

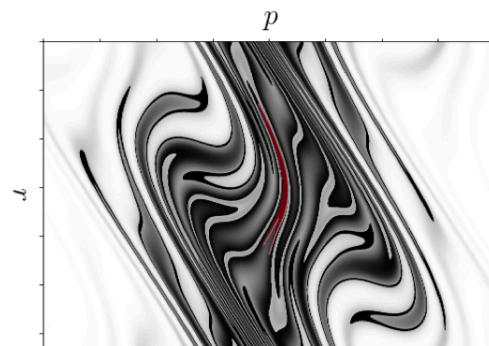
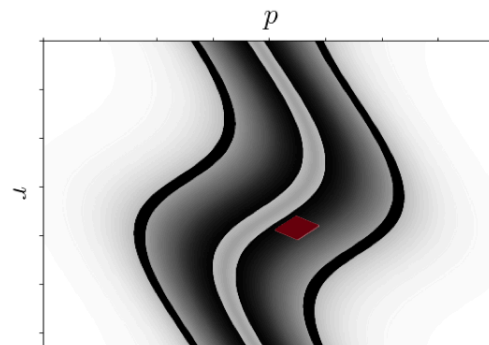
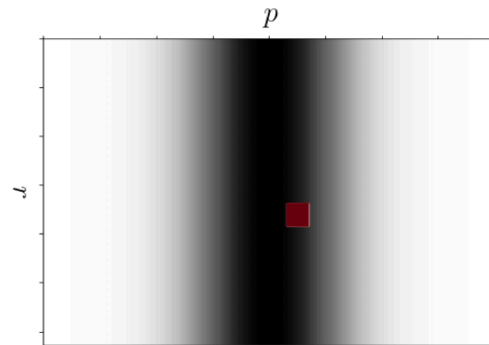
A photograph of a snowy mountain landscape. In the foreground, a cable car cable runs diagonally from the top left. The middle ground shows a dense forest of snow-covered evergreen trees. In the background, a large, snow-capped mountain peak rises against a clear blue sky. The right side of the image is a solid red background with white text.

Ground states: constraints from the field equations and unresolved issues

Funded by NWO-Rubicon

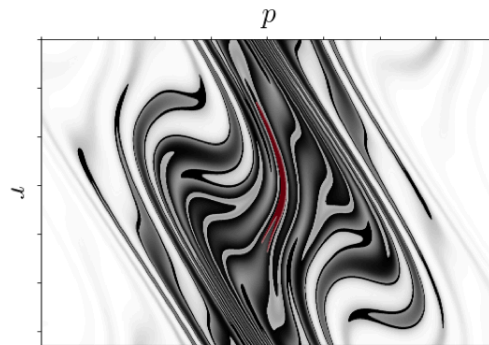
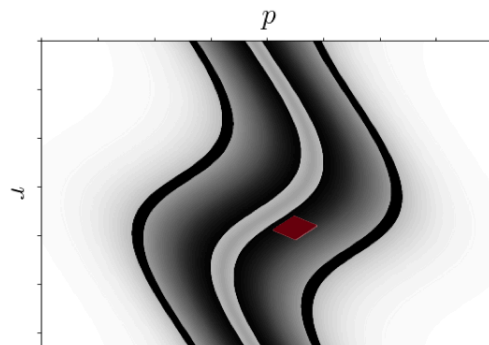
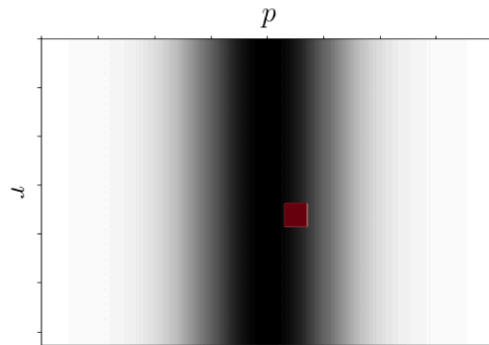
R.J.J. Mackenbach & R.J. Ewart

Introduction



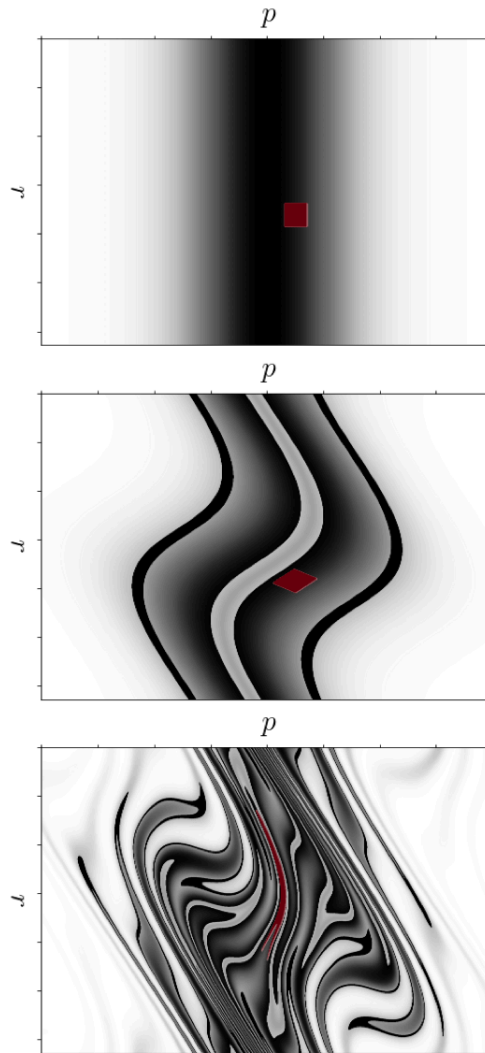
Thermodynamic perspective:

- Don't attempt to describe all the dynamics
 - look at energy/entropy



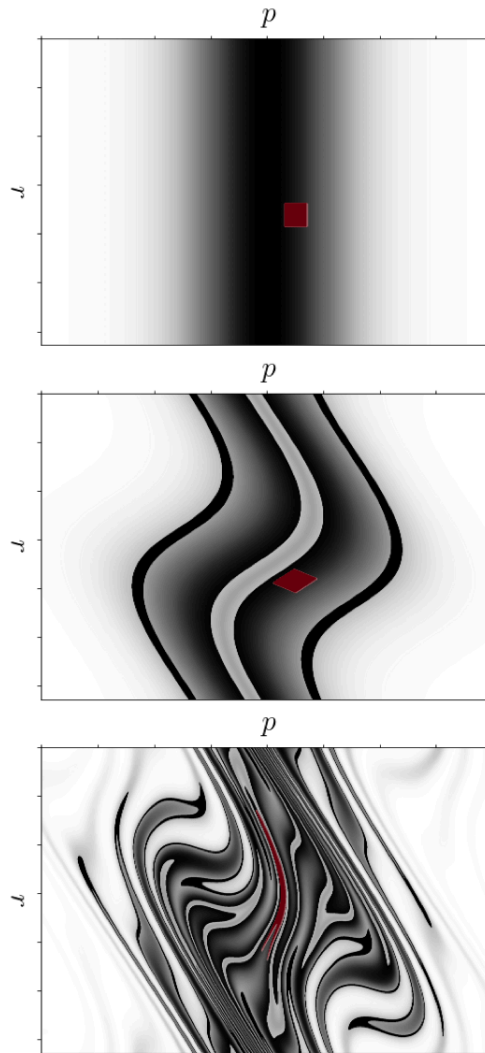
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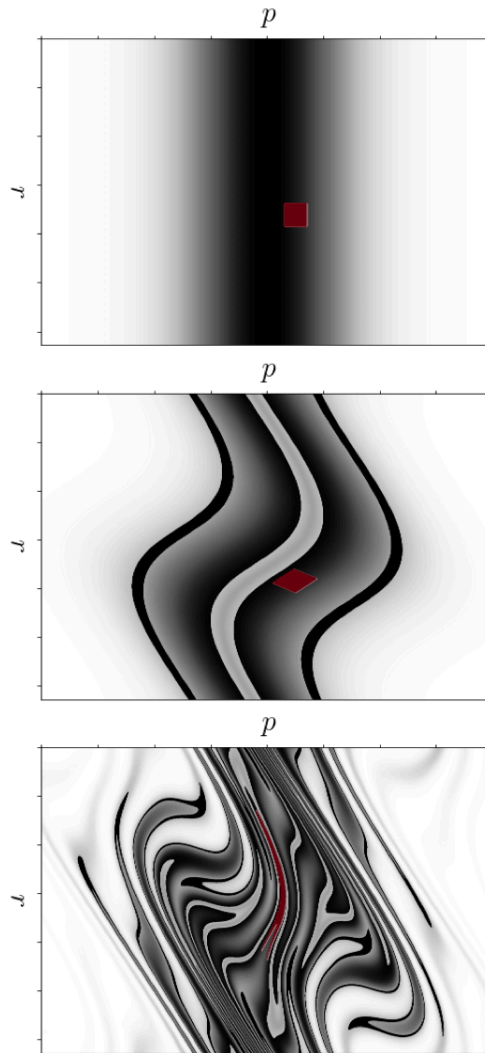
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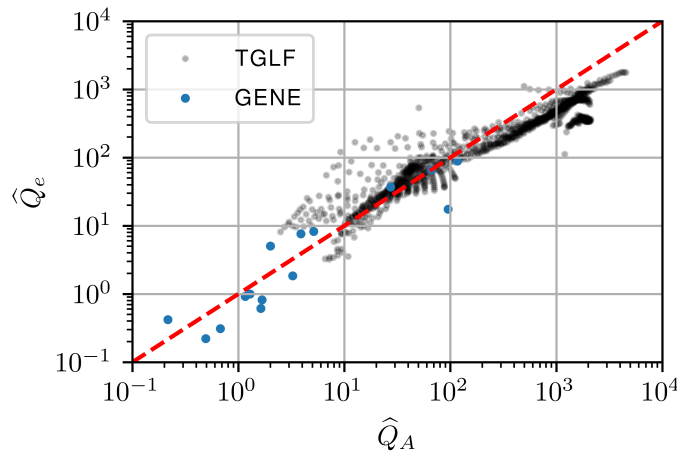
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Universal power laws [Ewart *et al.*]



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Universal power laws [Ewart *et al.*]
 - **Available Energy:**
Bounds on field energy, TEM available energy correlates with heat-flux [Mackenbach *et al.* 2022]



Available Energy

What is it?

EPFL Available Energy

What is it?

- Assumption:
Plasma described by Vlasov

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0$$

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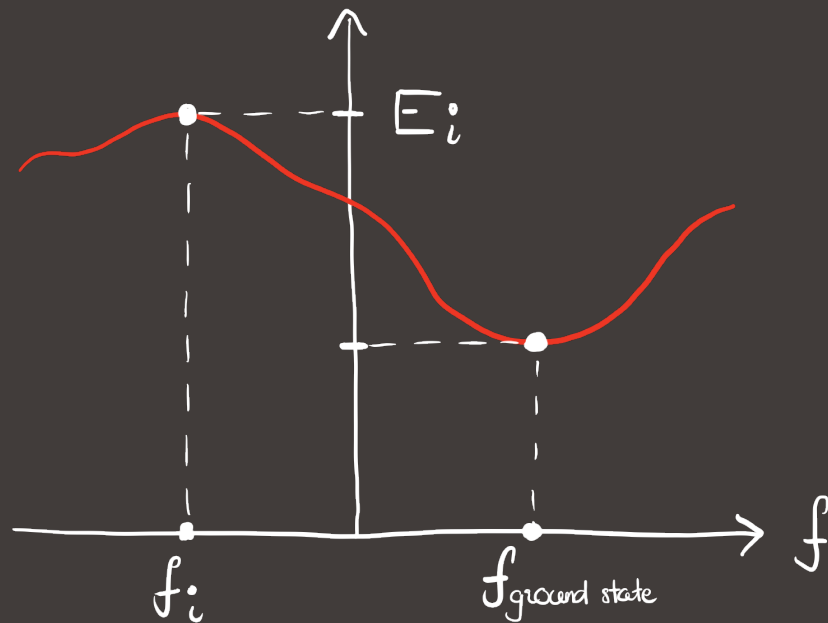
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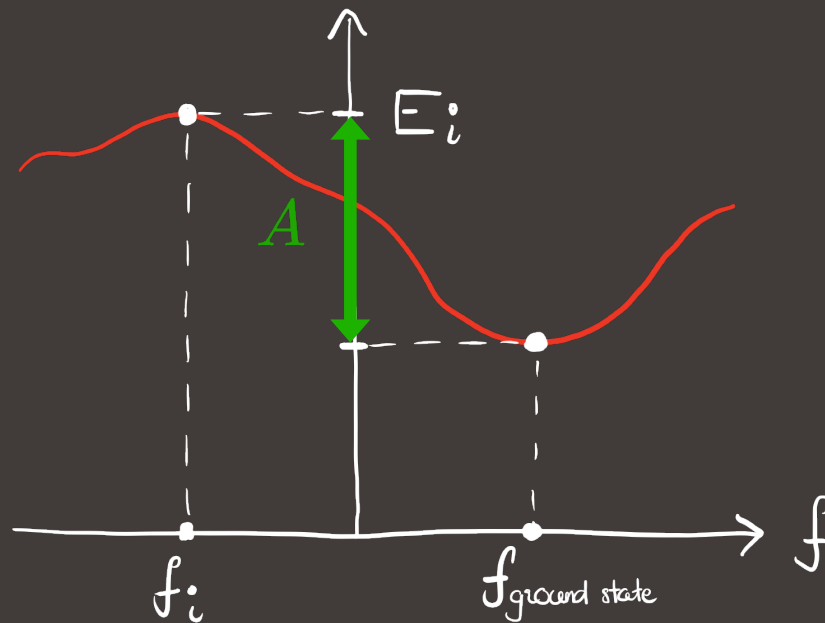
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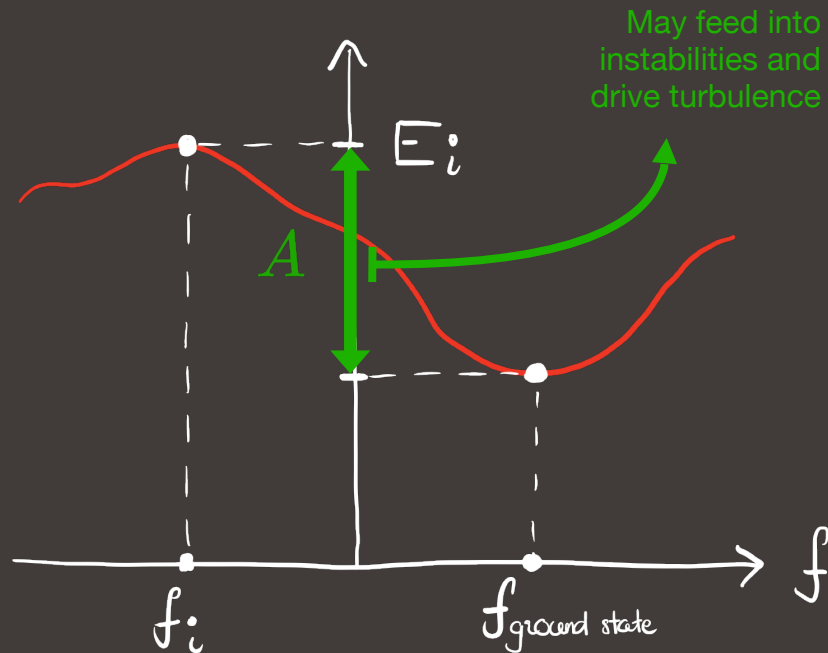
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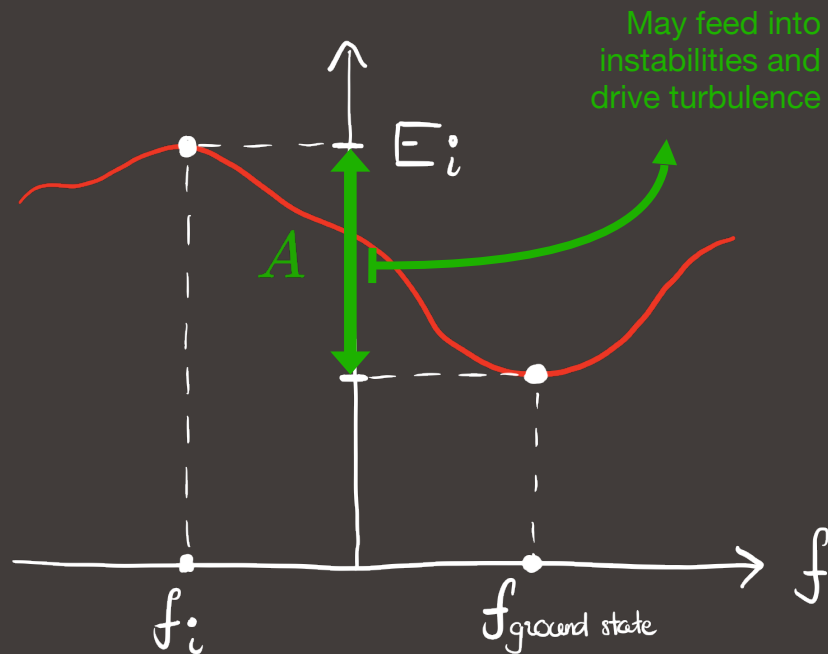
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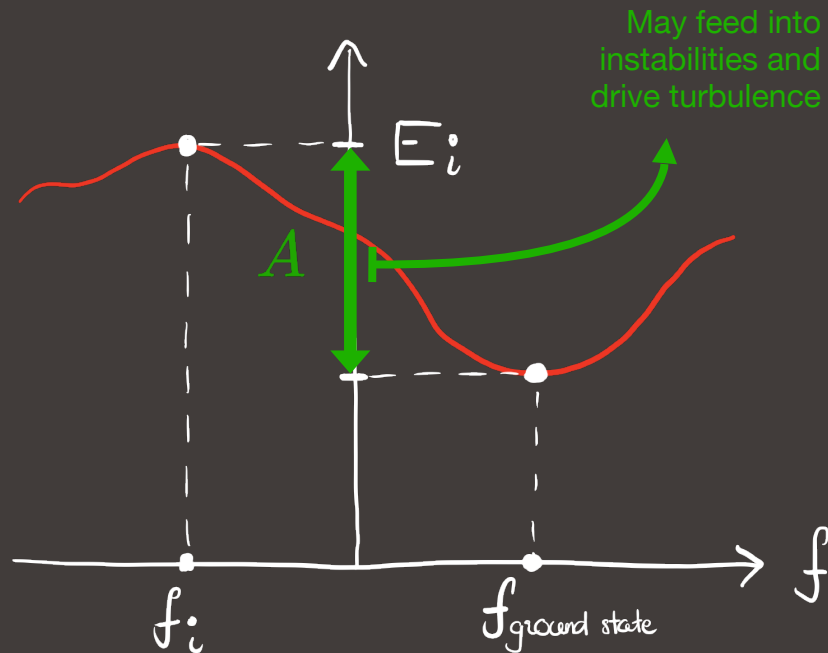
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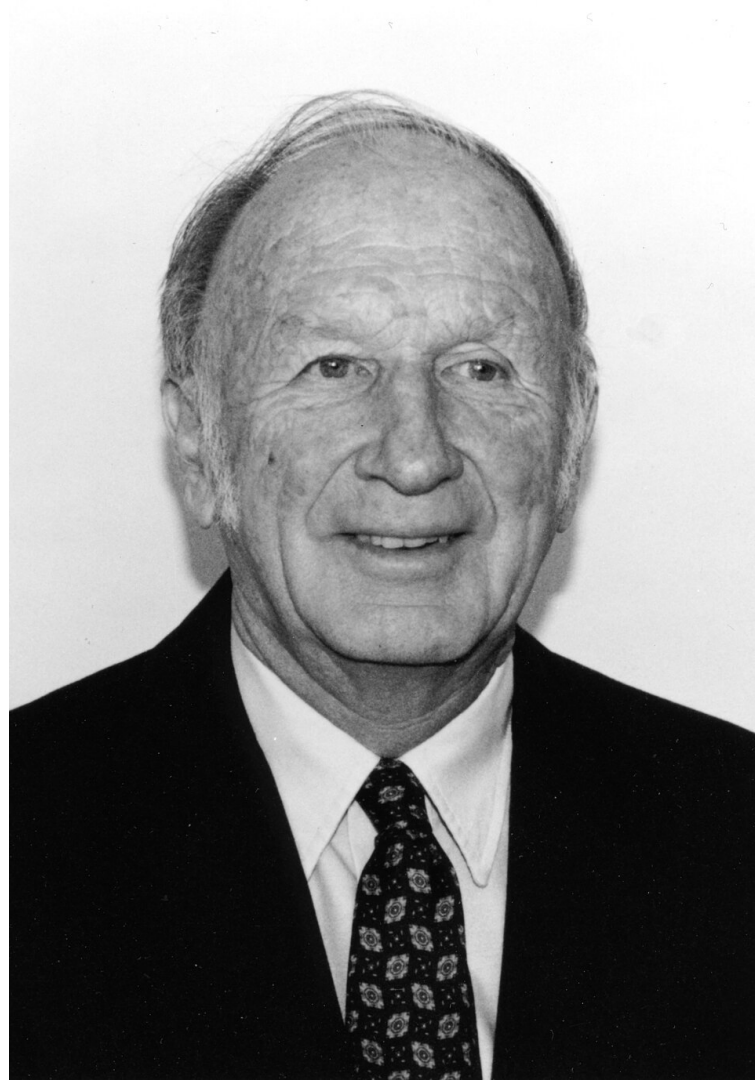
- Very cheap to compute!
- Maps initial distribution function, geometry, ..., onto scalar

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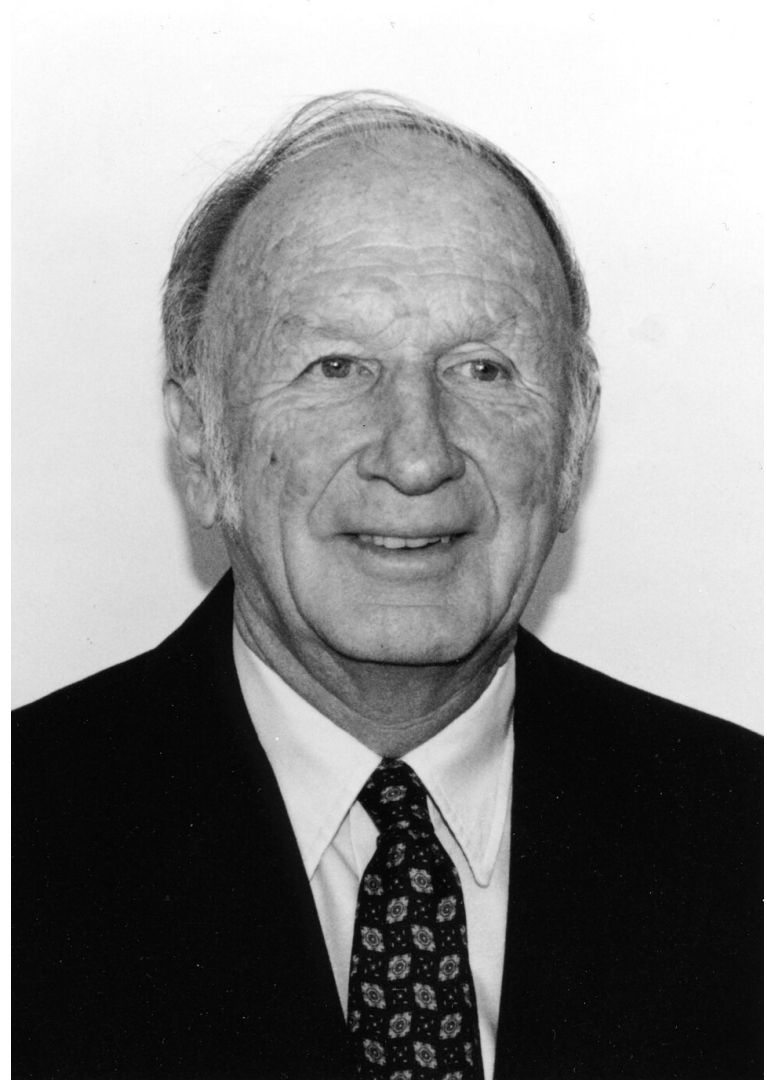
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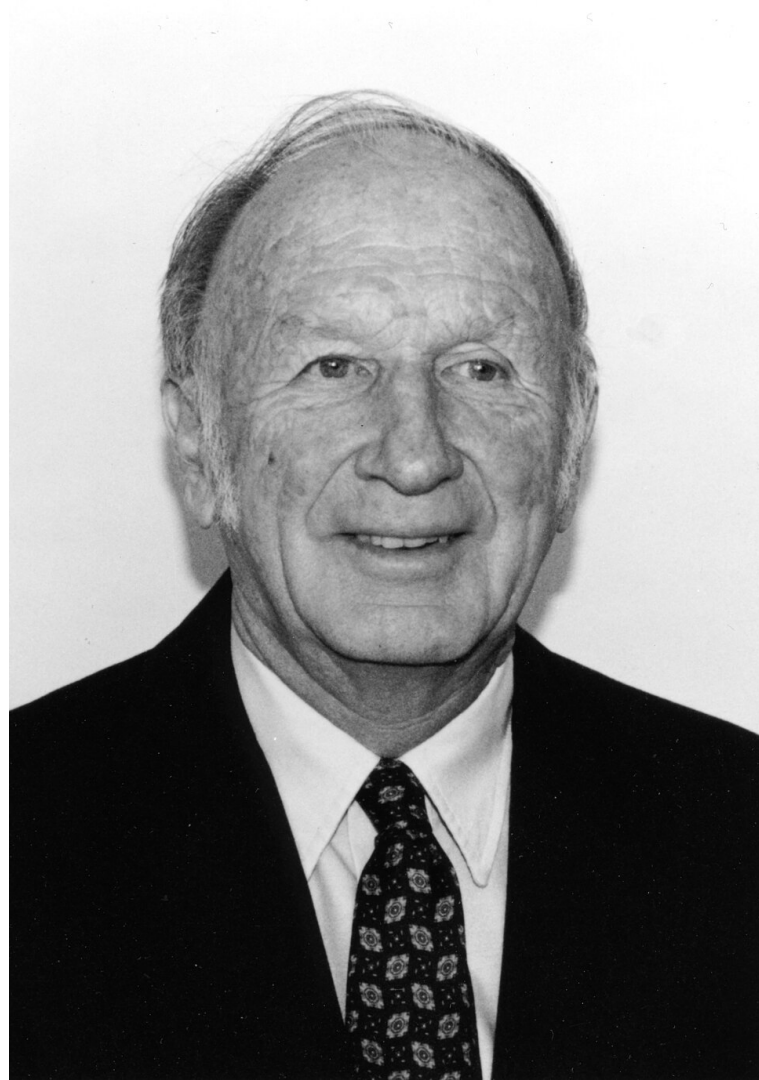
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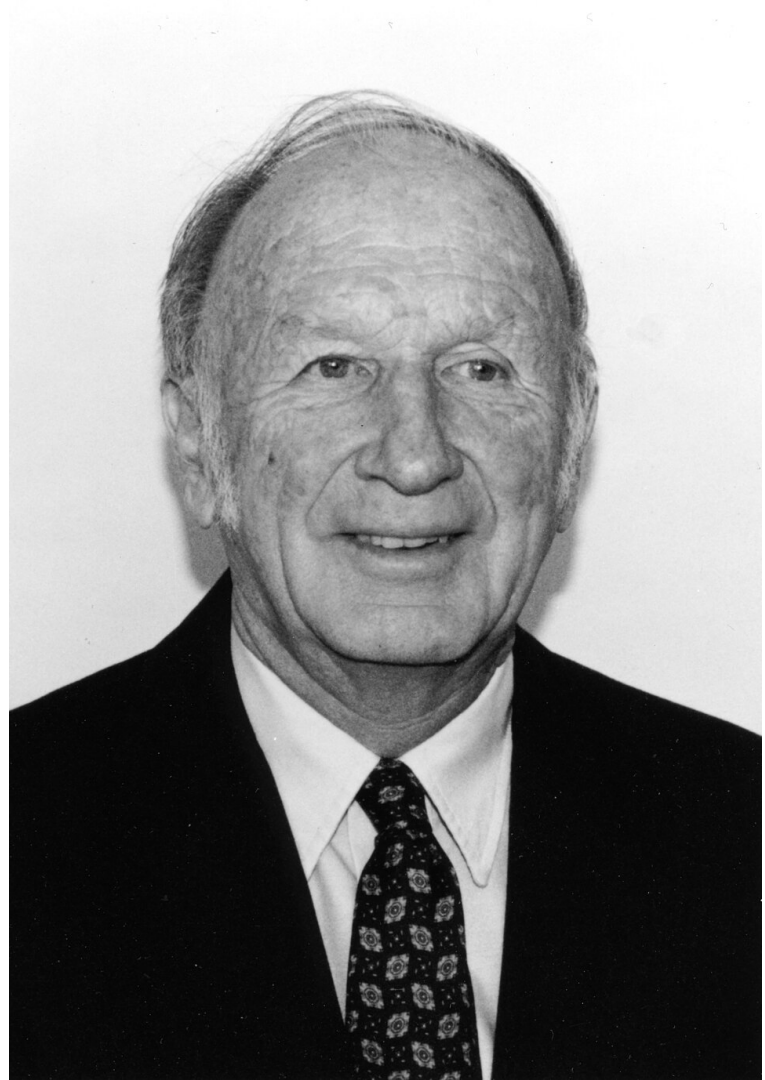
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- Has been studied extensively in **quantum systems** too (quantum ergotropy)



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 - This talk: something new...

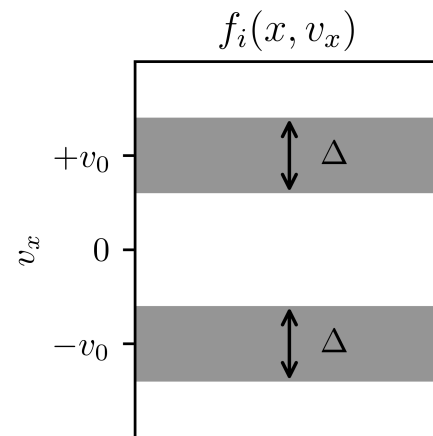
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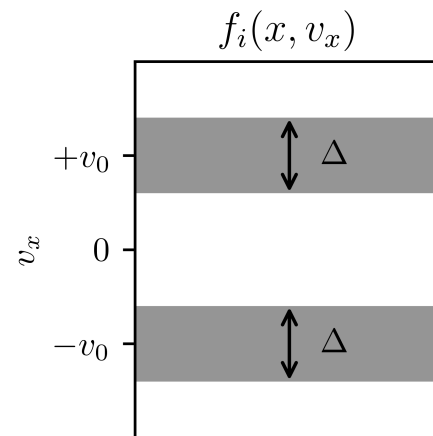
Let's go back to basics...

Pictorial representation



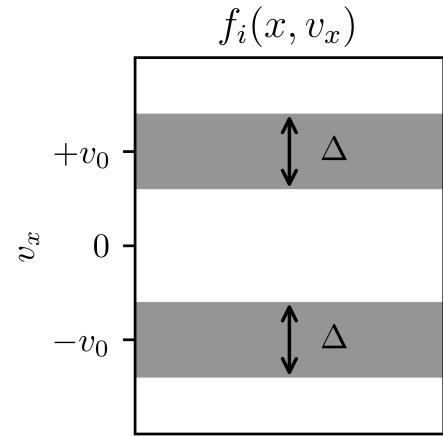
Pictorial representation

- Two-stream instability, waterbag



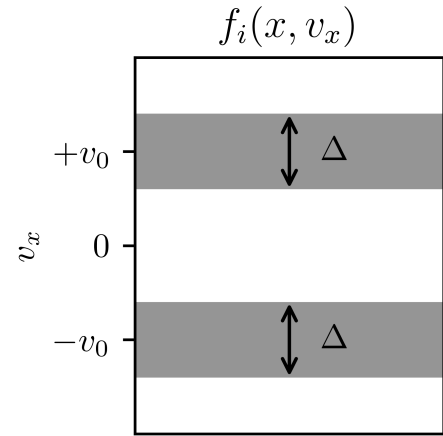
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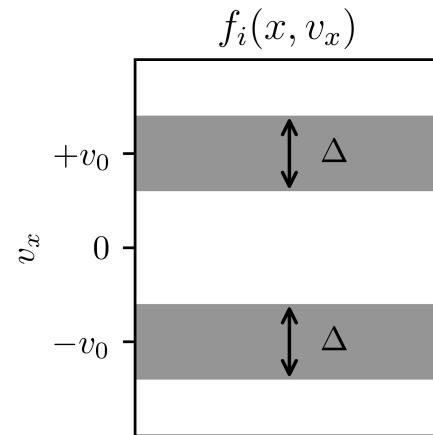
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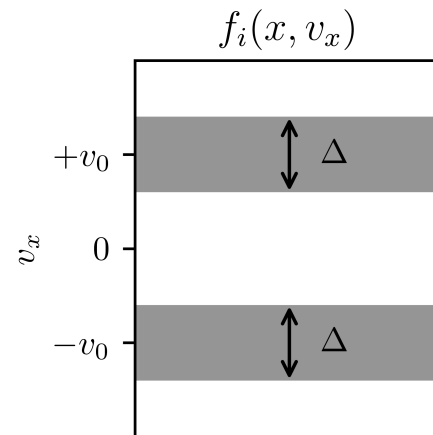
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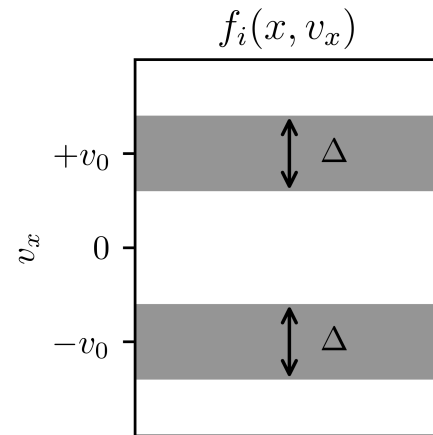
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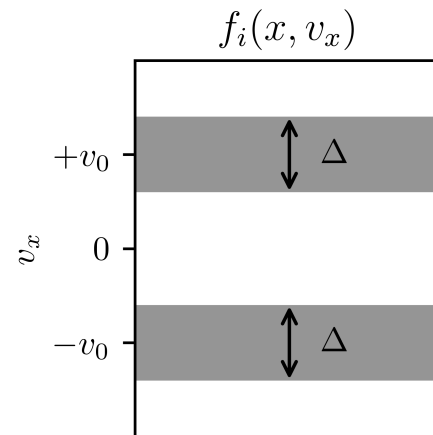
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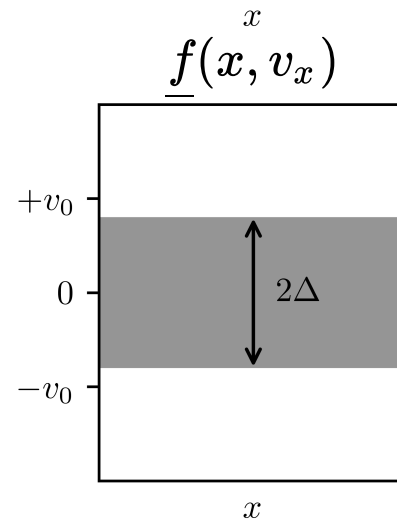
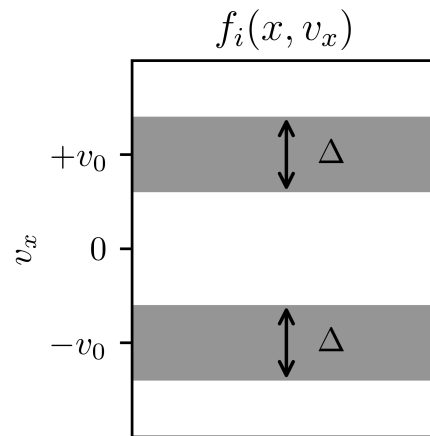
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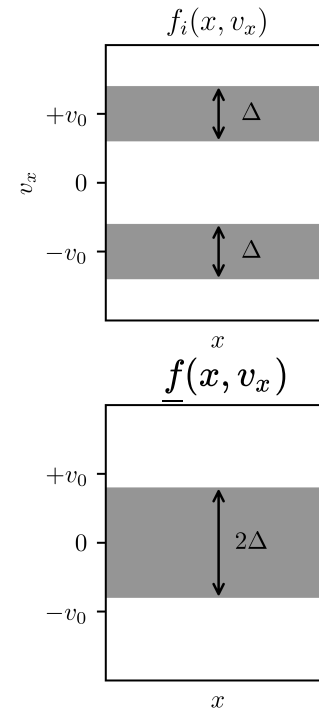
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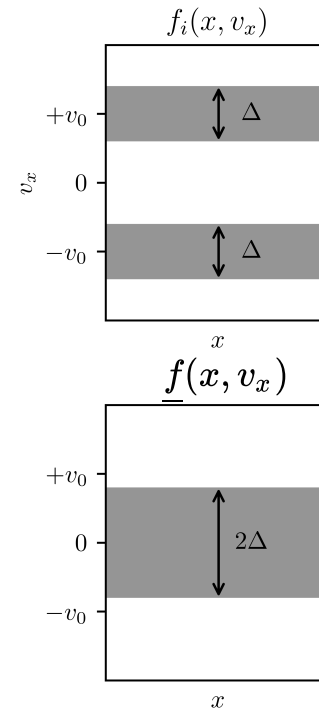
A fundamental issue

- Our simple example:
 - Ground state is homogeneous: **no electric field** - **PROBLEM!**



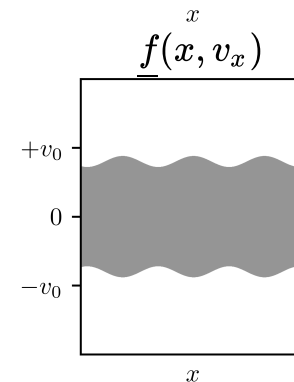
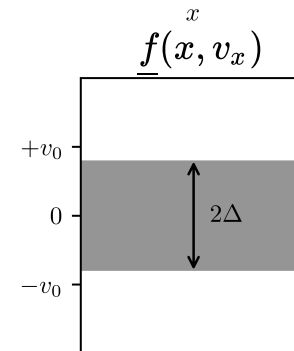
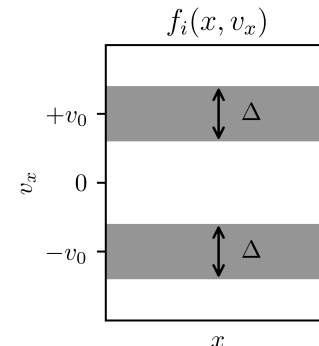
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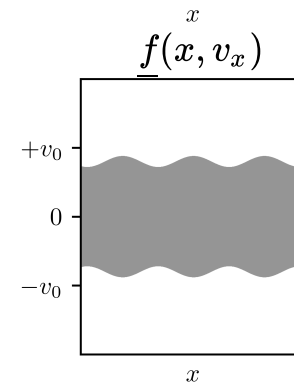
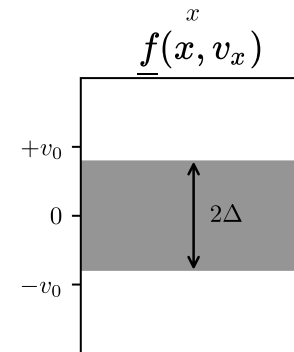
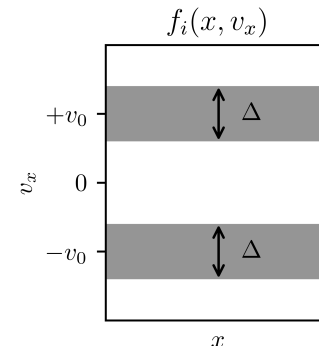
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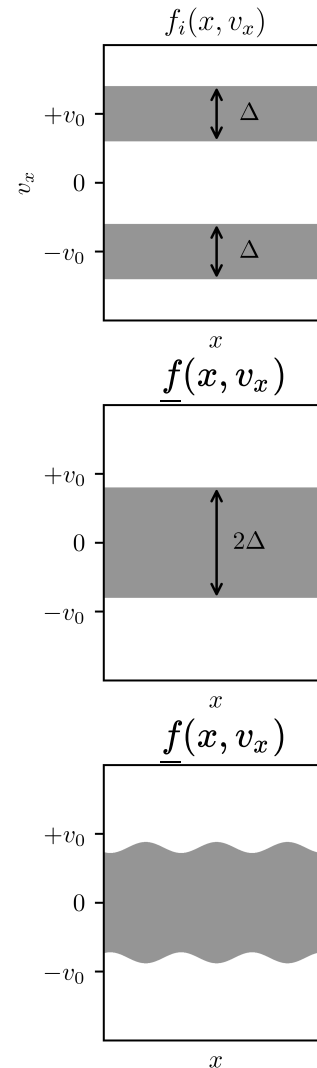
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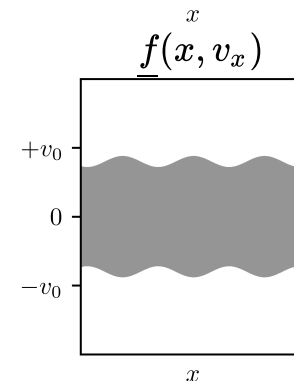
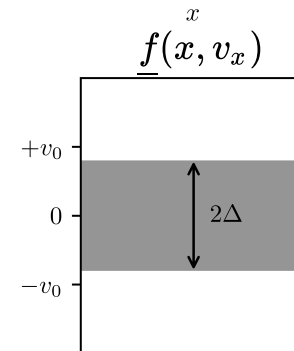
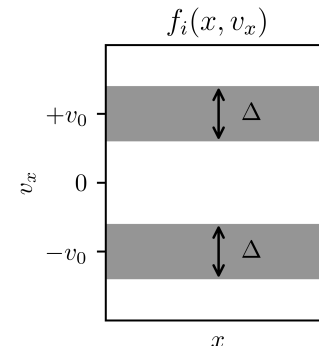
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- Problem posed generally: What is the ground state with a **given** density profile? (converts problem to question about given profile)
- Main goal:** investigate how density constraints change ground state and free energy

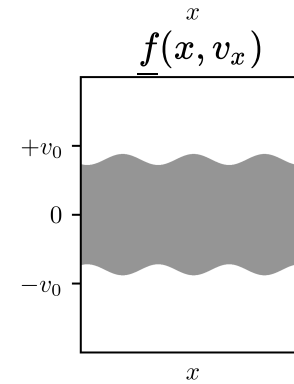
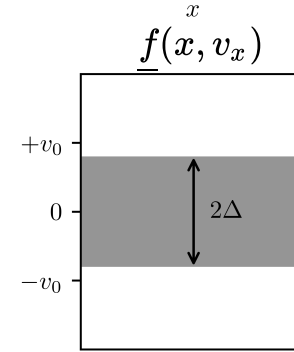
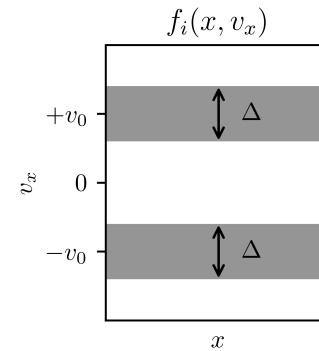


Disclaimer:

The maths can get aesthetically awful—I focus on physics and sketching calculations here.

Theory:
*Two-stream
waterbags &
phase-
transitions*

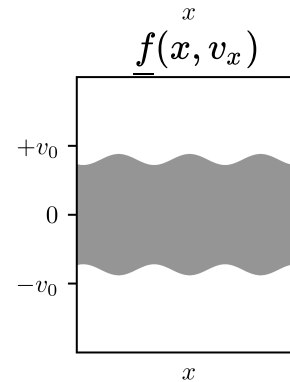
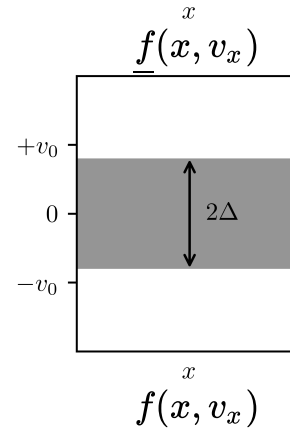
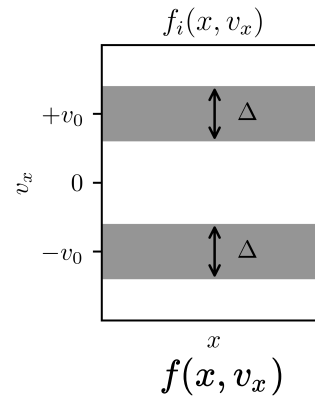
EPFL The set-up



The set-up

- Available energy with density profile in d -dimensions:

$$A = E_{\text{init}} - E_{\text{G}} \left[\frac{n(\mathbf{r})}{n_0} \right]^{\frac{d+2}{d}}$$



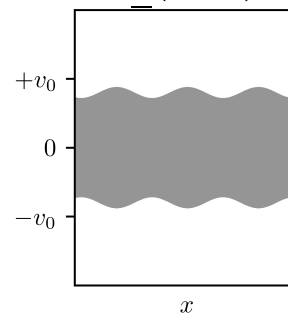
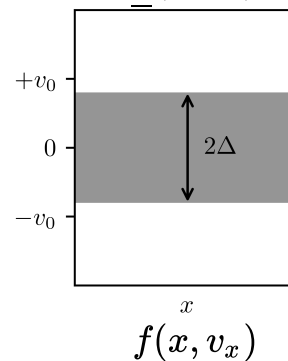
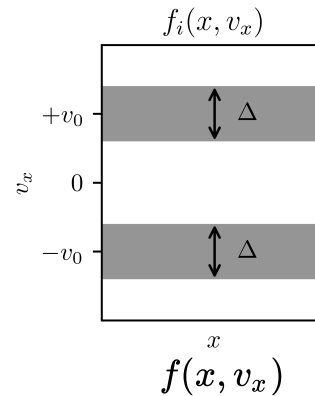
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- The energy of the electric field:

$$E_{\phi} = \int d\mathbf{r} \frac{|\nabla\phi|^2}{8\pi}$$



The set-up

- Available energy with density profile in d -dimensions:

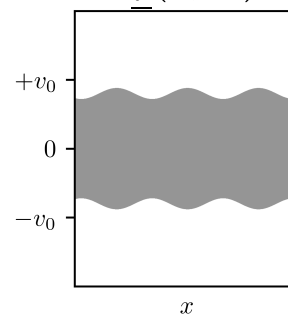
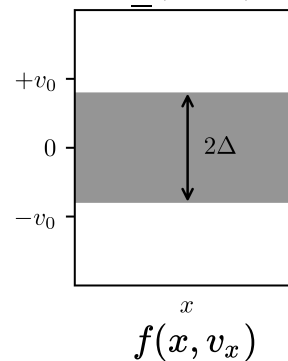
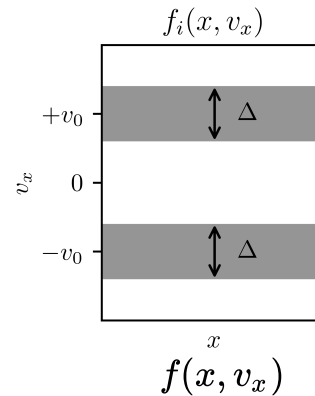
$$A = E_{\text{init}} - E_G \left[\frac{n(\mathbf{r})}{n_0} \right]^{\frac{d+2}{d}}$$

- The energy of the electric field:

$$E_\phi = \int d\mathbf{r} \frac{|\nabla\phi|^2}{8\pi}$$

- Density & potential via Poisson:

$$\lambda_D^2 \nabla^2 \phi = \frac{\delta n_e}{n_0} - \frac{\delta n_i}{n_0}$$



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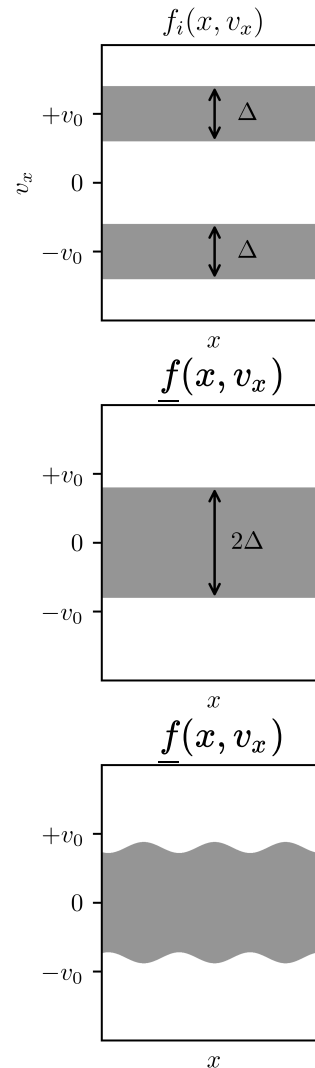
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- Goal: maximise A , and liberated energy must reside in the field

$$A = E_\phi$$



A sketch of what rolls out

Three important ingredients...

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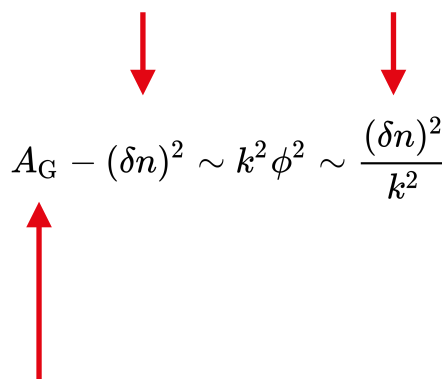
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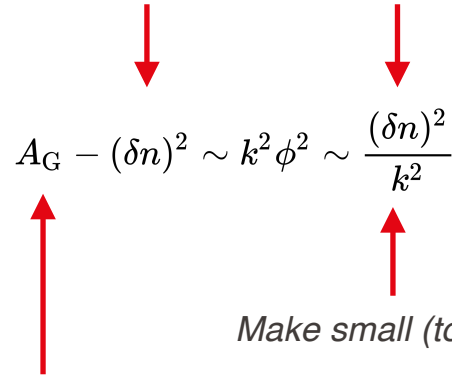
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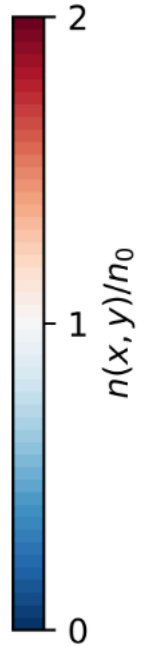
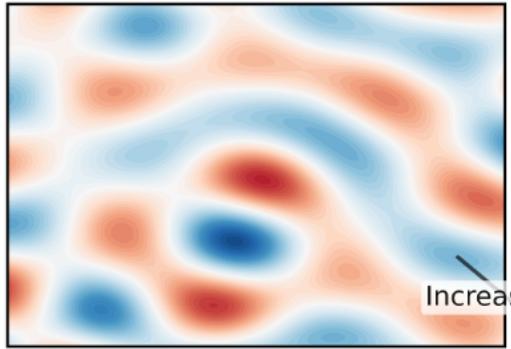
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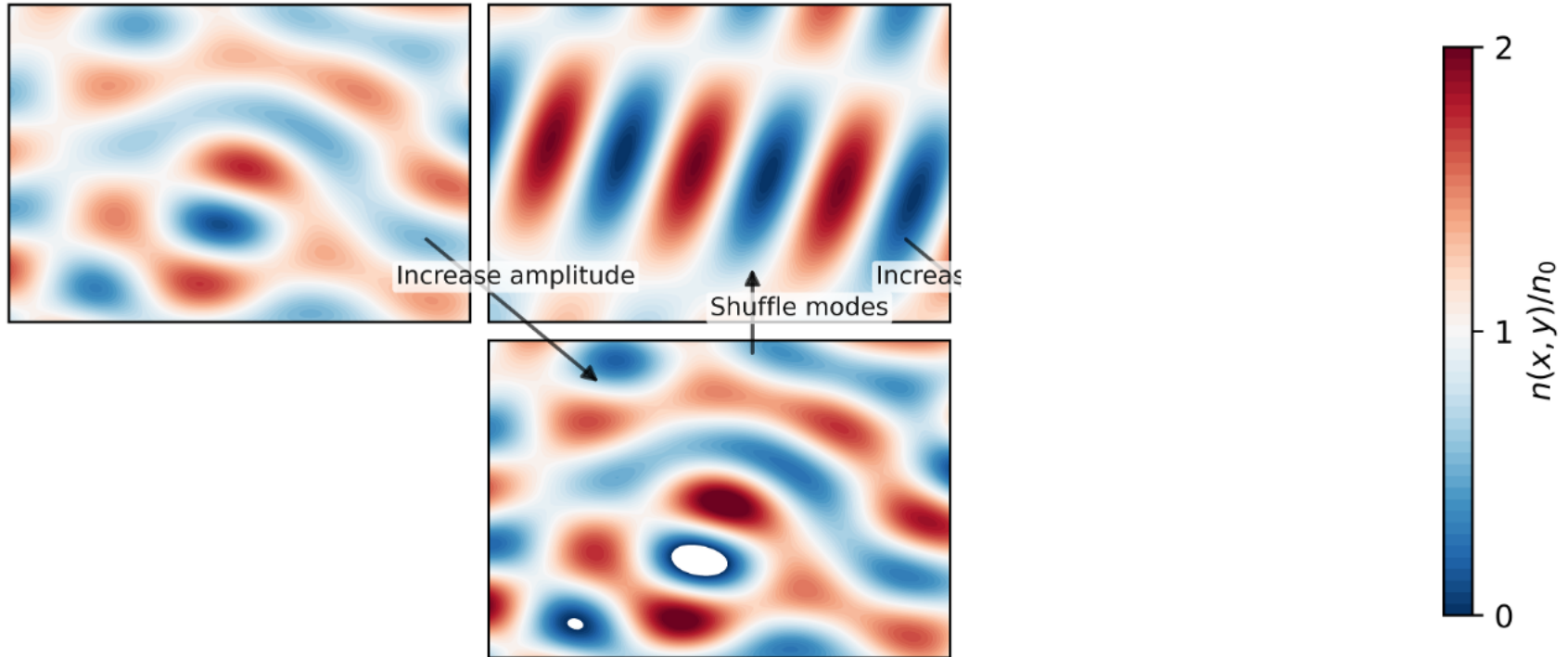
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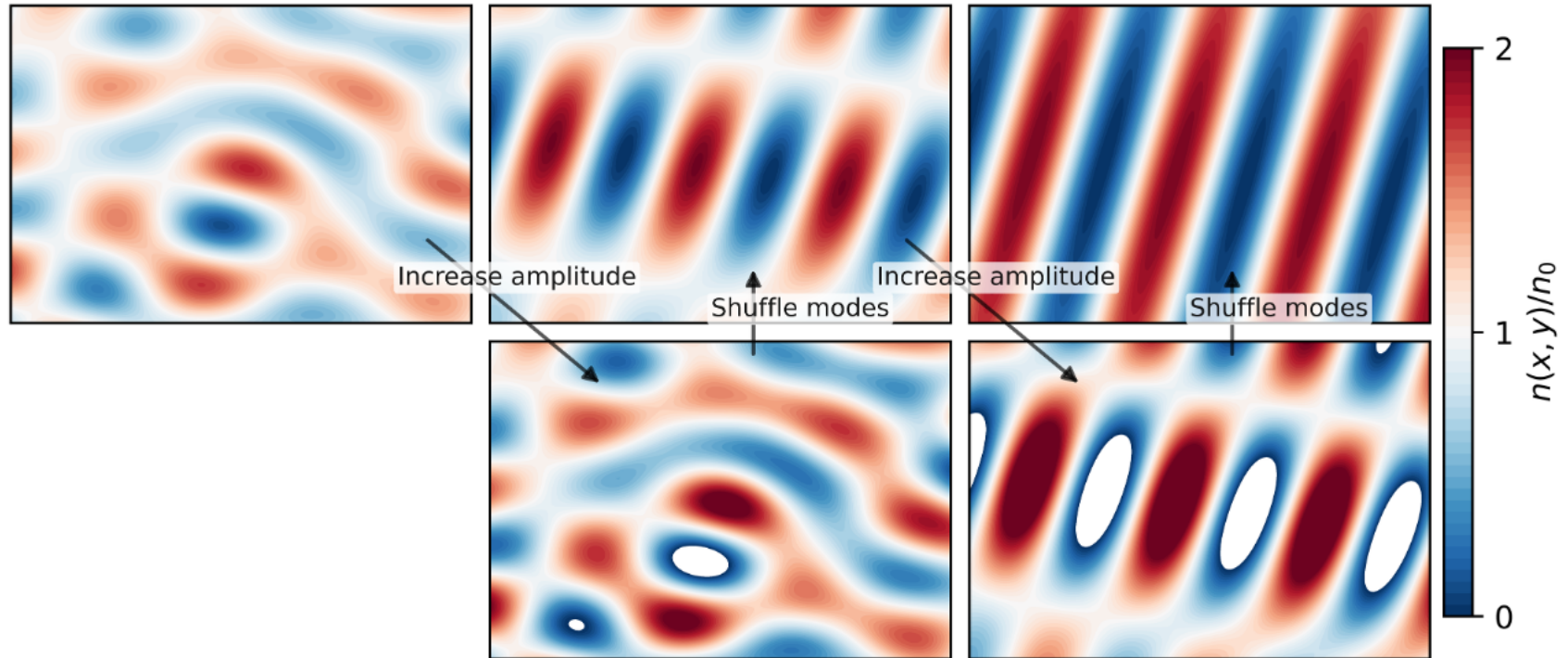
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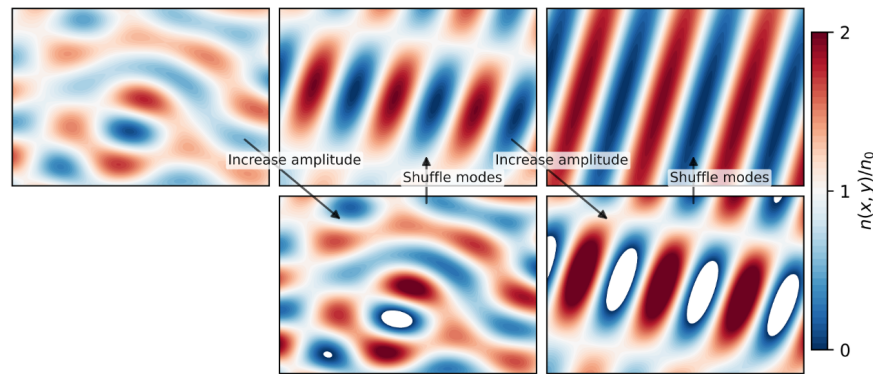
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Phase-transition-like behaviour of ground state

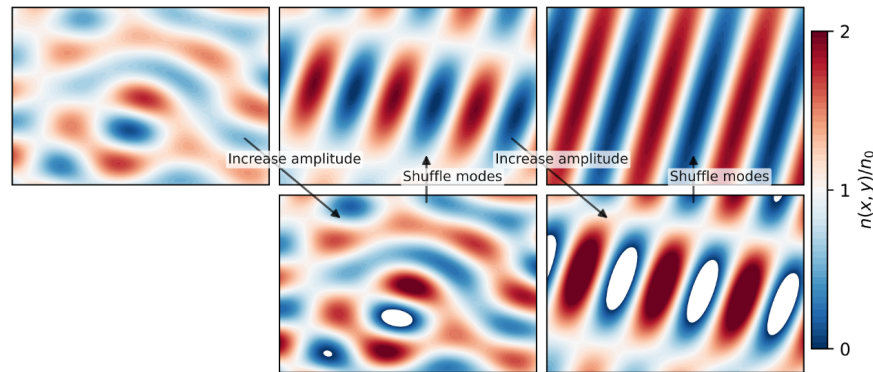
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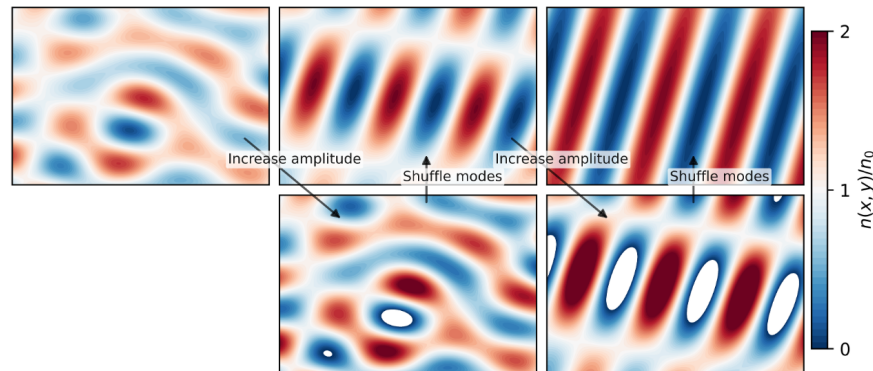
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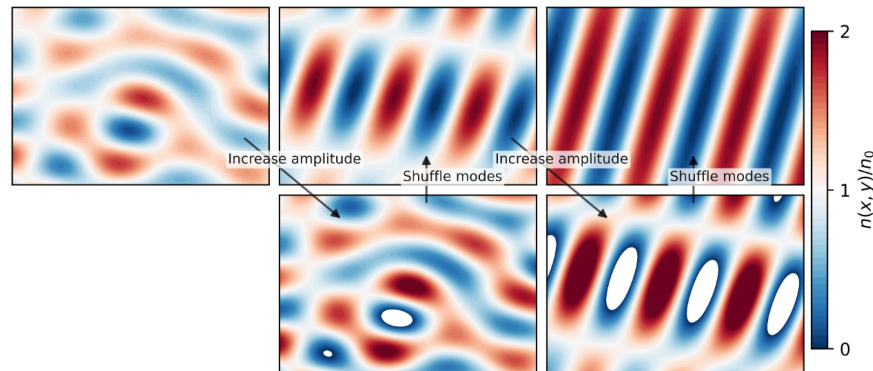
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Phase-transition-like behaviour of ground state

Take-aways...

- Only three simple effects required (conserved energy+Poisson, Debye screening, positive density)
- More free energy buys less field (density starts spending it on shape rather than amplitude)
- Related to cellularisation of high-amplitude turbulence?



Theory:
*Asymptotic
expansion*

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$$\int d\mathbf{z} \Theta[\underline{f} - \phi] \sqrt{g} = \int d\mathbf{z} \Theta[f_i - \phi] \sqrt{g}$$

$$\underline{f} = \underline{f}[\epsilon + \kappa(\mathbf{r}), \mathbf{y}]$$

- Construct functional with

- Invariants of motion: $\mathbf{y}$

- Remainder of phase space irrelevant

- Jacobian: $\int d\mathbf{z} \delta \mathbf{z} = \int d\mathbf{z}$

- Given ground-state density: $\underline{n}(\mathbf{r})$

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Very difficult to solve in general!
Let's look at asymptotics...

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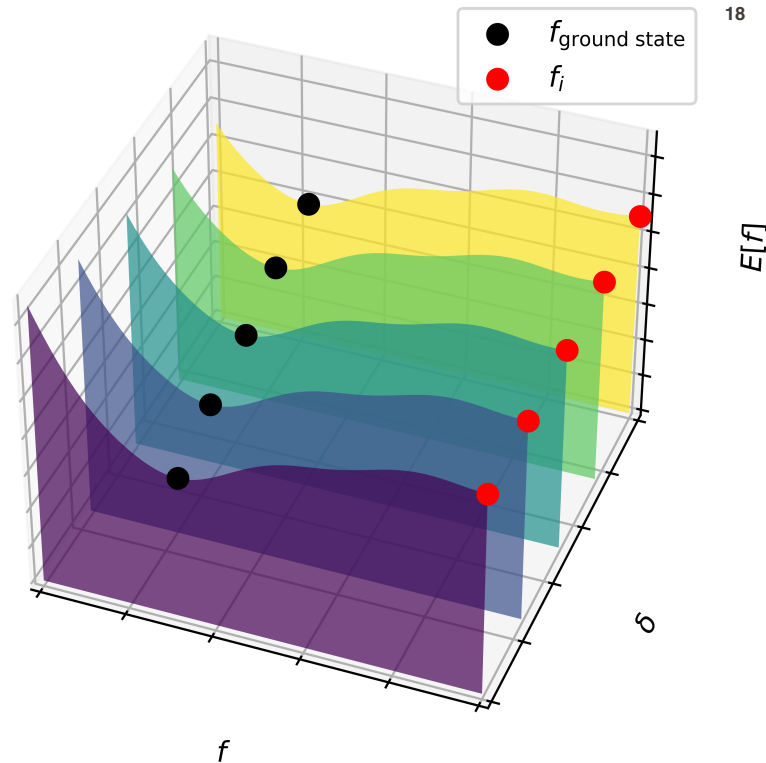
Asymptotic analysis

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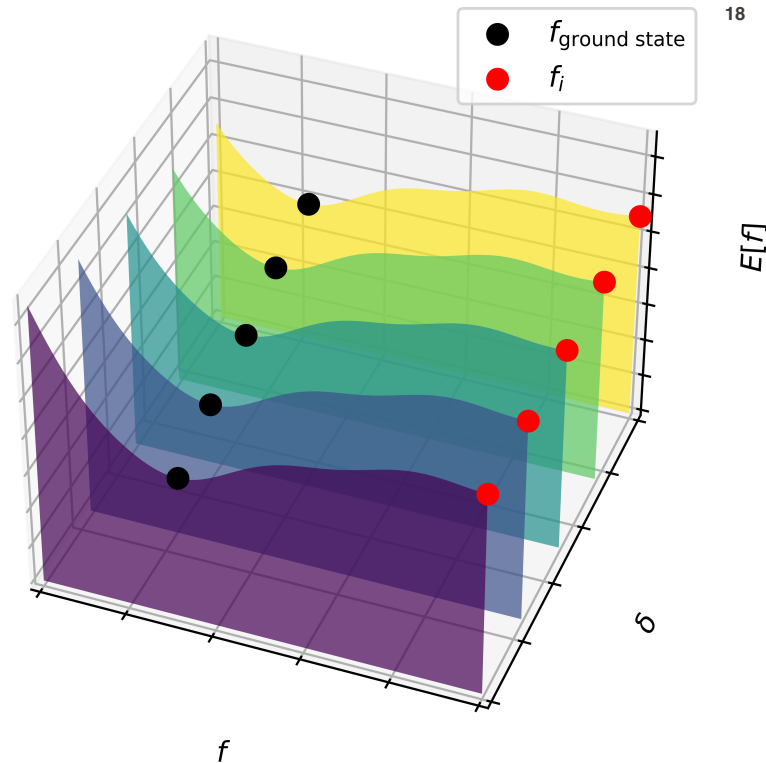
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$$\int d\mathbf{v} \, \underline{f}[\epsilon + \kappa, \mathbf{y}] = \underline{\delta n_1}(\mathbf{r}), \quad \kappa(\mathbf{r}) = \sum_{n=1}^{\infty} \delta^n \kappa_n(\mathbf{r})$$



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R.J.J. Mackenbach and R.J. Ewart

and taking the first and second derivative of this equation with respect to w , one finds that

$$\delta[f_0(\mathbf{x}) - F_0(w, \mathbf{y})] = -\frac{\delta[w - \epsilon]}{F'_0(w, \mathbf{y})} \quad (4.13)$$

and

$$\delta'[f_0(\mathbf{x}) - F_0(w, \mathbf{y})] = \frac{1}{F'_0(w, \mathbf{y})} \frac{\partial}{\partial w} \left(\frac{\delta[w - \epsilon]}{F'_0(w, \mathbf{y})} \right), \quad (4.14)$$

respectively.

The next order, $\mathcal{O}(\varepsilon)$, gives an equation involving derivatives of the Heaviside function, which is the Dirac-delta distribution $\delta[x]$, giving rise to

$$\int d\mathbf{z} [f_1(\mathbf{x}) - F_1(w, \mathbf{y})] \delta[f_0(\mathbf{x}) - F_0(w, \mathbf{y})] \sqrt{g} = - \int d\mathbf{z} \kappa_1(\mathbf{r}) \delta[w - \epsilon] \sqrt{g}, \quad (4.15)$$

and using (4.13) this may be written as

$$\int d\mathbf{z} (f_1 - F_1) \delta[w - \epsilon] \sqrt{g} = F'_0(w, \mathbf{y}) \int d\mathbf{z} \kappa_1 \delta[w - \epsilon] \sqrt{g}. \quad (4.16)$$

Filling this into the leading-order result of the available energy (4.7), writing $\epsilon = w\delta[w - \epsilon]$, gives

$$\begin{aligned} A &\approx \varepsilon \int d\mathbf{y} d\mathbf{z} dw [f_1(\mathbf{x}) - F_1(w, \mathbf{y}) - \kappa_1(\mathbf{r}) F'_0(w, \mathbf{y})] w \delta[w - \epsilon] \sqrt{g} \\ &\approx 0. \end{aligned} \quad (4.17)$$

The available energy to order $\mathcal{O}(\varepsilon)$ thus vanishes and one is required to go to second order. It shall prove useful to rewrite (4.16) in terms of F_1 alone, using (4.9). One finds the following integral equation for F_1 ,

$$\int d\mathbf{z} \left(\frac{F_1}{F'_0} - \frac{\langle F_1 \rangle_{\mathbf{e}}}{\langle F'_0 \rangle_{\mathbf{e}}} \right) \delta[w - \epsilon] \sqrt{g} = \int d\mathbf{z} \left(\frac{f_1}{F'_0} - \frac{n_1}{\langle F'_0 \rangle_{\mathbf{e}}} \right) \delta[w - \epsilon] \sqrt{g}, \quad (4.18)$$

having defined $\langle \dots \rangle_{\mathbf{e}} = \int d\mathbf{v} \dots$

To second order $\mathcal{O}(\varepsilon^2)$, the ground-state equation becomes

$$\begin{aligned} \int d\mathbf{z} \{ [f_1(\mathbf{x}) - F_1(w, \mathbf{y})]^2 \delta[f_0(\mathbf{x}) - F_0(w, \mathbf{y})] - F_2(w, \mathbf{y}) \delta[f_0(\mathbf{x}) - F_0(w, \mathbf{y})] \} \sqrt{g} \\ = \int d\mathbf{z} (\kappa_1(\mathbf{r})^2 \delta[w - \epsilon] - \kappa_2(\mathbf{r}) \delta[w - \epsilon]) \sqrt{g}. \end{aligned} \quad (4.19)$$

Using (4.13) and (4.14) we find that

$$\begin{aligned} \int d\mathbf{z} \left\{ (f_1 - F_1)^2 \frac{\partial}{\partial w} \left(\frac{\delta[w - \epsilon]}{F'_0} \right) - \kappa_1^2 \frac{\partial F'_0 \delta[w - \epsilon]}{\partial w} \right\} \sqrt{g} \\ = - \int d\mathbf{z} [F_2 + \kappa_1^2 F''_0 + \kappa_2 F'_0] \delta[w - \epsilon] \sqrt{g}. \end{aligned} \quad (4.20)$$

Density-constrained ground states

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Substituting this into the available energy, it may be written as

$$\begin{aligned} A &\approx -\frac{\varepsilon^2}{2} \int d\mathbf{y} d\mathbf{z} dw [F_2(w, \mathbf{y}) + \kappa_1^2 F''_0(w, \mathbf{y}) + \kappa_2 F'_0(w, \mathbf{y}) \\ &\quad + 2\kappa_1 F'_1(w, \mathbf{y})] w \delta[w - \epsilon] \sqrt{g} \\ &\approx \frac{\varepsilon^2}{2} \int d\mathbf{y} d\mathbf{z} dw \left([f_1 - F_1]^2 \frac{\partial}{\partial w} \left(\frac{\delta[w - \epsilon]}{F'_0} \right) - \kappa_1^2 \frac{\partial (F'_0 \delta[w - \epsilon])}{\partial w} \right. \\ &\quad \left. - 2\kappa_1 F'_1 \delta[w - \epsilon] \right) w \sqrt{g}. \end{aligned} \quad (4.21)$$

We integrate the terms that have a derivative with respect to w by parts, giving

$$\int dw w (f_1 - F_1)^2 \frac{\partial}{\partial w} \left(\frac{\delta[w - \epsilon]}{F'_0} \right) = - \int dw [(f_1 - F_1)^2 - 2w(f_1 - F_1) F'_1] \frac{\delta[w - \epsilon]}{F'_0} \quad (4.22)$$

and

$$\int dw w \frac{\partial (F'_0 \delta[w - \epsilon])}{\partial w} = - \int dw F'_0 \delta[w - \epsilon]. \quad (4.23)$$

Combining all results, we find that

$$A \approx \frac{\varepsilon^2}{2} \int d\mathbf{y} d\mathbf{z} dw \left[-\frac{(f_1 - F_1)^2}{F'_0} + \kappa_1^2 F'_0 + 2w F'_1 \left(\frac{f_1 - F_1}{F'_0} - \kappa_1 \right) \right] \delta[w - \epsilon] \sqrt{g}, \quad (4.24)$$

where the last term vanishes due to (4.16).

All in all, we have thus shown that

$$A \approx \frac{\varepsilon^2}{2} \int d\mathbf{x} \frac{(f_1(\mathbf{x}) - F_1[\epsilon, \mathbf{y}])^2}{|F'_0[\epsilon, \mathbf{y}]|} - \frac{\varepsilon^2}{2} \int d\mathbf{x} \kappa_1^2(\mathbf{r}) |F'_0[\epsilon, \mathbf{y}]| + \mathcal{O}(\varepsilon^3). \quad (4.25)$$

The first term is identical to the result in Helander & Mackenbach (2024) and is closely related to the notion of Helmholtz free energy. This term is furthermore always positive definite, so that if $\kappa = 0$, the available energy is always greater than or equal to zero. The second term encoding the constraint on the density can only serve to decrease the available energy, which is manifestly true as a more constrained calculation of it can only lower the value. It thus encodes what the energetic penalty is of some perturbation to the ground-state density, n_1 , which couples to κ_1 via (4.8) and (4.16).

Another useful form of the available energy may be obtained by substituting in (4.8), so that the result becomes

$$A \approx \frac{\varepsilon^2}{2} \int d\mathbf{x} \left[\frac{(f_1(\mathbf{x}) - F_1[\epsilon, \mathbf{y}])^2}{|F'_0[\epsilon, \mathbf{y}]|} - \left(\frac{n_1(\mathbf{r}) - \langle F_1 \rangle_{\mathbf{e}}}{|F'_0[\epsilon, \mathbf{y}]|} \right)^2 F'_0[\epsilon, \mathbf{y}] \right]. \quad (4.26)$$

This can be simplified further by noting that $\int F'_0[\epsilon, \mathbf{y}] d\mathbf{x} = \int \langle F'_0 \rangle_{\mathbf{e}} d\mathbf{r}$, so that we have⁷

$$A \approx \frac{\varepsilon^2}{2} \int d\mathbf{x} \frac{(f_1(\mathbf{x}) - F_1[\epsilon, \mathbf{y}])^2}{|F'_0[\epsilon, \mathbf{y}]|} - \frac{\varepsilon^2}{2} \int d\mathbf{r} \frac{(n_1(\mathbf{r}) - \langle F_1 \rangle_{\mathbf{e}})^2}{|F'_0[\epsilon, \mathbf{y}]|}. \quad (4.27)$$

⁷The second integral is ≥ 0 as a consequence of Hölder's inequality, since $\int d\mathbf{r} d\mathbf{v} Q^2/B \geq \int d\mathbf{r} (\int d\mathbf{v} Q)^2 / \int d\mathbf{v} B$ for $B > 0$ and $Q \geq 0$.

Yielding opaque equations...

Ground state:

$$\int dz f_1 \delta[w - \epsilon] \sqrt{g} = \int dz \left(\underline{f_1} - \kappa_1 f'_0 \right) \delta[w - \epsilon] \sqrt{g}$$

Density constraint:

$$\int dv \left(\underline{f_1} + \kappa_1 f'_0 \right) = n_1$$

Available energy:

$$A \approx -\frac{\delta^2}{2} \int d\mathbf{x} \frac{(f_1 - \underline{f_1})^2}{f'_0} + \frac{\delta^2}{2} \int d\mathbf{x} \kappa_1^2 f'_0$$

We do learn something...

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[Helander & Mackenbach 2024]

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Cost of imposed profile
(new!)

Analytically solvable in certain cases:

Case (1): A unique property of the Maxwellian

EPFL **A unique property of the Maxwellian**
Nonlinear stability

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- Suppose **density remains fixed** (e.g. adiabatic electrons)

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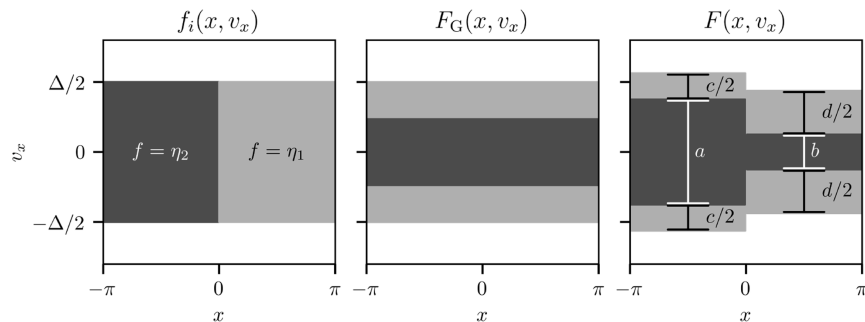
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Case (2): “Drift kinetic” ions in a constant magnetic field

Maxwellian ions in a constant magnetic field

Maxwellian ions in a constant magnetic field

- Initial f Maxwellian with varying density/temperature

$$n = n_0(1 + \nu)$$

$$T = T_0(1 + \tau)$$

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$$A = \frac{\delta^2 n_0 T_0 V}{2} \left\langle \nu^2 - \tilde{\nu}^2 + \frac{3}{2} \tau^2 \right\rangle$$

- Available energy generalises [Helander 2020]

Couple with field equations!

- Invariant energy [Dubin 1983]

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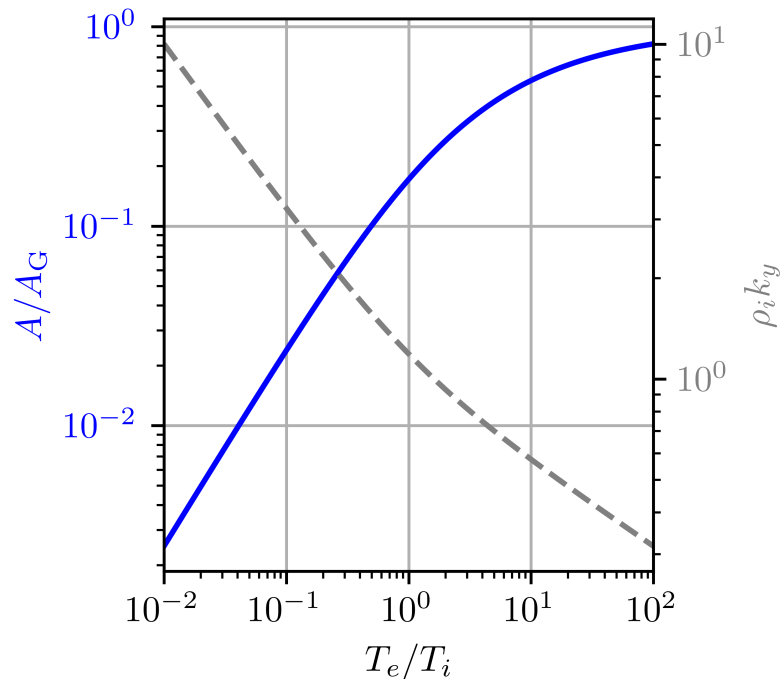
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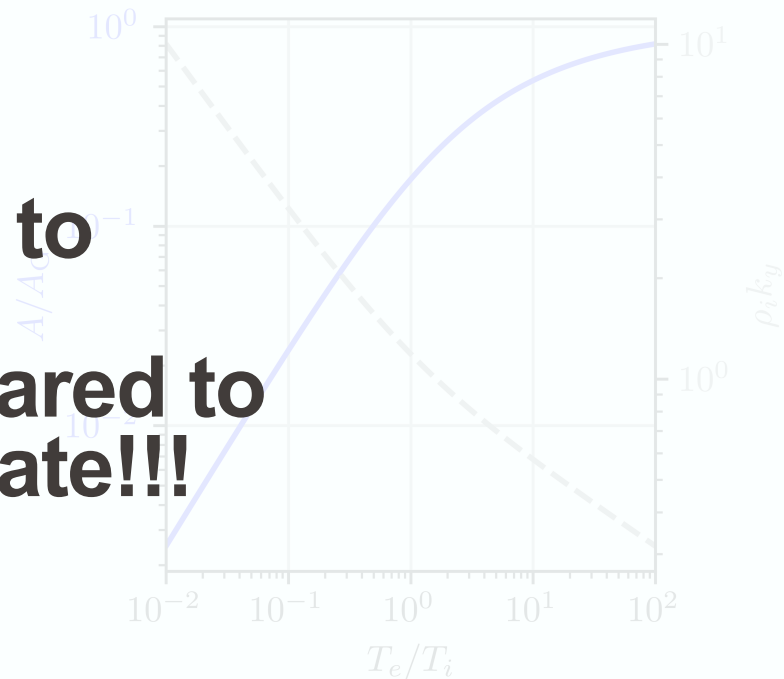
**$T_e = T_i$ leads to
83%
decrease compared to
“naïve” estimate!!!**

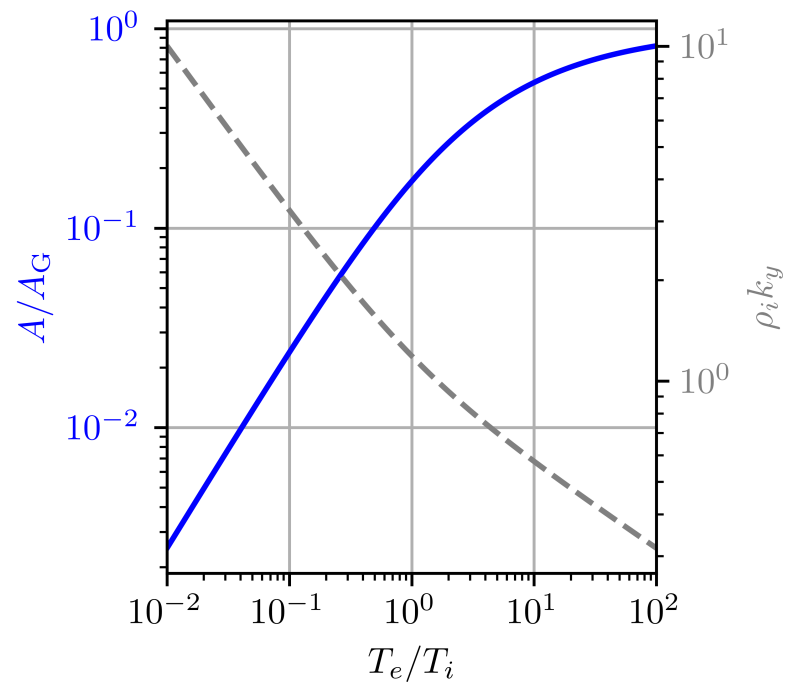
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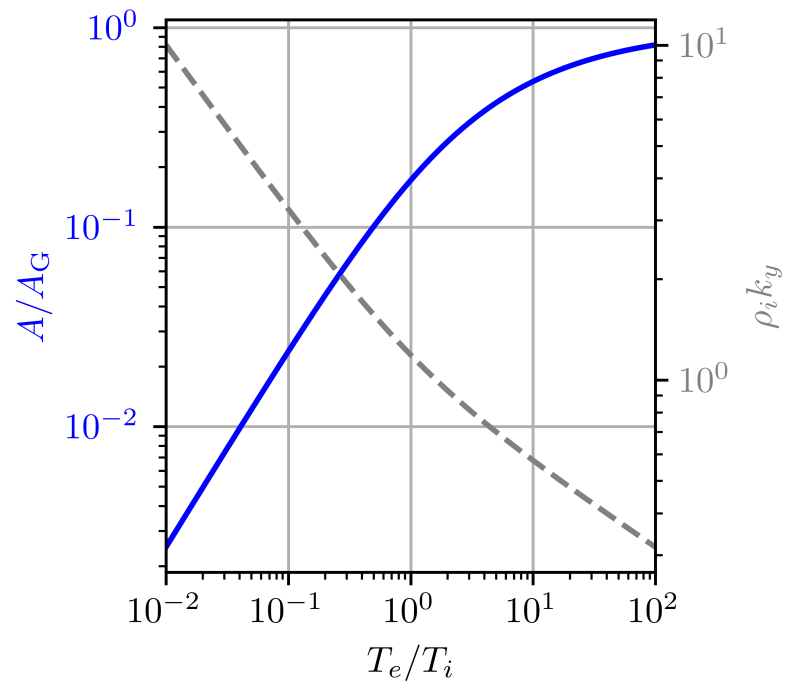
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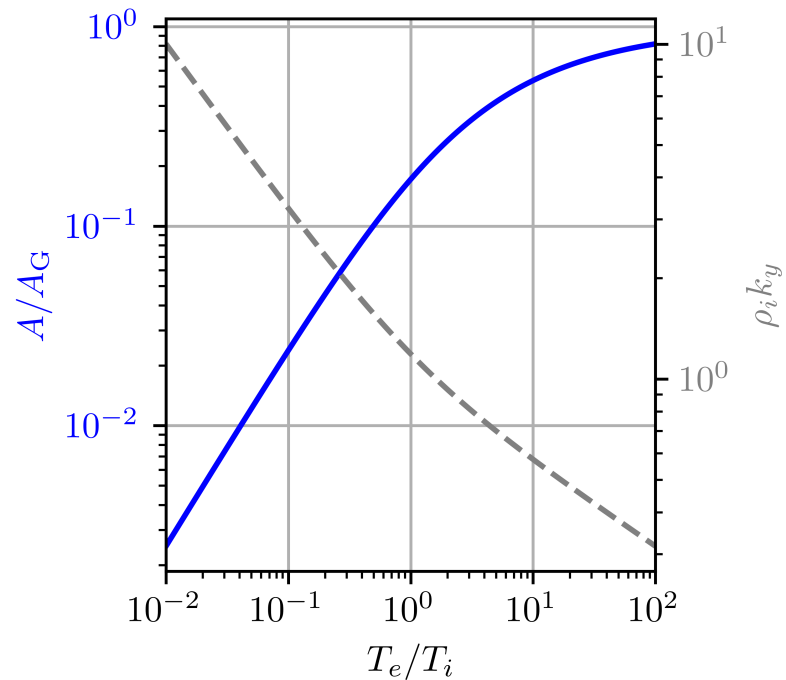


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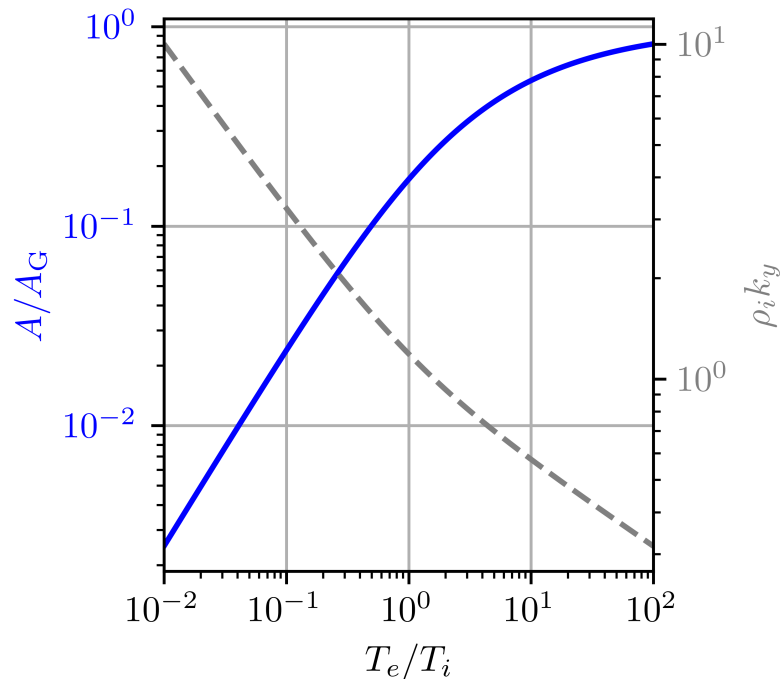
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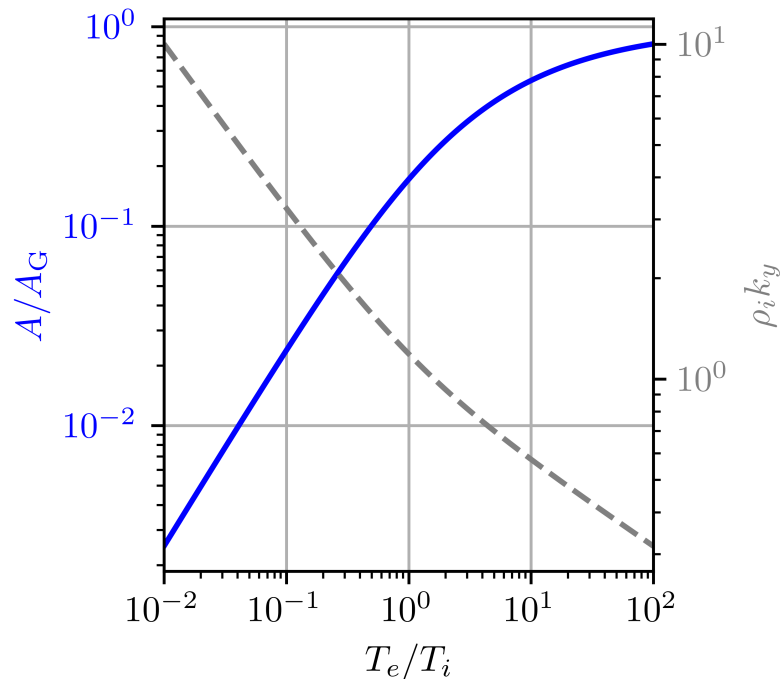
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Case (2.5): The effect of zonal flows

Corrected adiabatic response

Accounting for zonal flows

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- Do ask after talk :)

**Conclusion +
open ends**

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- The end-station: ground-states with flow, coupled to Vlasov-Maxwell/electromagnetic GK?

Are you interested in plasma thermodynamics?

- Workshop on this topic 5–7 October, 2026!
- In Lausanne, Switzerland
- Please let me know if you'd like to participate!
- <https://indico.epfl.ch/event/12/>
- ralf.mackenbach@epfl.ch



Extra slides

Available Energy

Curvature driven ITG

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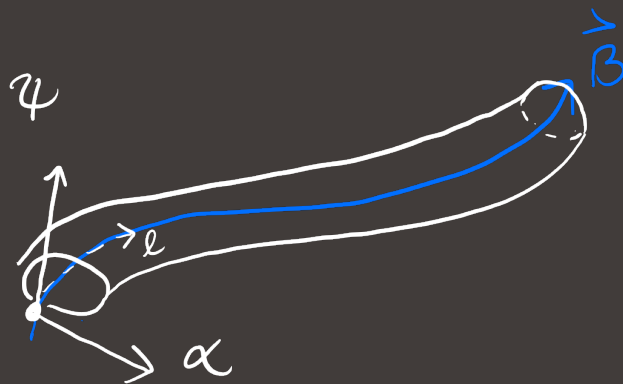
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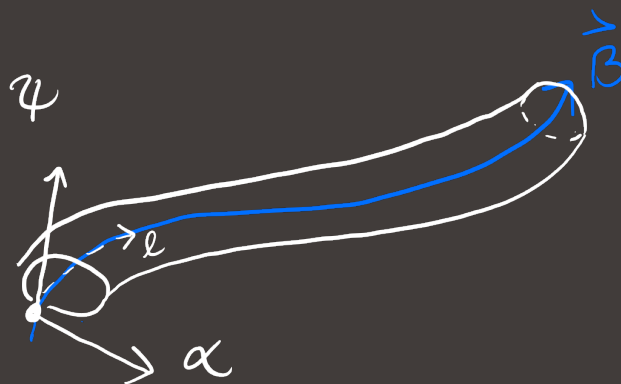
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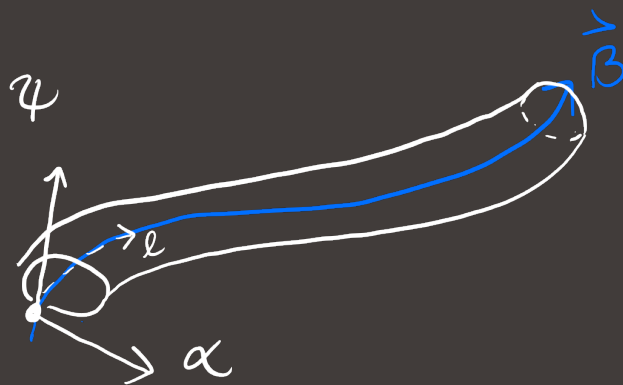
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$$\omega_{\psi} = \mathbf{v}_D \cdot \nabla \psi$$

$$\omega_{\alpha} = \mathbf{v}_D \cdot \nabla \alpha$$

$$\omega_*^T = + \frac{T_0}{q_e} \left(\frac{\partial \ln n}{\partial \psi} + \frac{\partial \ln T}{\partial \psi} \left[\frac{\epsilon_0}{T_0} - \frac{3}{2} \right] \right)$$



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Density constraint:

$$\int \frac{f_{M,0}}{T_0} (-\kappa_\alpha - \omega_\alpha) G(\kappa_\psi, \kappa_\alpha) d\mu dv_\parallel = \int \frac{f_{M,0}}{T_0} (\omega_*^T - \omega_\alpha) d\mu dv_\parallel,$$
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Available energy:

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Geometry Profiles

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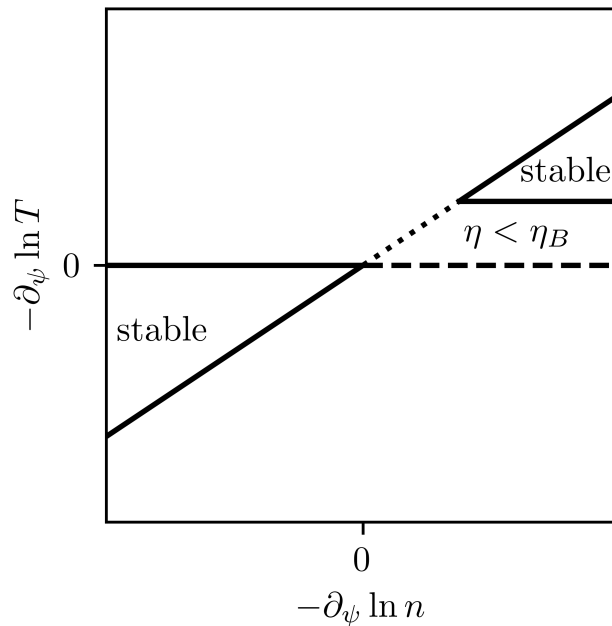
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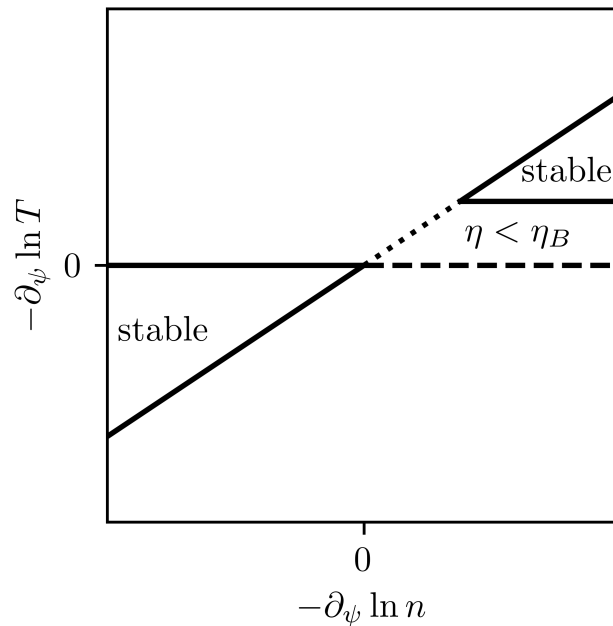
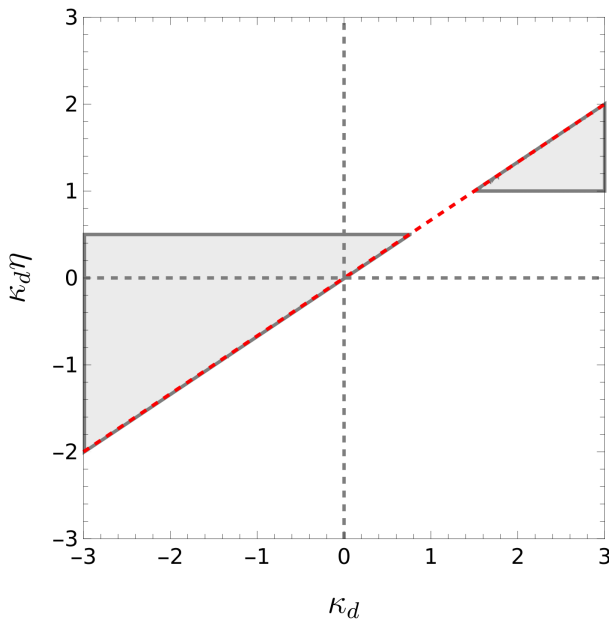
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In a device with everywhere bad curvature



Close to nonlinear stability bounds [Costello 2025]!



When is there no available energy; cont.

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- Available energy vanishes for **any geometry** (to leading order)
iff

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**Comparison
to nonlinear
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Comparison to nonlinear gyrokinetic simulations

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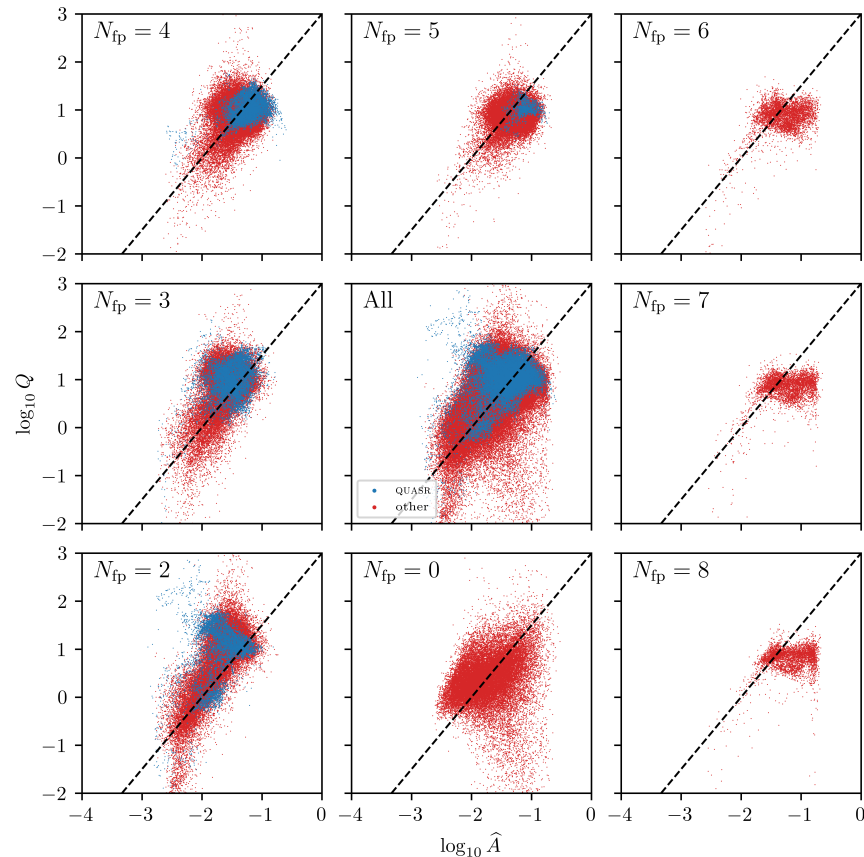
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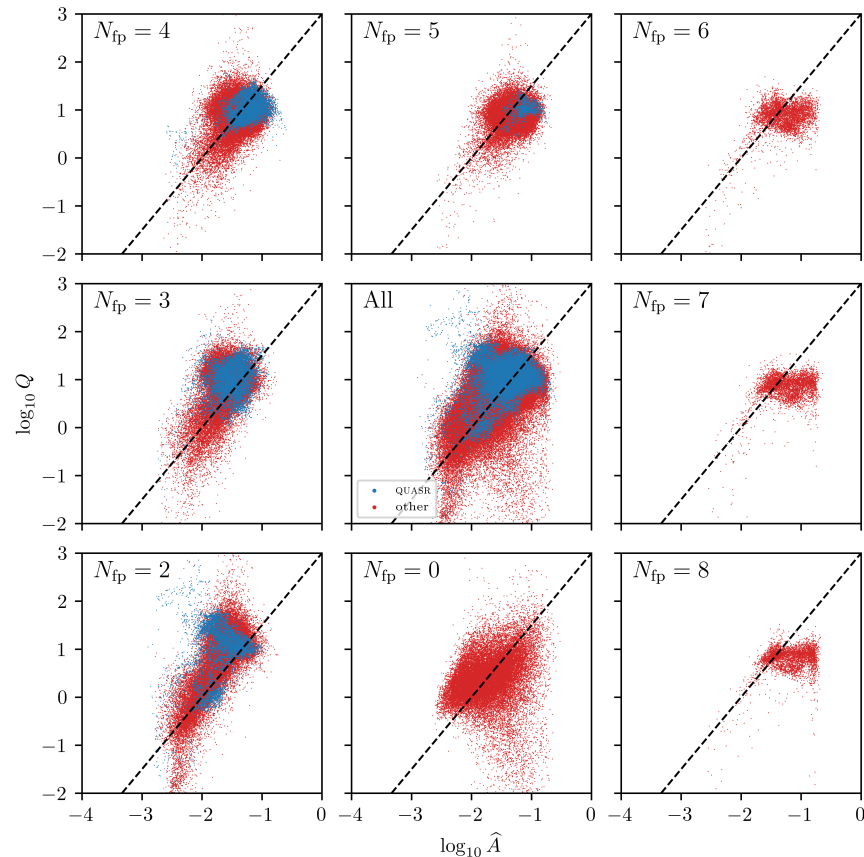
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- Multiple NFP, vacuum, & finite beta



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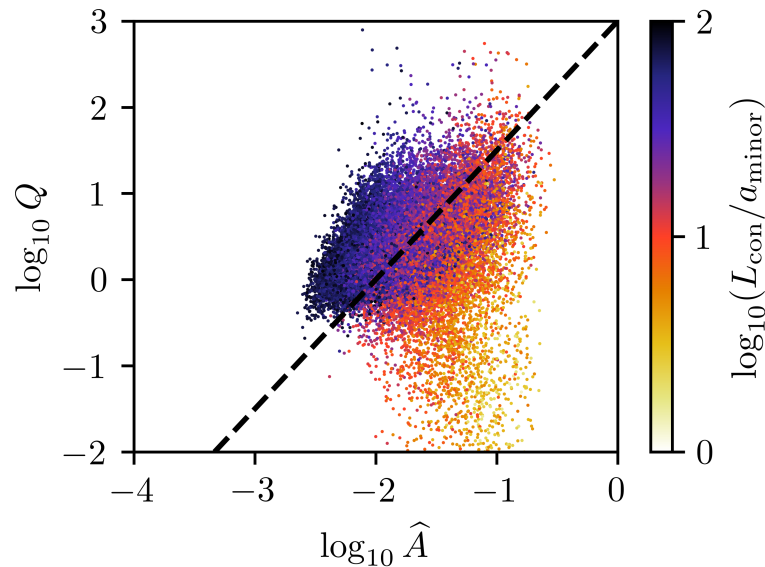
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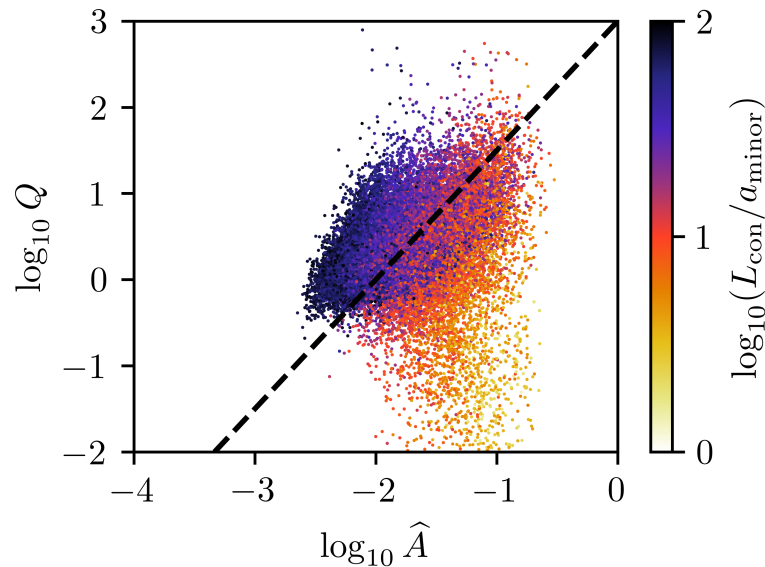


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- Test hypothesis in tokamaks (unambiguous connection length)
- Points with short connection length deviate most strongly



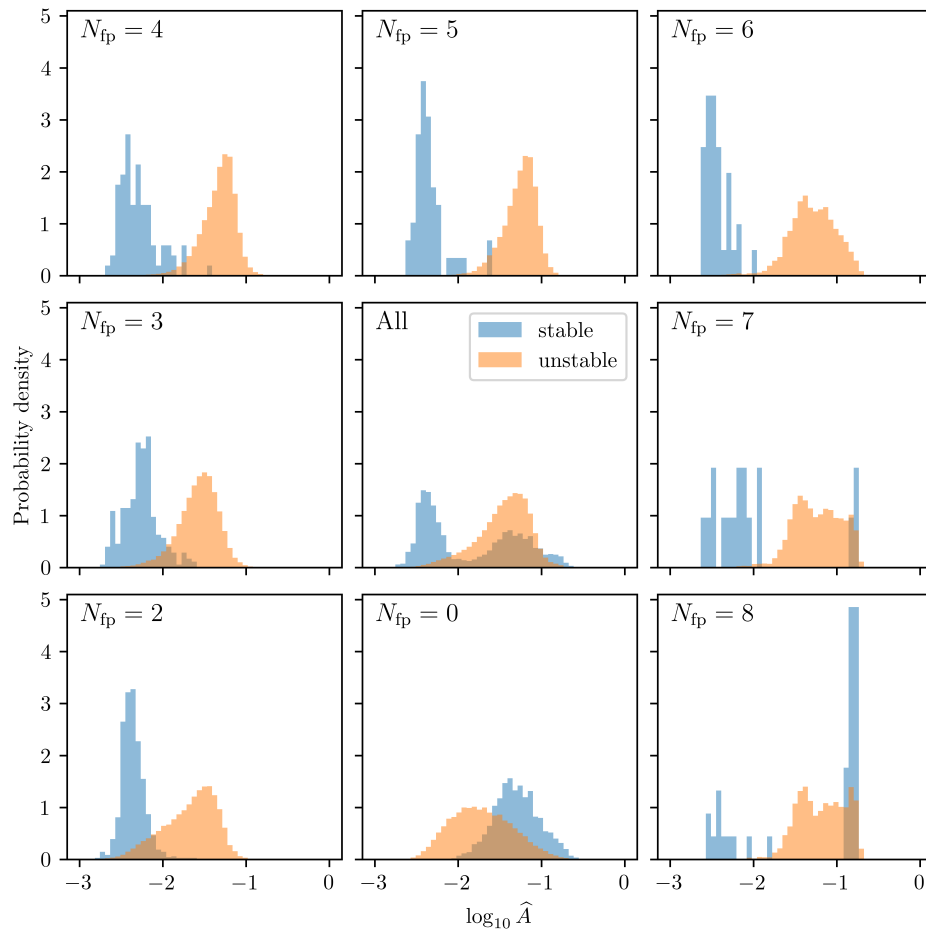
A positive: stability

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- Defining $Q < 0.1$ as stability

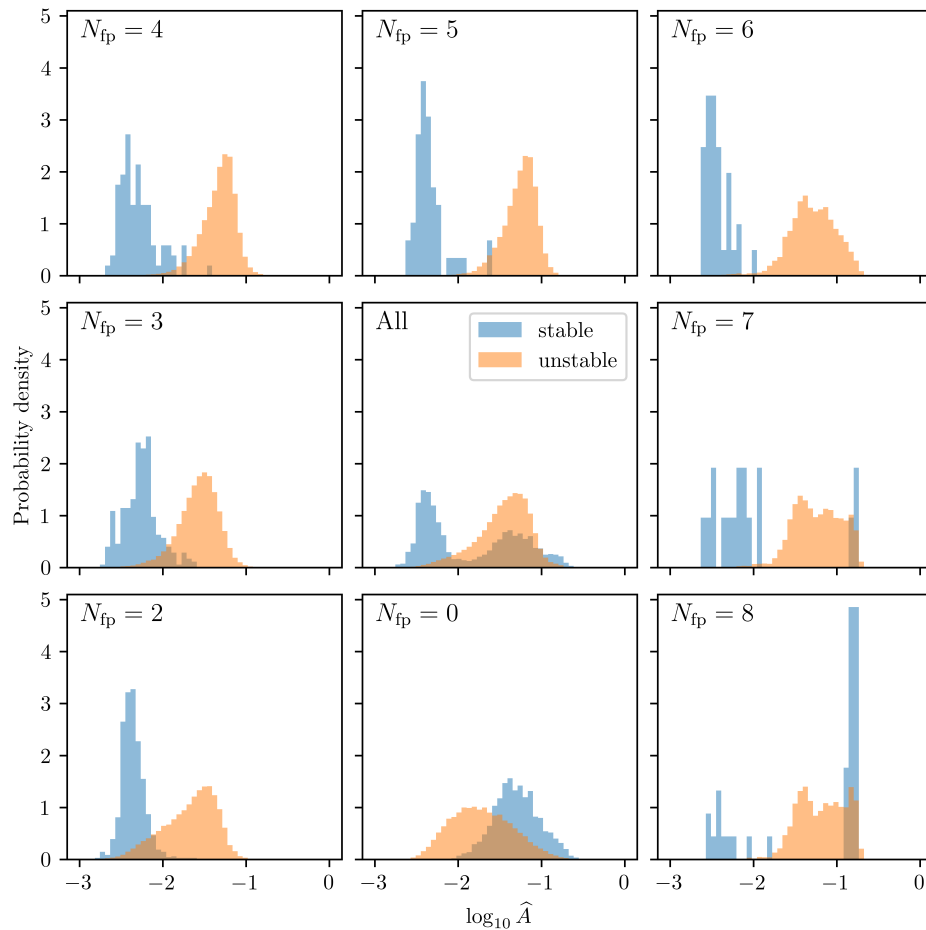
A positive: stability

- Defining $Q < 0.1$ as stability
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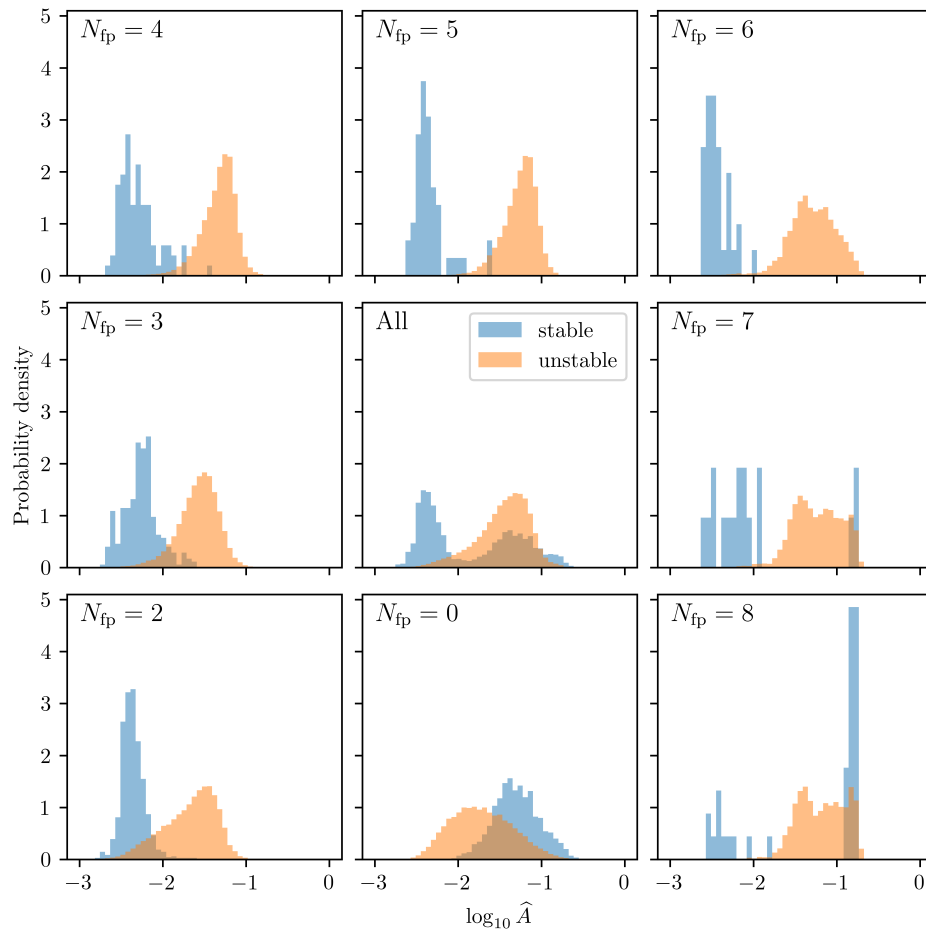
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A positive: stability

- Defining $Q < 0.1$ as stability
- Distribution of stable available energies left of unstable ones for stellarators
- Thus: low AE implies stability
- **Not** for tokamaks!



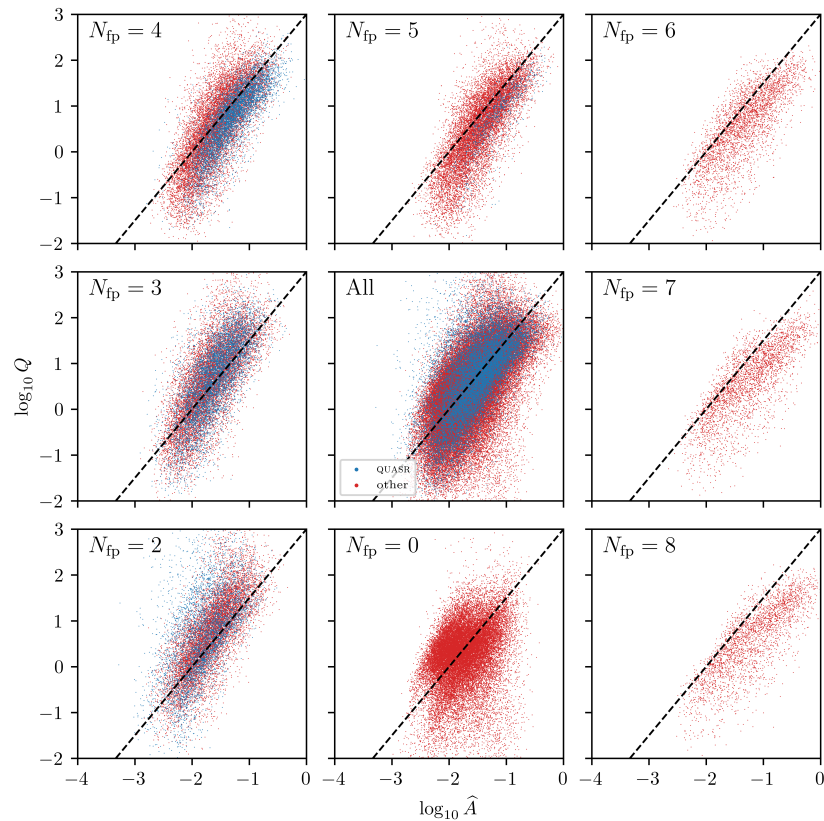
Comparison to nonlinear gyrokinetic simulations

Comparison to nonlinear gyrokinetic simulations

- Same exercise but with **randomly** chosen gradients

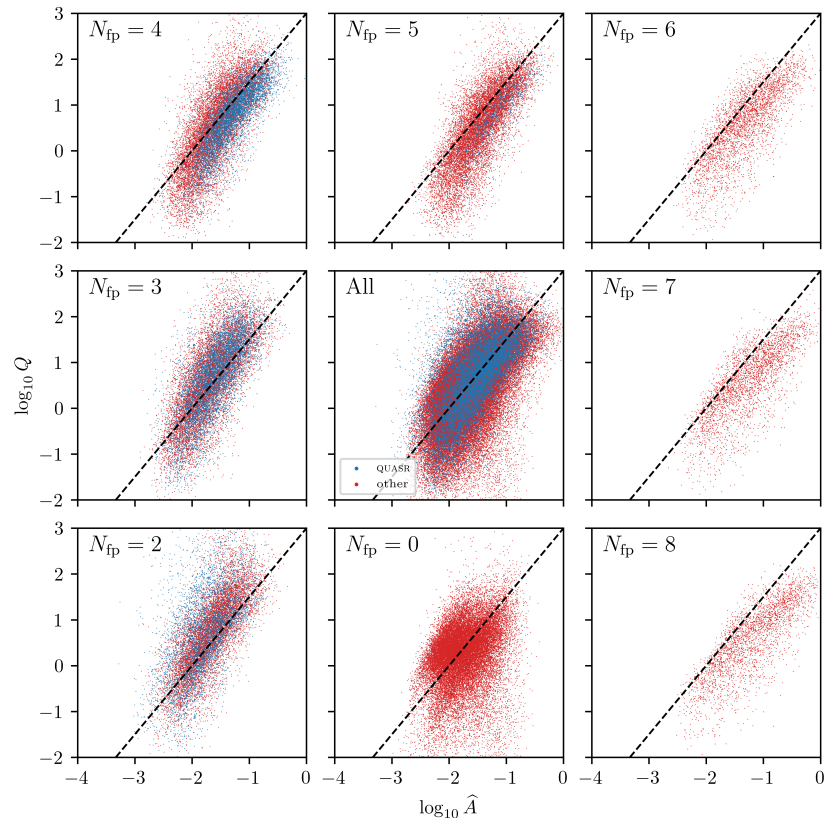
Comparison to nonlinear gyrokinetic simulations

- Same exercise but with **randomly** chosen gradients



Comparison to nonlinear gyrokinetic simulations

- Same exercise but with **randomly** chosen gradients
- Correlation more pronounced



Comparison to nonlinear gyrokinetic simulations

- Same exercise but with **randomly** chosen gradients
- Correlation more pronounced
- Captures variability due to gradients well

