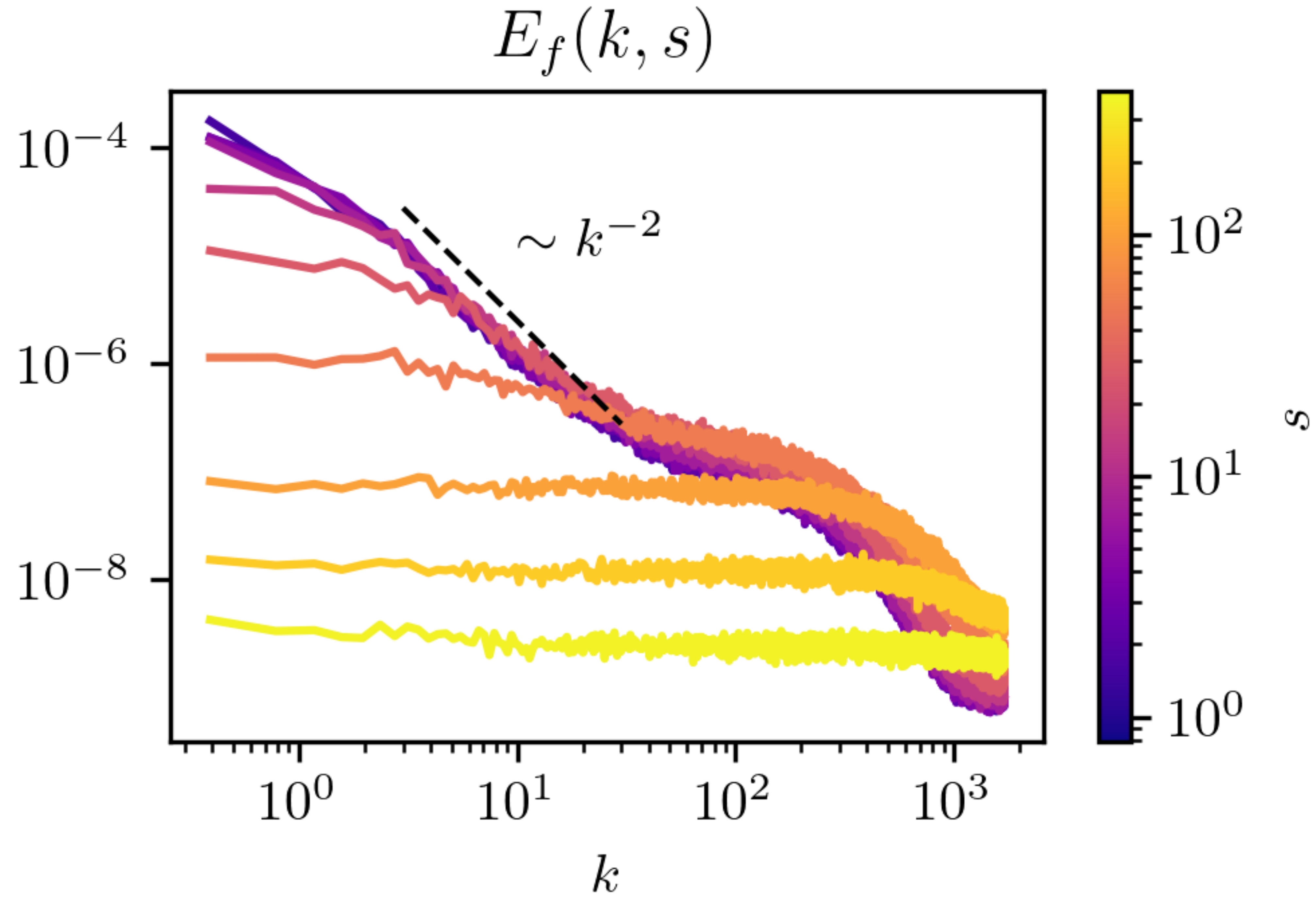


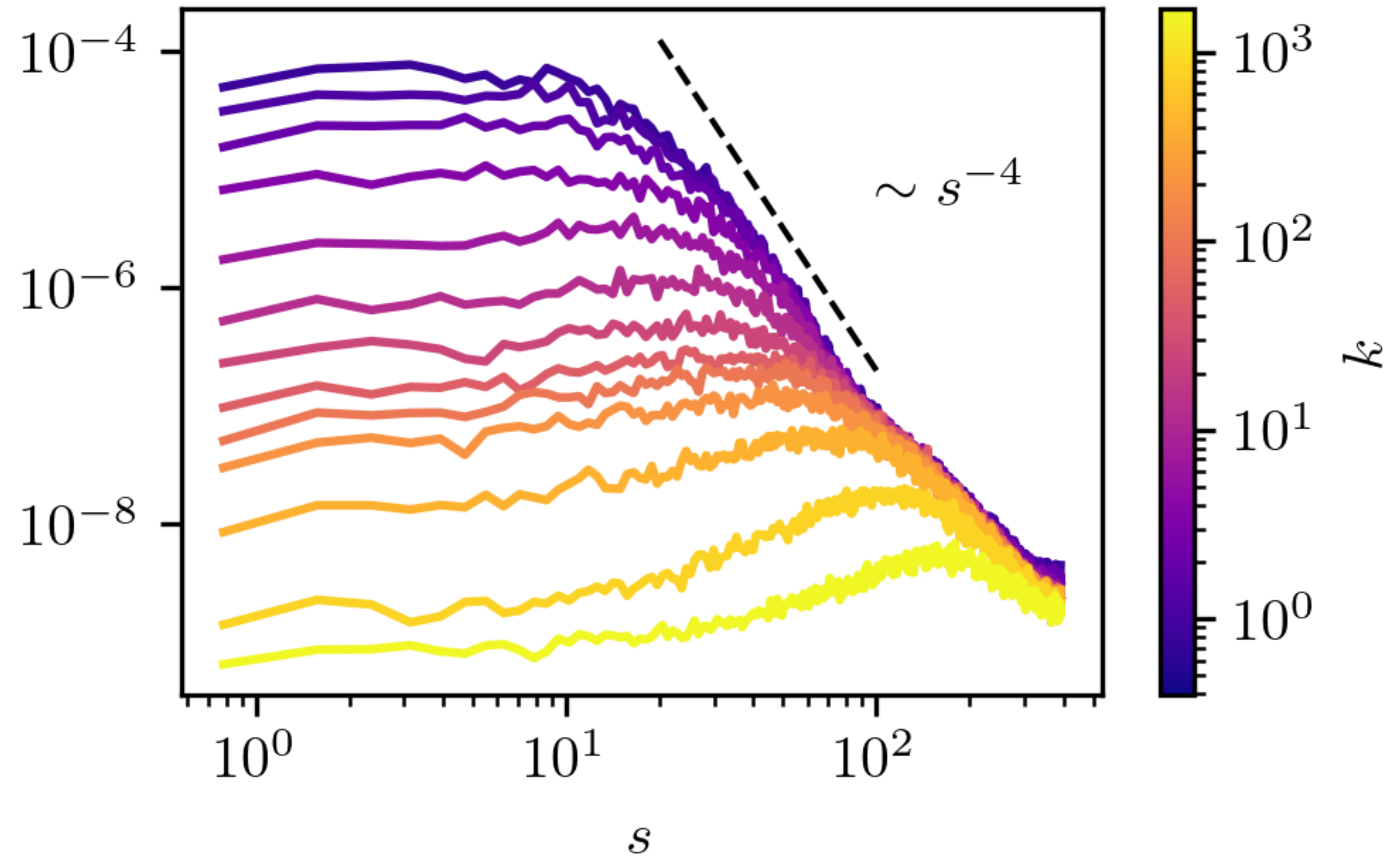
Entropy cascade

- Critically-balanced cascade of δf^2 (Nastac *et al.* 2025)
- Suppose $\tau_{nl} \sim u$
- $\tau_c \sim \tau_{nl}$ at each scale balances with $\tau_l \sim \ell / u$ $\implies \ell \sim u^2$
- Entropy cascade: $\epsilon \sim \dfrac{\delta f^2}{\tau_c}$
- Predicts:

$$E_f(k,s) = \begin{cases} k^{-2} & k \gg \sim s^2 \\ s^{-4} & k \ll \sim s^4 \end{cases}$$

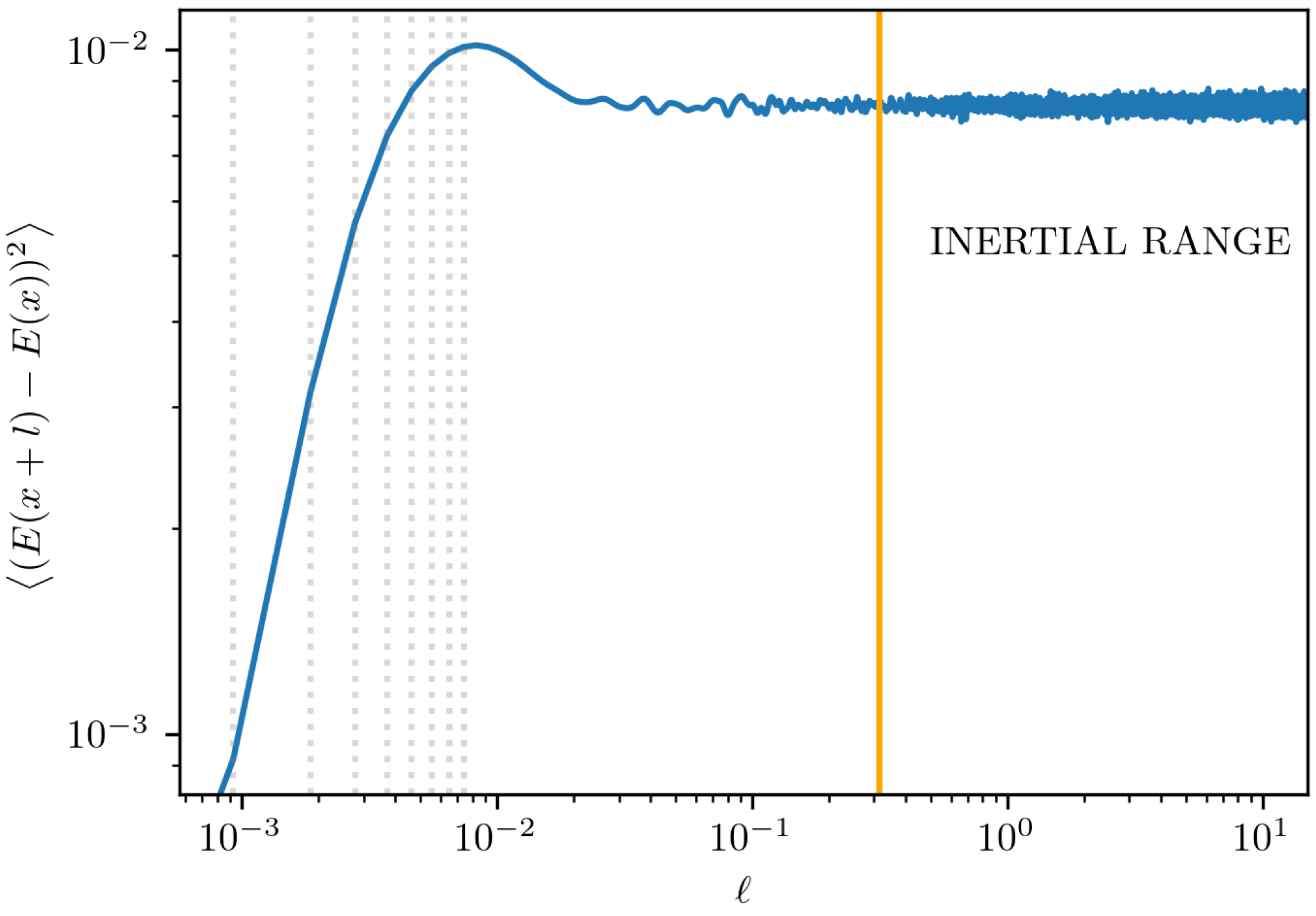
- Why $\tau_{nl} \sim u$?





The obvious solution, and its obvious problem

- From $E\partial_v f$ term in Vlasov, $\tau_{nl} \sim u/\delta E$
- δE is constant with scale
- This is because δE dominated by smallest-scale modes
- However:
  - Option 1: smallest scales decohere quickly, but then should get diffusive behaviour at large-scales
  - Option 2: small-scales are just coherent oscillations. But then, how is separation achieved at all?



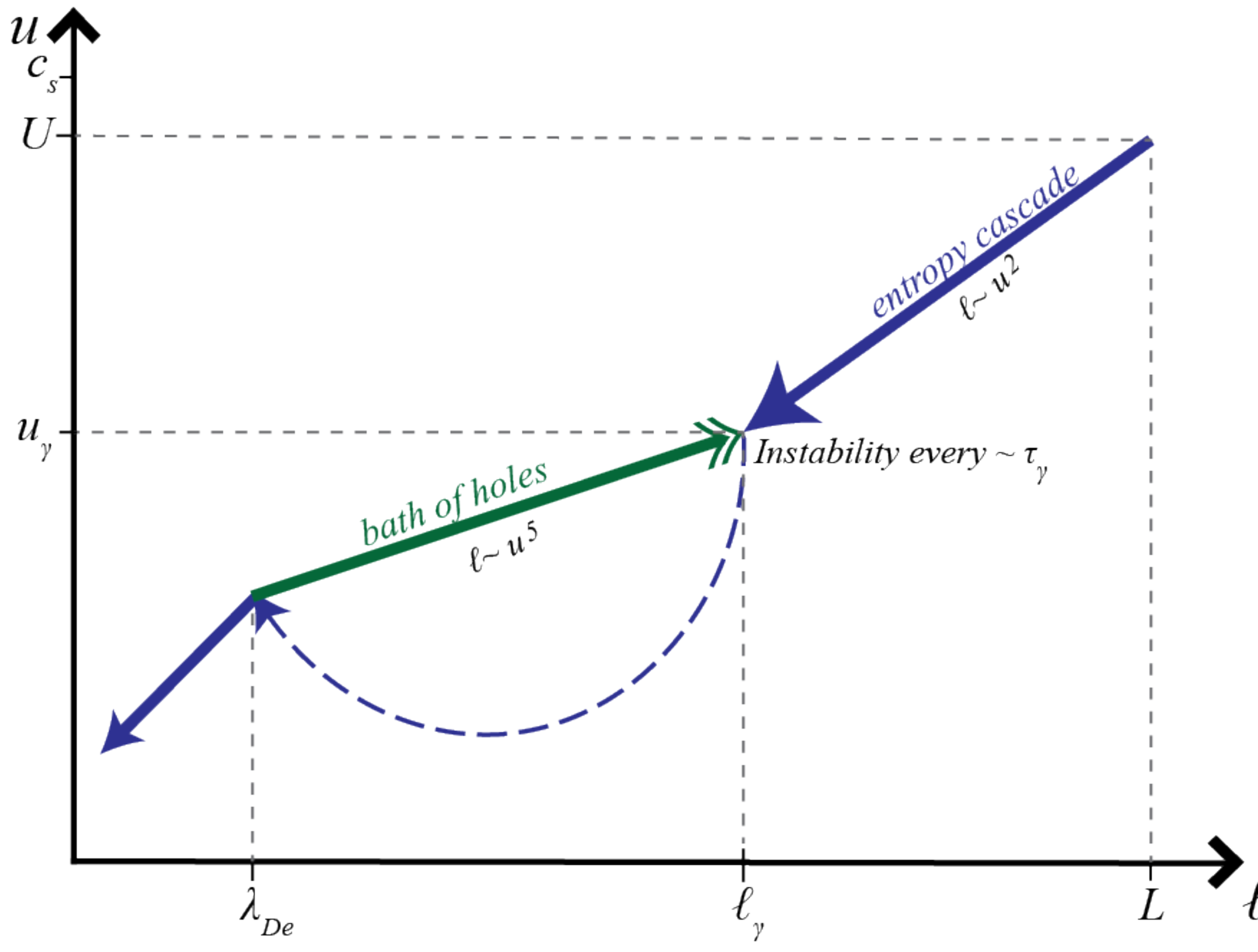
Possible reason for $\tau_{nl} \sim u$?

- Can interpret $\tau_{nl} \sim u$ through binary particle separation: $\Delta v \sim t$
- Suppose modes appear *suddenly* and then stick around
- Each time a mode appears, non-resonant particles jump to a different 'energy', experiencing a net kick of $\delta v \sim \dfrac{\phi}{v - v_{ph}}$
- If $\mathbb{P}(v - v_{ph}) \sim$ const, then $\mathbb{P}(\delta v) \sim 1/\delta v^2$ — heavy-tailed!
- So, dominated by Lévy flights: $\Delta v \sim \sum \delta v \sim t \log t$
- Avoids previous dilemma
- PROBLEM: $\Delta v \gg \sqrt{\phi_\gamma}$ in macroscale cascade, so not in this regime

Kolmogorov scale

- $n_b :\sim u_\gamma \delta f_{u_\gamma}$ is density of phase-space structures at unstable scale
- Then, marginally unstable modes where $u_\gamma \sim \sqrt{\frac{n_b}{n_0}} c_s$
- Free energy of beams matches injection energy into modes: $e\phi_\gamma / m_i \sim u_\gamma^2$
- Can show that (negative) entropy is transferred to Debye scales:

$$\delta f_\gamma \sim \delta f_{\text{micro}} \sim \frac{n_b}{\sqrt{\phi_\gamma}}$$

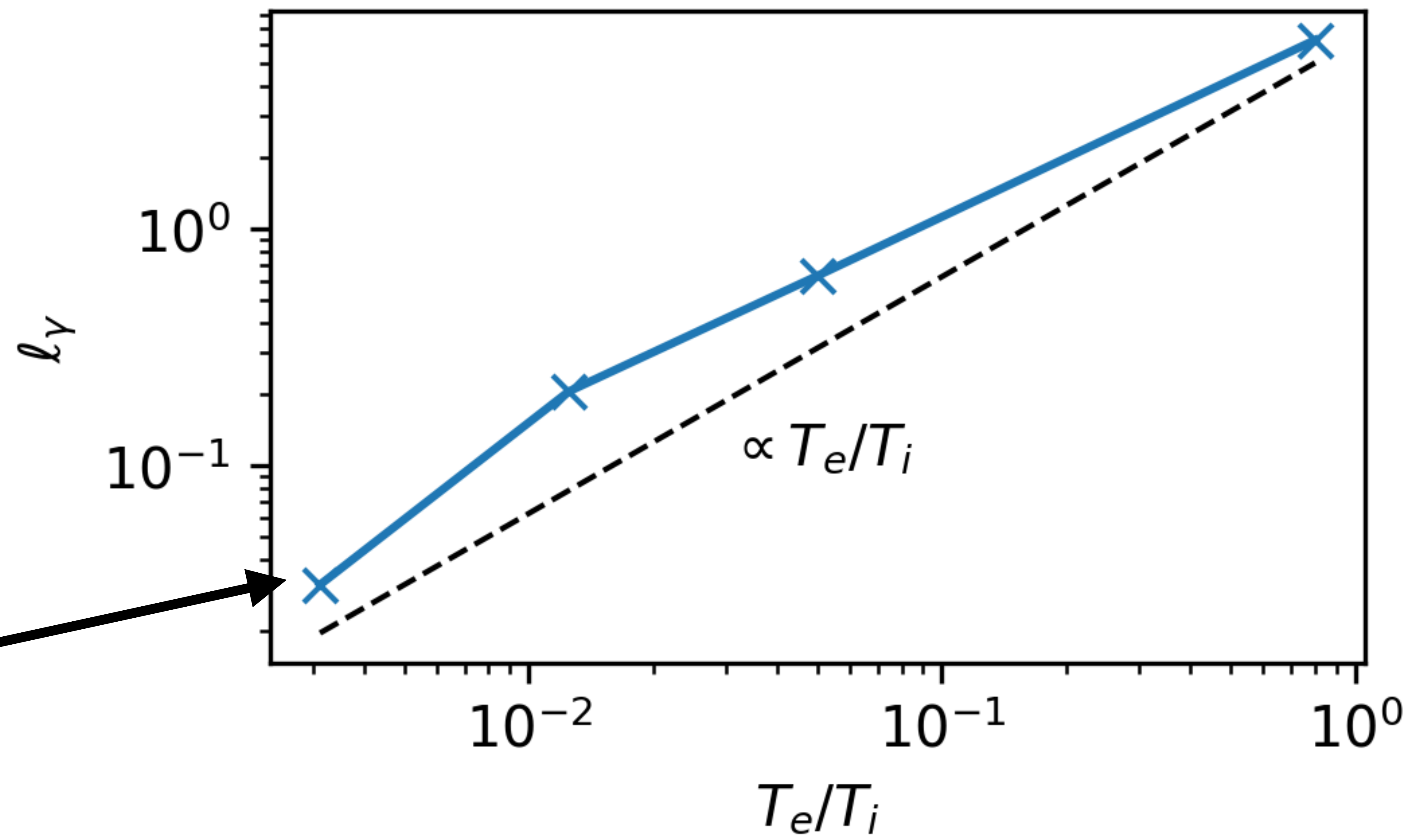


Connecting the scales

1. Injection of free energy at Debye length balanced by holes merging out of Debye length: $\frac{\phi_\gamma^2}{\tau_\gamma} \sim \frac{E_\phi/L}{\tau_b}\Big|_{k\sim\lambda_{De}^{-1}}$ where $\tau_b \sim 1/k(E_\phi/L)^{1/4}$ is the bounce frequency

2. $E_{\phi,macro} \sim E_{\phi,micro}$ at transition scale $k_\gamma \sim 1/\ell_\gamma$

- Predicts $\ell_\gamma \sim c_s^2/v_{th}^2 \sim T_e/T_i$



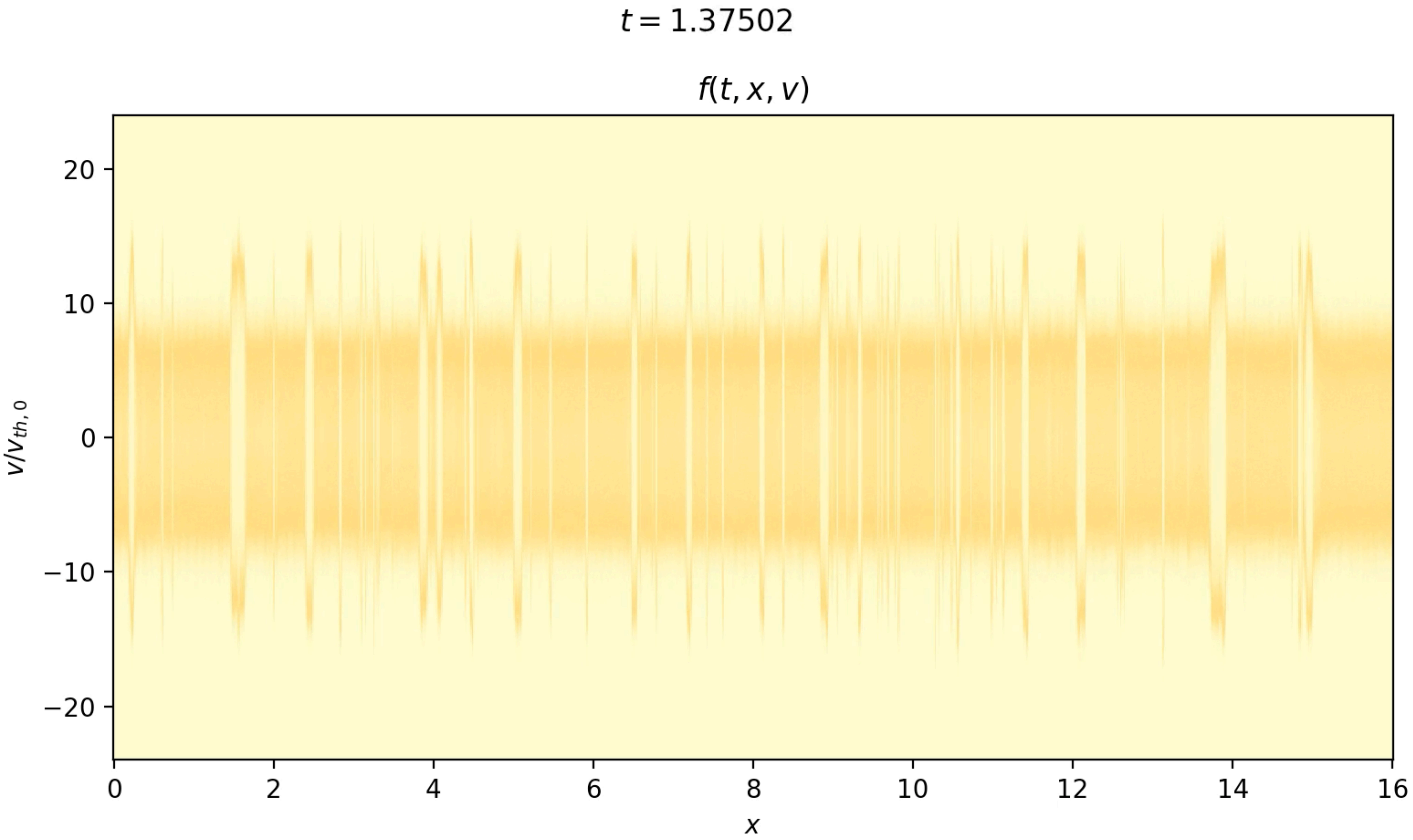
P.S. switches to Batchelor

Conclusions

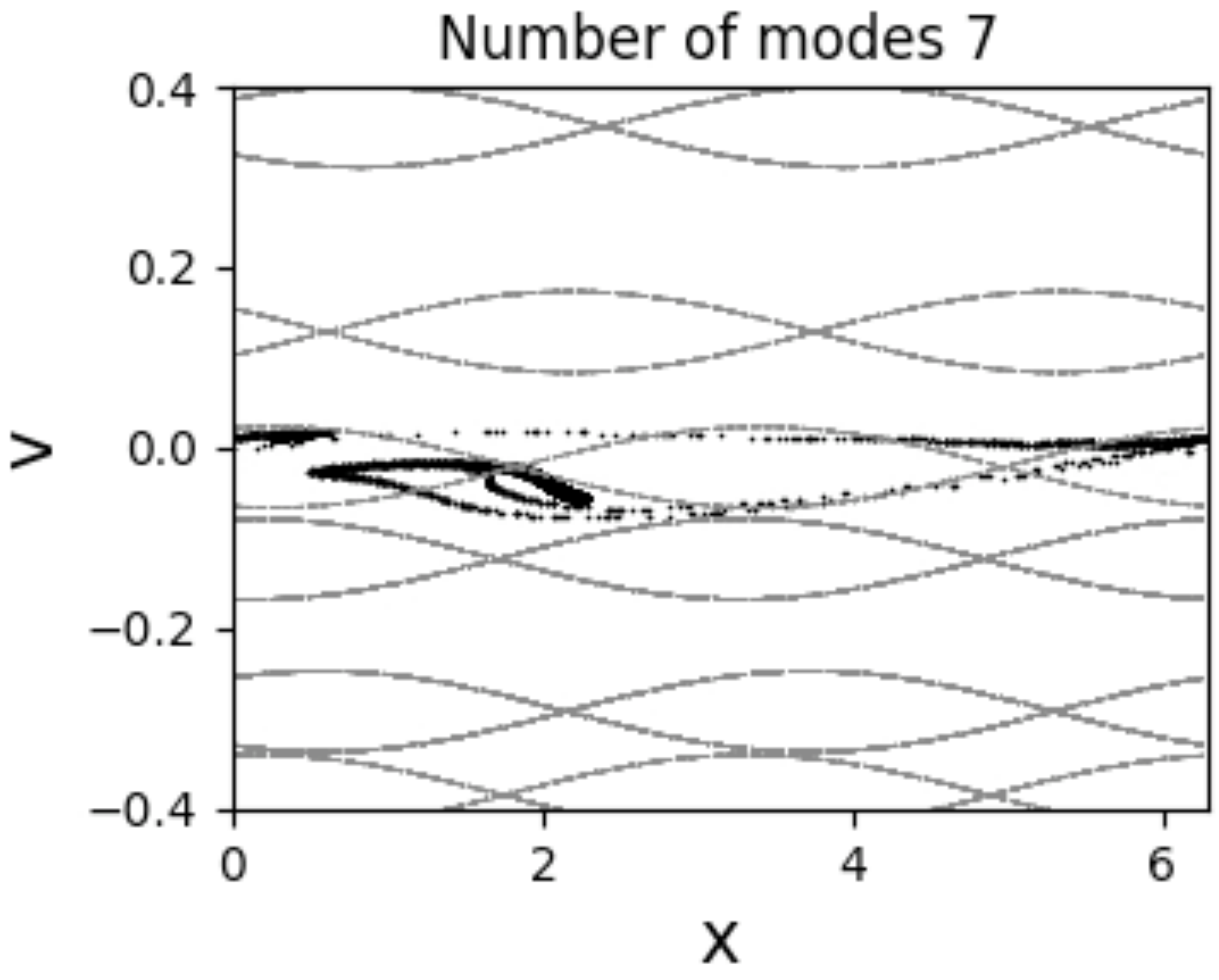
- Macroscale entropy cascade creates phase-space structures that go unstable to the two-stream instability
- Mode saturates as BGK holes
- A steady-state spectrum of BGK holes is formed with $E_\phi \sim k^{-4/5}$
- Macroscale entropy cascade is set by critical balance between $\tau_l \sim \ell/u$ and $\tau_{nl} \sim u$
- Kolmogorov scale of macroscale cascade when phase-space structures are unstable
- Remaining question: why $\tau_{nl} \sim u$?

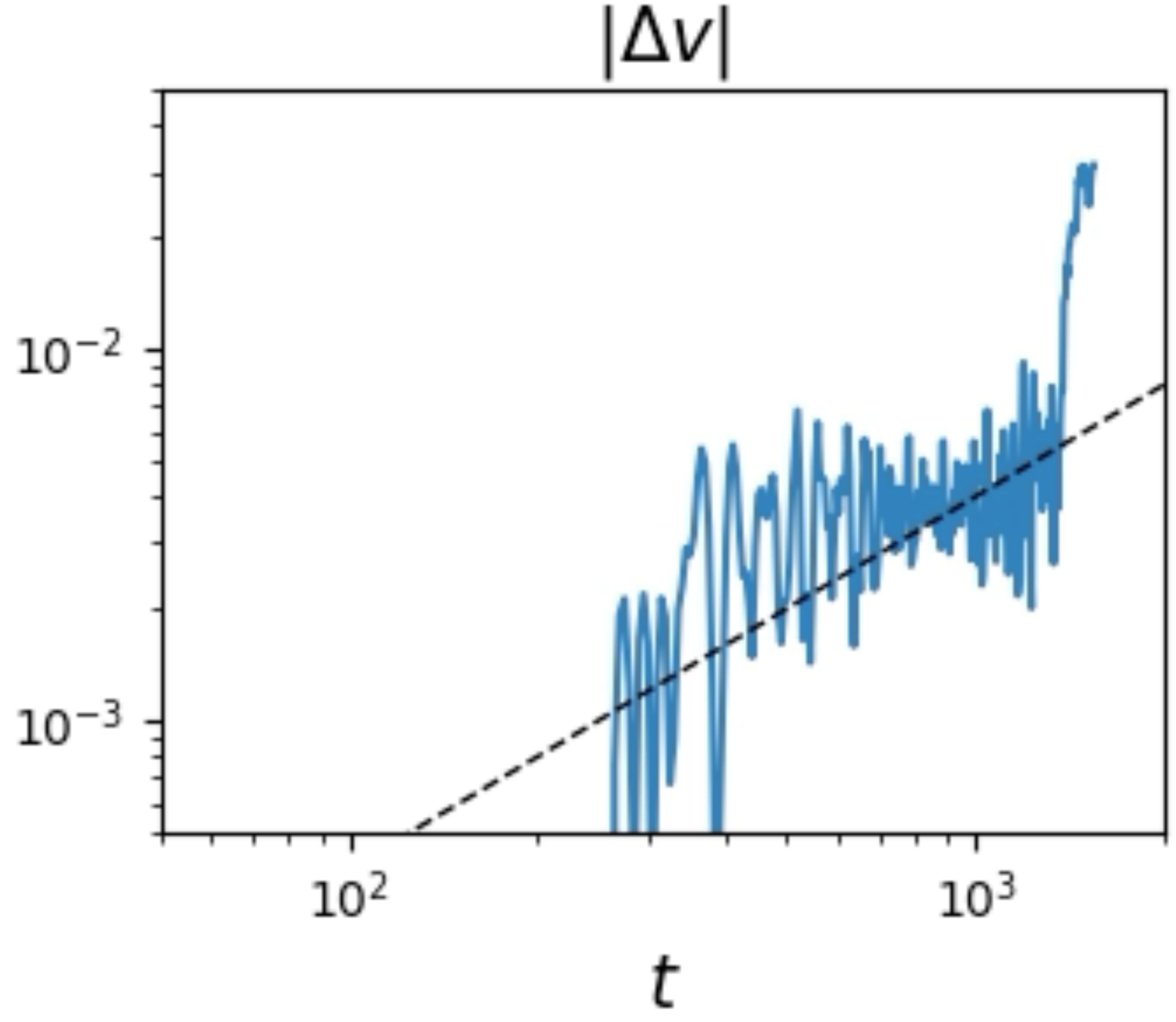
Backup slides

Isolated two-stream

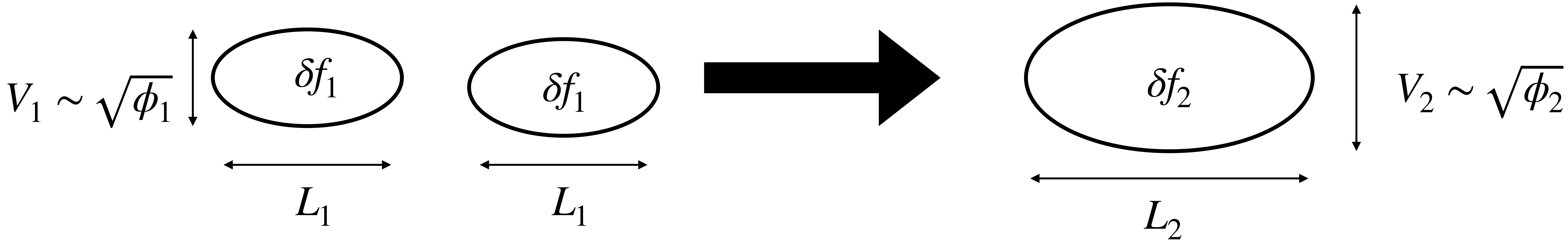


Visualization of Lévy flights

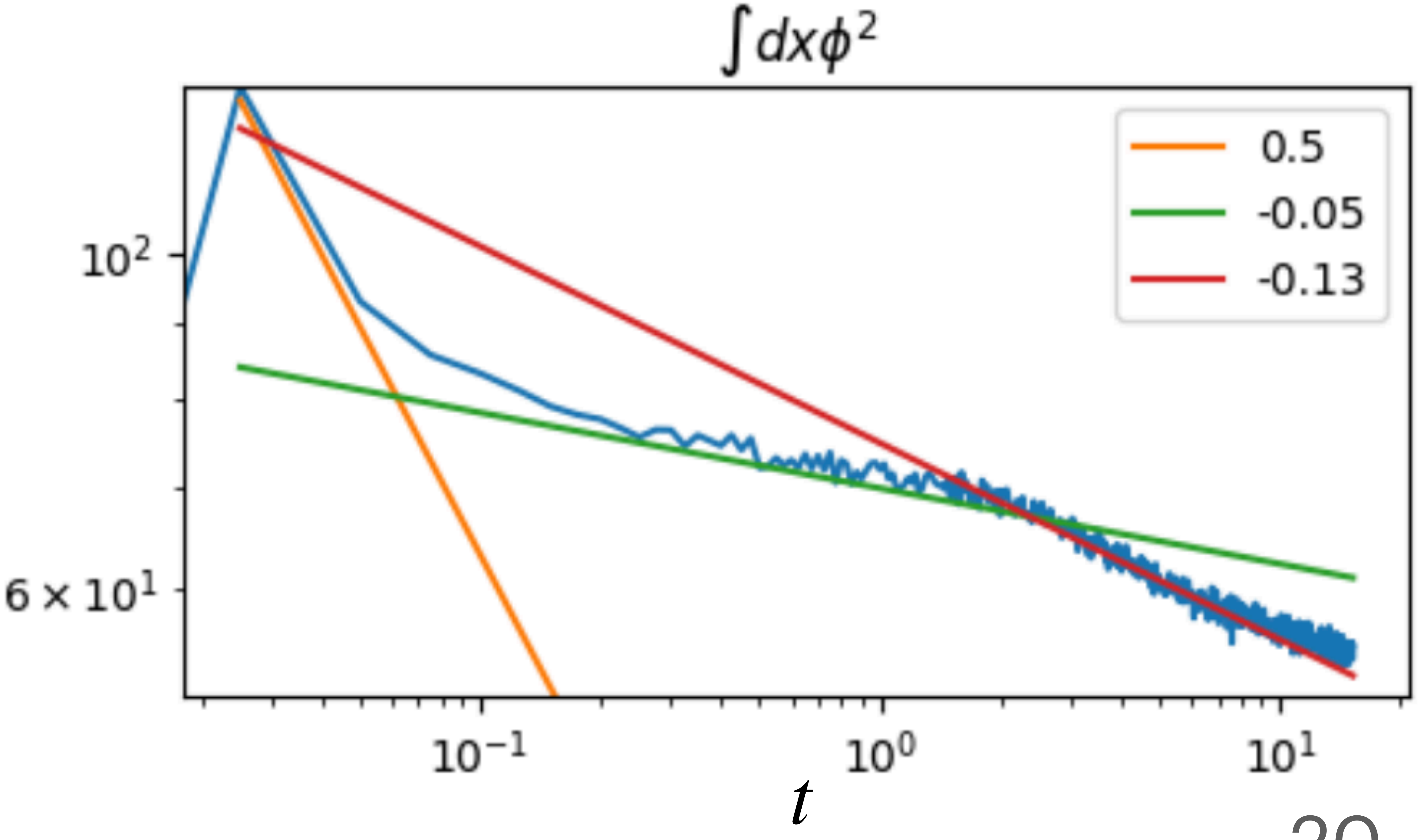




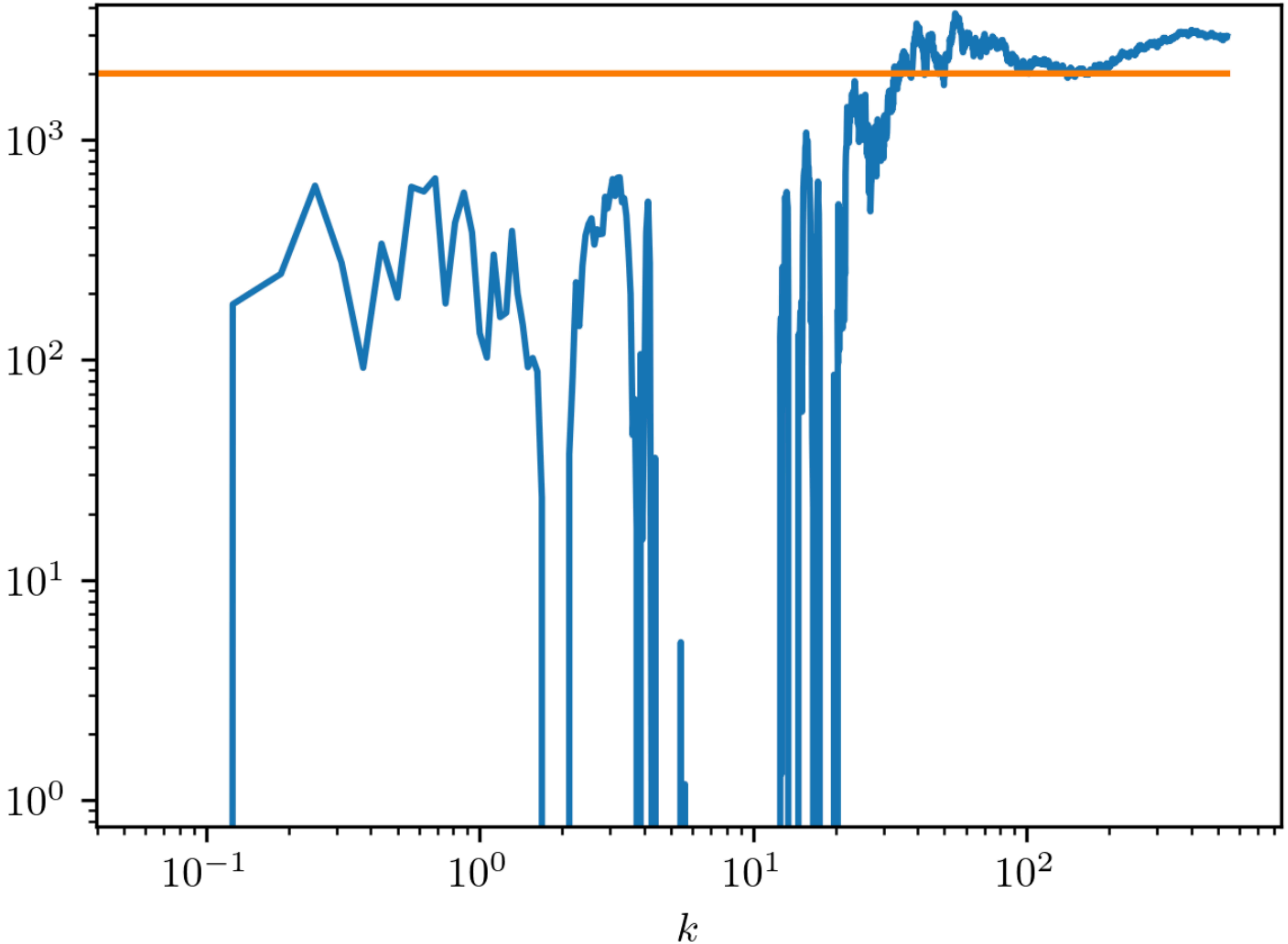
Merging hole theory



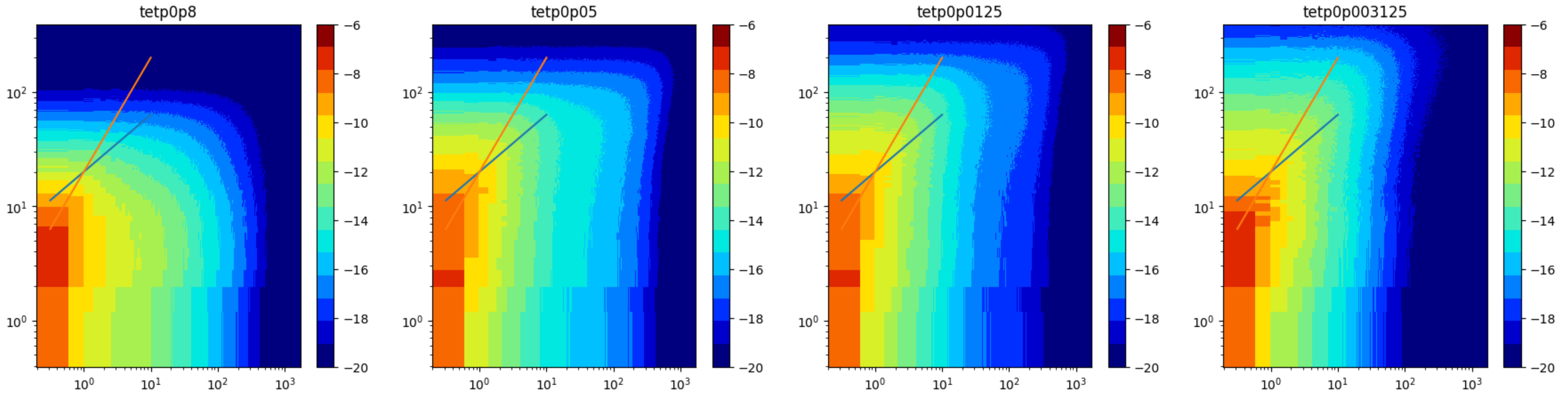
- Trapped particles conserved: $2V_1L_1\delta f_1 \sim V_2L_2\delta f_2$
- BGK condition: $\delta f \sim 1/\sqrt{\phi}$
- Boltzmann-quasi-neutrality: $\phi \sim \delta f V$
- Field energy is $\varepsilon_1 \sim 2L_1\phi_1^2$, $\varepsilon_2 \sim L_2\phi_2^2$
- $\Longrightarrow \varepsilon_1 \sim \varepsilon_2$



Negative Flux of ϕ^2



T_e/T_i scan



tetp0p8
tetp0p05
tetp0p0125
tetp0p003125
10^0
10^1
10^2
10^3
−6
−8
−10
−12
−14
−16
−18
−20