Quadratic Invariants and the Helicity Barrier

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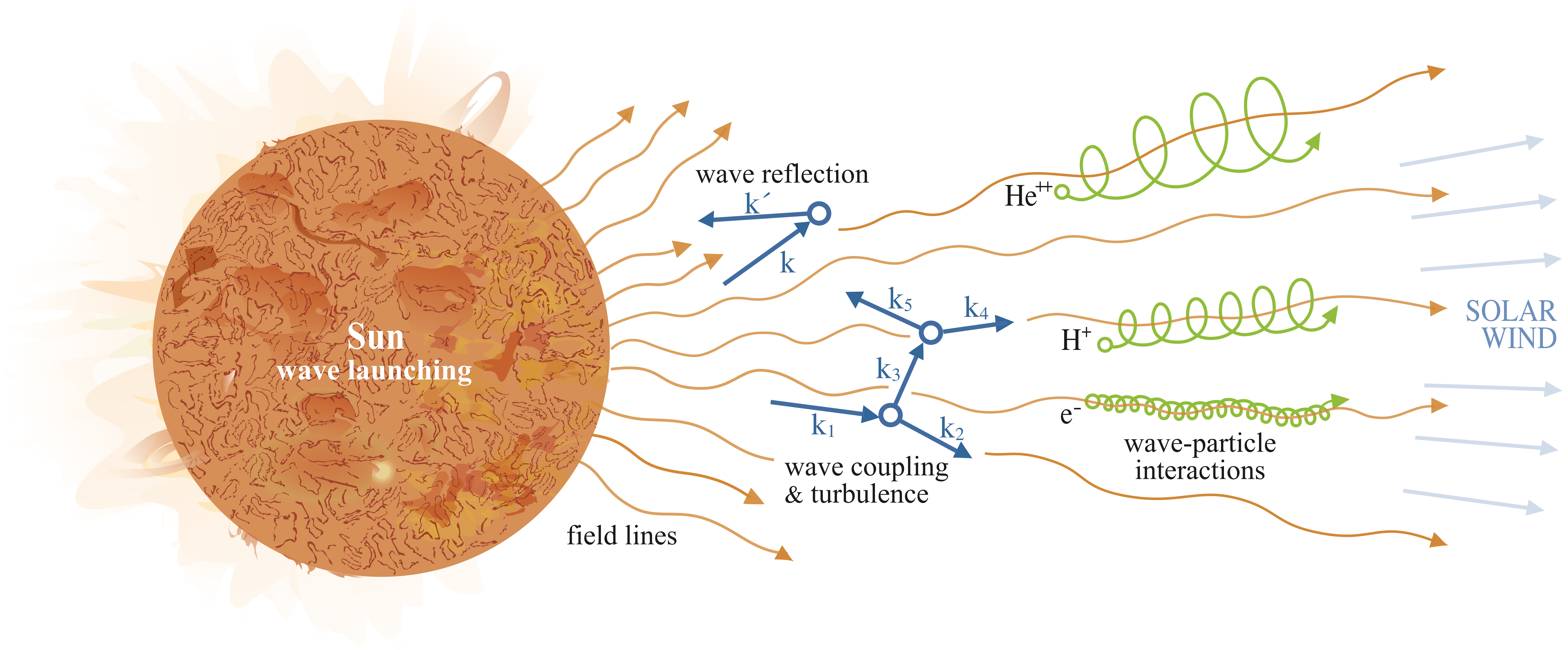
Thanks to my co-authors, Alfred Mallet and Romain Meyrand

Leading Paradigm for Solar Wind's Origin

(E.g., Velli et al 1989, Matthaeus et al 1999, Dmitruk et al 2002, Perez & Chandran 2013, Meyrand et al 2015)



- Photospheric motions and/or reconnection launch Alfven waves along open magnetic-field lines
- Outward-propagating waves undergo partial reflection because of the Alfvén-speed gradient
- Counter-propagating waves interact to produce turbulence. Wave energy cascades to small scales and heats the plasma, leading to solar-wind acceleration.
- Works pretty well in RMHD, but not in FLR-MHD

Reduced Magnetohydrodynamics (RMHD)

- Elsasser variables: $\boldsymbol{z}^{\pm} = \delta \boldsymbol{v}_{\perp}^{\pm} \pm \dfrac{\delta \boldsymbol{B}_{\perp}}{\sqrt{4\pi n m_{\mathrm{p}}}}$. Near Sun, $z_{\mathrm{rms}}^{+} \gg z_{\mathrm{rms}}^{-}$

- Two quadratic invariants: the energy $\mathscr{E} = \int \mathrm{d}^3x \left[\left(z^{+}\right)^2 + (z^{-})^2 \right]$ and cross helicity $\mathscr{H}_{\mathrm{c}} = \int \mathrm{d}^3x \left[\left(z^{+}\right)^2 - (z^{-})^2 \right]$, or, equivalently, $\mathscr{E}^{+} = \int \mathrm{d}^3x \, \left(z^{+}\right)^2$ and $\mathscr{E}^{-} = \int \mathrm{d}^3x \, (z^{-})^2$. These all cascade to smaller scales, leading to turbulent heating.

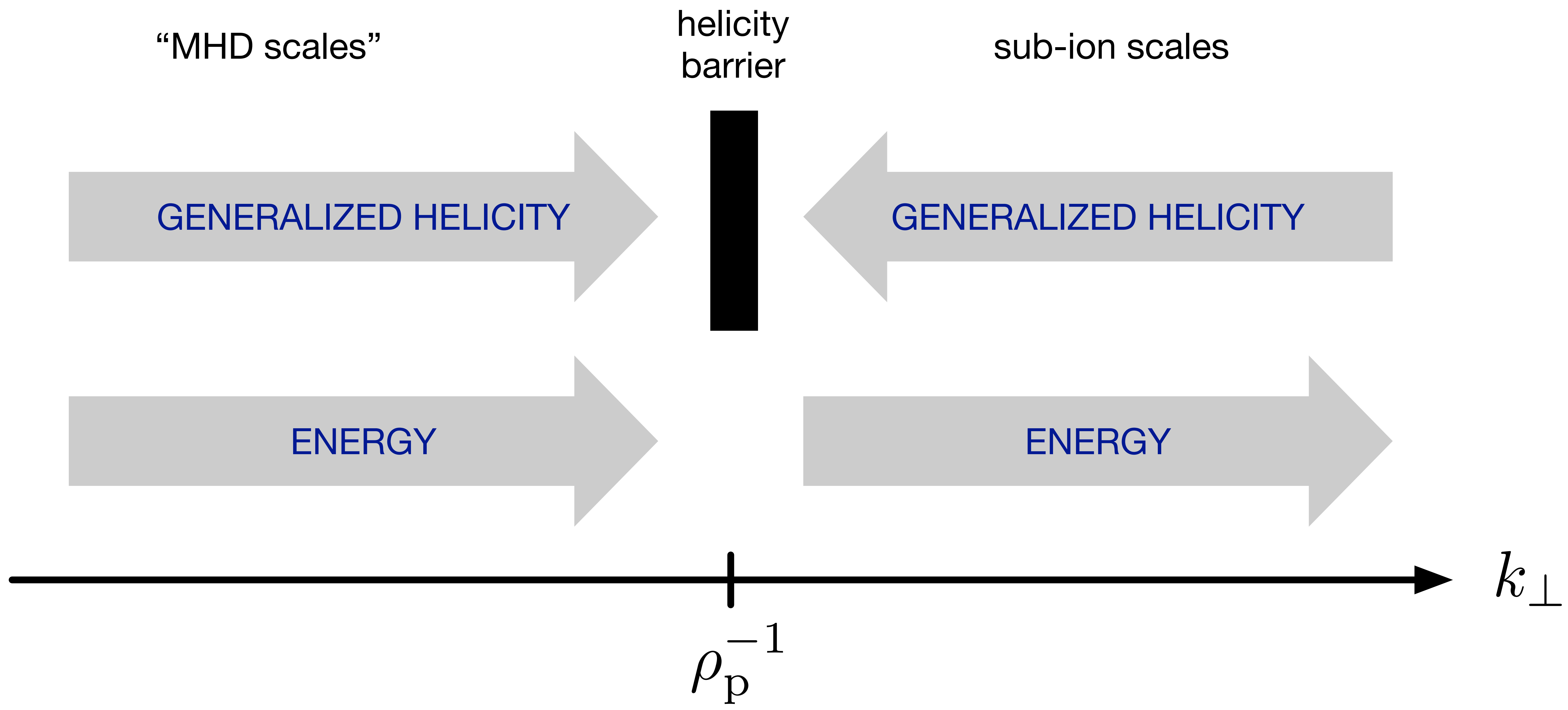
- Cascade power and turbulent heating rate: $\epsilon_{\mathrm{RMHD}} = \dfrac{\left(z_{\mathrm{rms}}^{+}\right)^2 z_{\mathrm{rms}}^{-} + \left(z_{\mathrm{rms}}^{-}\right)^2 z_{\mathrm{rms}}^{+}}{L_{\perp}}$

Finite Larmor Radius Magnetohydrodynamics (FLR-MHD)

(Meyrand et al 2021)

- Subsidiary expansion of astrophysical gyrokinetics (i.e., uniform equilibrium) in which $m_e/m_i \ll \beta \ll 1$, electrons are an isothermal fluid, and ion fluctuations are Alfvénic. Uniformly valid at $k_\perp \rho_i < 1$ and $k_\perp \rho_i > 1$.

- Quadratic invariants are the energy and generalized helicity $\mathscr{H}_{\rm gen}$. $\mathscr{H}_{\rm gen}$ reduces to the RMHD cross helicity at $k_\perp \rho_i \ll 1$ and the magnetic helicity at $k_\perp \rho_i \gg 1$, which cascades inversely, leading to the helicity barrier.



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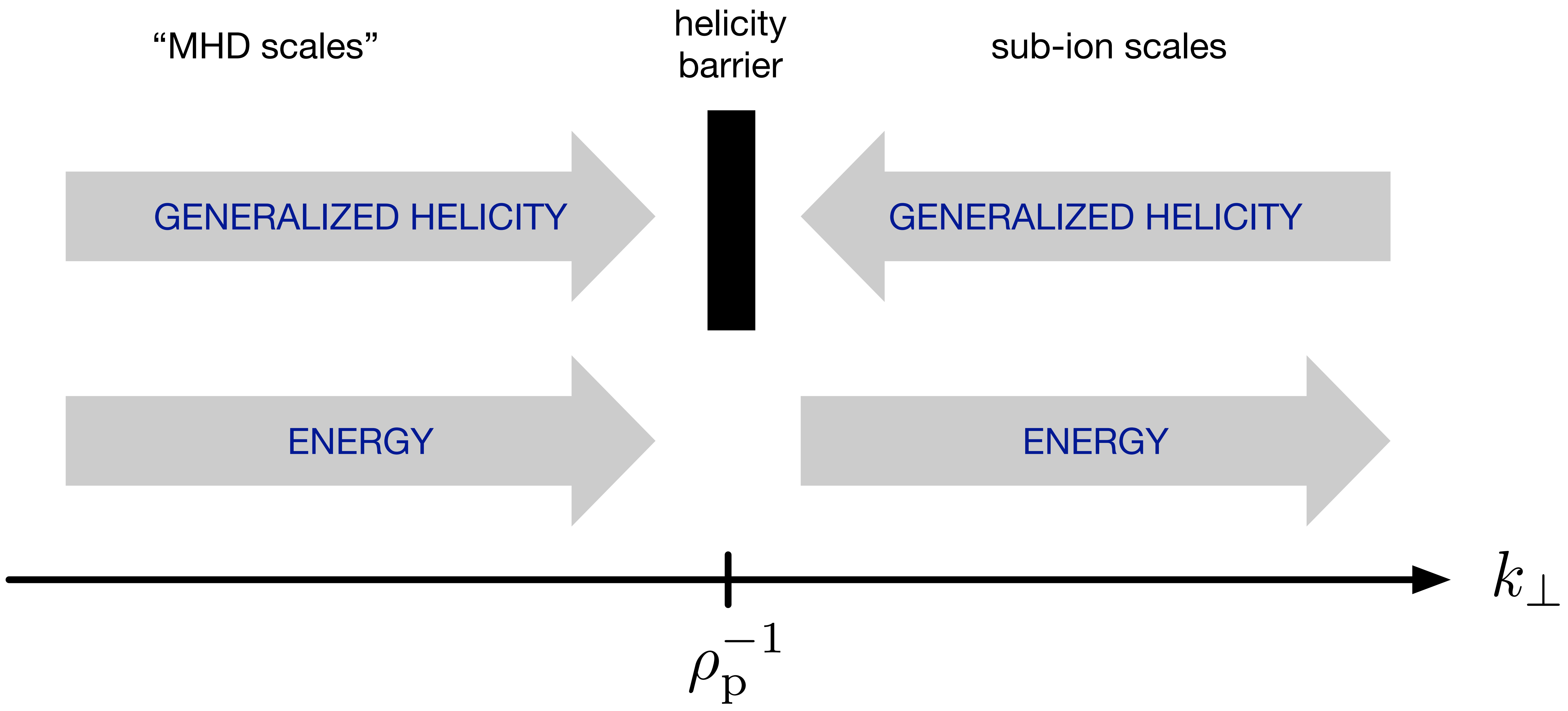
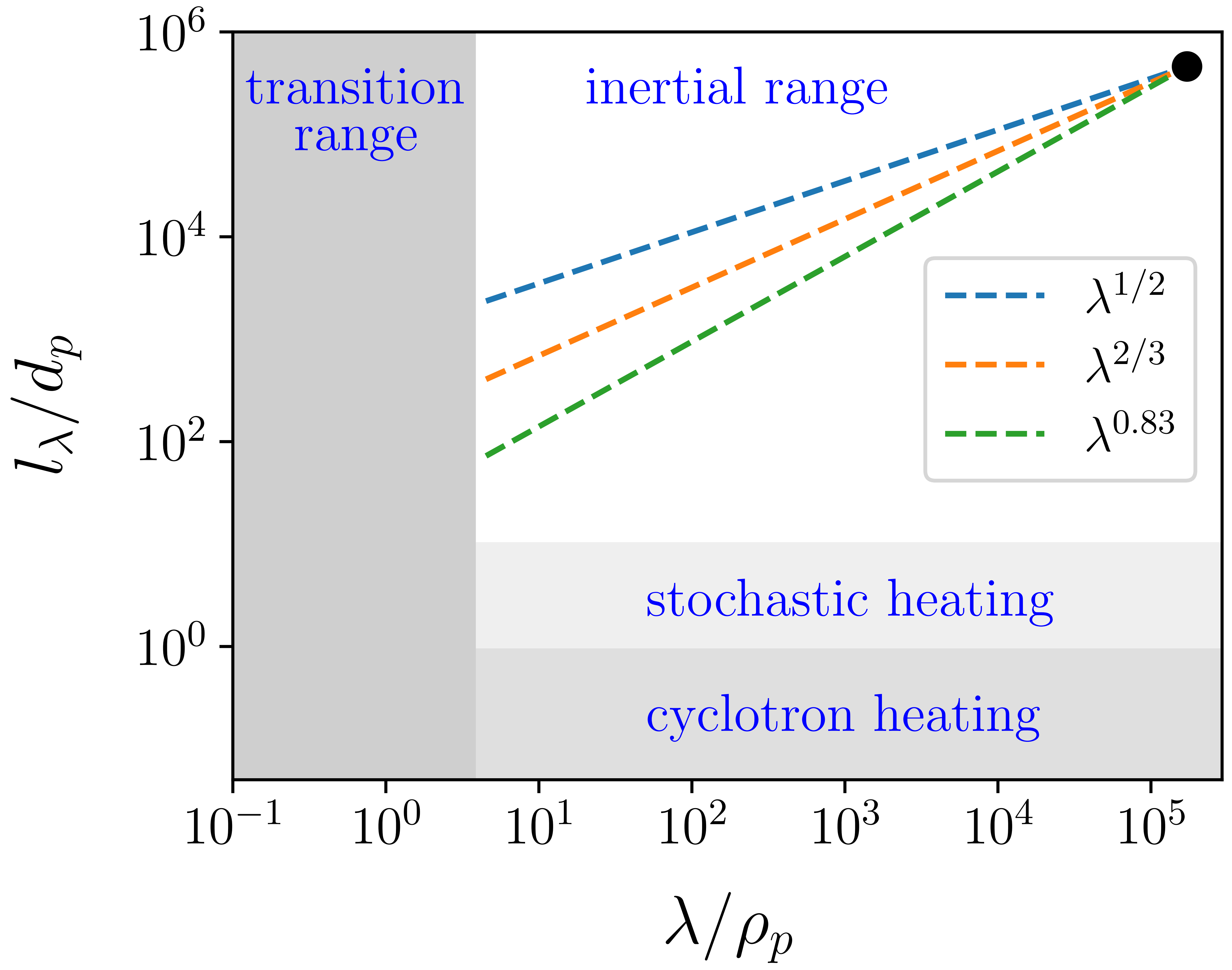


- Any imbalance between z^+ and z^- manifests as generalized helicity $\mathscr{H}_{\rm gen}$. In absence of dissipation at $k_\perp \rho_i < 1$, only balanced amounts of z^+ and z^- energy can cascade to smaller scales.

- Cascade power and turbulent heating rate: $\epsilon_{\rm FLR-MHD} = \dfrac{2\left(z^-_{\rm rms}\right)^2 z^+_{\rm rms}}{L_\perp} + D_{\rm k_\perp \rho_i < 1}$

FLR-MHD Helicity Barrier Breaks Leading Paradigm for Solar Wind's Origin



● = outer scale at $r = 2R_{\mathrm{sun}}$

λ = scale $\perp$ to $\boldsymbol{B}$

l_λ = scale || to $\boldsymbol{B}_0$

$d_{\mathrm{p}} = v_{\mathrm{A}}/\Omega_{\mathrm{p}}$

- Ion-scale heating rate, $D_{k_\perp\rho_i<1}$, is small because of anisotropy and transition range
- $\epsilon_{\mathrm{FLR-MHD}} = \dfrac{2\left(z^-_{\mathrm{rms}}\right)^2 z^+_{\mathrm{rms}}}{L_\perp} + D_{\mathrm{k_\perp\rho_i<1}} \ll \epsilon_{\mathrm{RMHD}}$ in coronal holes where $z^-_{\mathrm{rms}} \ll z^+_{\mathrm{rms}}$

Could Non-Alfvénic Ion Fluctuations Fix SW Models?

- Low corona is filled with large-amplitude density variations (Raymond et al 2014).

- Could non-Alfvenic ion fluctuations `puncture' the helicity barrier? I.e., act as a sink of $\mathscr{H}_{\rm gen}$ at $k_\perp \rho_i \sim 1$, so that the $\mathscr{E}^+$ cascade rate could exceed the $\mathscr{E}^-$ cascade rate, restoring the turbulent heating rate towards its RMHD value?

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- Is there a total conserved total helicity in full astrophysical gyrokinetics???

Astrophysical Gyrokinetics

(Schekochihin et al 2009)

$\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$

$n_{0s}, T_{0s}, B_0 =$ constants

$$\delta f_s(\boldsymbol{r},\boldsymbol{v},t) = -\frac{q_s\phi(\boldsymbol{r},t)}{T_{0s}}F_{0s}(v) + h_s(\boldsymbol{R}_s, v_\perp, v_\parallel, t)$$

Gyrokinetic equation:

$$\frac{\partial h_s}{\partial t} + v_\parallel\frac{\partial h_s}{\partial z} + \frac{c}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\} = \frac{q_s}{T_{0s}}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t}F_{0s} + \left(\frac{\partial h_s}{\partial t}\right)_{\mathrm{c}}$$

$$\boldsymbol{R}_{\mathrm{s}} = \boldsymbol{r} + \frac{\boldsymbol{v}_\perp\times\hat{\boldsymbol{z}}}{\Omega_{\mathrm{s}}} \qquad \chi = \phi - \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{c} \qquad \{f,g\} \equiv \hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}f\times\boldsymbol{\nabla}g)$$

$$= (\hat{z}\times\nabla_\perp f)\cdot\nabla_\perp g:$$

Quasineutrality:

$$\sum_s q_s\delta n_s = \sum_s q_s\left(-\frac{q_s n_{0s}}{T_{0s}}\phi + \int \mathrm{d}^3\boldsymbol{v}\langle h_s\rangle_{\boldsymbol{r}}\right) = 0$$

Ampere's law:

$$\sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\langle h_s\boldsymbol{v}\rangle_{\boldsymbol{r}} = -\frac{c}{4\pi}\nabla_\perp^2\boldsymbol{A}$$

Free energy:

$$W = \int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\frac{|\delta\boldsymbol{B}|^2}{8\pi} + \sum_s\int \mathrm{d}^3\boldsymbol{v}\frac{T_{0s}(\delta f_s)^2}{2F_{0s}}\right]$$

consider the quantity

$$\mathcal{H} = \lim_{\epsilon \to 0^+} \sum_s T_{0s} \int \frac{\mathrm{d}^3 \boldsymbol{R}_s}{V} \int \mathrm{d}^3 \boldsymbol{v} \frac{\left(h_s - q_s T_{0s}^{-1} \langle \chi \rangle_{\boldsymbol{R}_s} F_{0s}\right)^2}{2(v_\parallel - \mathrm{i}\epsilon v_{\mathrm{th}s}) F_{0s}}$$

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$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \lim_{\epsilon \to 0^+} \sum_s T_{0s} \int \frac{\mathrm{d}^3 \boldsymbol{R}_s}{V} \int \mathrm{d}^3 \boldsymbol{v} \frac{\left(h_s - q_s T_{0s}^{-1} \langle \chi \rangle_{\boldsymbol{R}_s} F_{0s}\right)}{(v_\parallel - \mathrm{i}\epsilon v_{\mathrm{th}s}) F_{0s}} \frac{\partial}{\partial t} \left(h_s - q_s T_s^{-1} \langle \chi \rangle_{\boldsymbol{R}_s} F_{0s}\right)$$

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Gyrokinetic equation: $\frac{\partial}{\partial t}\left(h_s - q_s T_s^{-1} \langle\chi\rangle_{\boldsymbol{R}_s} F_{0s}\right) = -\frac{c}{B_0}\left\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\right\} - v_\parallel \frac{\partial h_s}{\partial z}$

$$\int \frac{\mathrm{d}^3\boldsymbol{r}}{V} f\{f, g\} = \int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\left(\hat{z} \times \nabla_\perp \frac{f^2}{2}\right) \cdot \nabla_\perp g = \int \frac{\mathrm{d}^3\boldsymbol{r}}{V} \nabla_\perp \cdot \left[\left(\hat{z} \times \nabla_\perp \frac{f^2}{2}\right) g\right] \to 0 \text{ as } V \to \infty$$

consider the quantity

$$\mathcal{H} = \lim_{\epsilon\to 0^+} \sum_s T_{0s} \int \frac{\mathrm{d}^3\boldsymbol{R}_s}{V} \int \mathrm{d}^3\boldsymbol{v} \frac{\left(h_s - q_s T_{0s}^{-1} \langle\chi\rangle_{\boldsymbol{R}_s} F_{0s}\right)^2}{2(v_\parallel - \mathrm{i}\epsilon v_{\mathrm{th}s}) F_{0s}}$$

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$$= -\sum_s \int \frac{\mathrm{d}^3 \boldsymbol{R}_s}{V} \int \mathrm{d}^3 \boldsymbol{v}\, q_s h_s \frac{\partial}{\partial z} \left\langle \phi - \frac{\boldsymbol{v} \cdot \boldsymbol{A}}{c} \right\rangle_{\boldsymbol{R}_s}$$

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$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \lim_{\epsilon \to 0^+} \sum_s T_{0s} \int \frac{\mathrm{d}^3 \boldsymbol{R}_s}{V} \int \mathrm{d}^3 \boldsymbol{v} \frac{\left(h_s - q_s T_{0s}^{-1} \langle \chi \rangle_{\boldsymbol{R}_s} F_{0s}\right)}{(v_\parallel - \mathrm{i}\epsilon v_{\mathrm{th}s}) F_{0s}} \left(-v_\parallel \frac{\partial h_s}{\partial z}\right) \\
&= -\sum_s \int \frac{\mathrm{d}^3 \boldsymbol{R}_s}{V} \int \mathrm{d}^3 \boldsymbol{v}\, q_s h_s \frac{\partial}{\partial z} \left\langle \phi - \frac{\boldsymbol{v} \cdot \boldsymbol{A}}{c} \right\rangle_{\boldsymbol{R}_s} \\
&= -\int \frac{\mathrm{d}^3 \boldsymbol{r}}{V} \left(\frac{\partial \phi}{\partial z} \sum_s q_s \int \mathrm{d}^3 \boldsymbol{v} \langle h_s \rangle_{\boldsymbol{r}} - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial z} \cdot \sum_s q_s \int \mathrm{d}^3 \boldsymbol{v} \langle h_s \boldsymbol{v} \rangle_{\boldsymbol{r}} \right)
\end{aligned}$$

consider the quantity

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&= -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V} \left(\frac{\partial\phi}{\partial z} \sum_s q_s \int \mathrm{d}^3\boldsymbol{v} \langle h_s\rangle_{\boldsymbol{r}} - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial z} \cdot \sum_s q_s \int \mathrm{d}^3\boldsymbol{v} \langle h_s \boldsymbol{v}\rangle_{\boldsymbol{r}} \right) \\
&= -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V} \left(\frac{\partial\phi}{\partial z} \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi + \frac{1}{4\pi} \frac{\partial\boldsymbol{A}}{\partial z} \cdot \nabla_\perp^2 \boldsymbol{A} \right)
\end{aligned}$$

A New 3D Quadratic Invariant - the `Gyrokinetic Helicity'

$$\mathcal{H} = \lim_{\epsilon\to 0^+} \sum_s T_{0s} \int \frac{\mathrm{d}^3\boldsymbol{R}_s}{V} \int \mathrm{d}^3\boldsymbol{v} \frac{\left(h_s - q_s T_{0s}^{-1}\langle\chi\rangle_{\boldsymbol{R}_s} F_{0s}\right)^2}{2(v_\parallel - \mathrm{i}\epsilon v_{\mathrm{th}s})F_{0s}}$$

$$\begin{aligned}
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&= -\sum_s \int \frac{\mathrm{d}^3\boldsymbol{R}_s}{V} \int \mathrm{d}^3\boldsymbol{v}\, q_s h_s \frac{\partial}{\partial z} \left\langle \phi - \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{c} \right\rangle_{\boldsymbol{R}_s} \\
&= -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V} \left(\frac{\partial \phi}{\partial z} \sum_s q_s \int \mathrm{d}^3\boldsymbol{v} \langle h_s \rangle_{\boldsymbol{r}} - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial z} \cdot \sum_s q_s \int \mathrm{d}^3\boldsymbol{v} \langle h_s \boldsymbol{v}\rangle_{\boldsymbol{r}} \right) \\
&= -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V} \left(\frac{\partial \phi}{\partial z} \sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \phi + \frac{1}{4\pi} \frac{\partial \boldsymbol{A}}{\partial z} \cdot \nabla_\perp^2 \boldsymbol{A} \right) \\
&= 0.
\end{aligned}$$

(Chandran, Mallet, & Meyrand, submitted)

Singular or Non-Singular?

$$\mathcal{H} = \sum_s \int \frac{\mathrm{d}^3\boldsymbol{R}_s}{V} \int_0^\infty \mathrm{d}v_\perp \pi v_\perp \left[\mathrm{P}\int_{-\infty}^{\infty} \frac{T_{0s}\psi_s^2}{v_\parallel F_{0s}} \mathrm{d}v_\parallel + \mathrm{i}\pi \left(\frac{T_{0s}\psi_s^2}{F_{0s}} \right)_{v_\parallel = 0} \right]$$

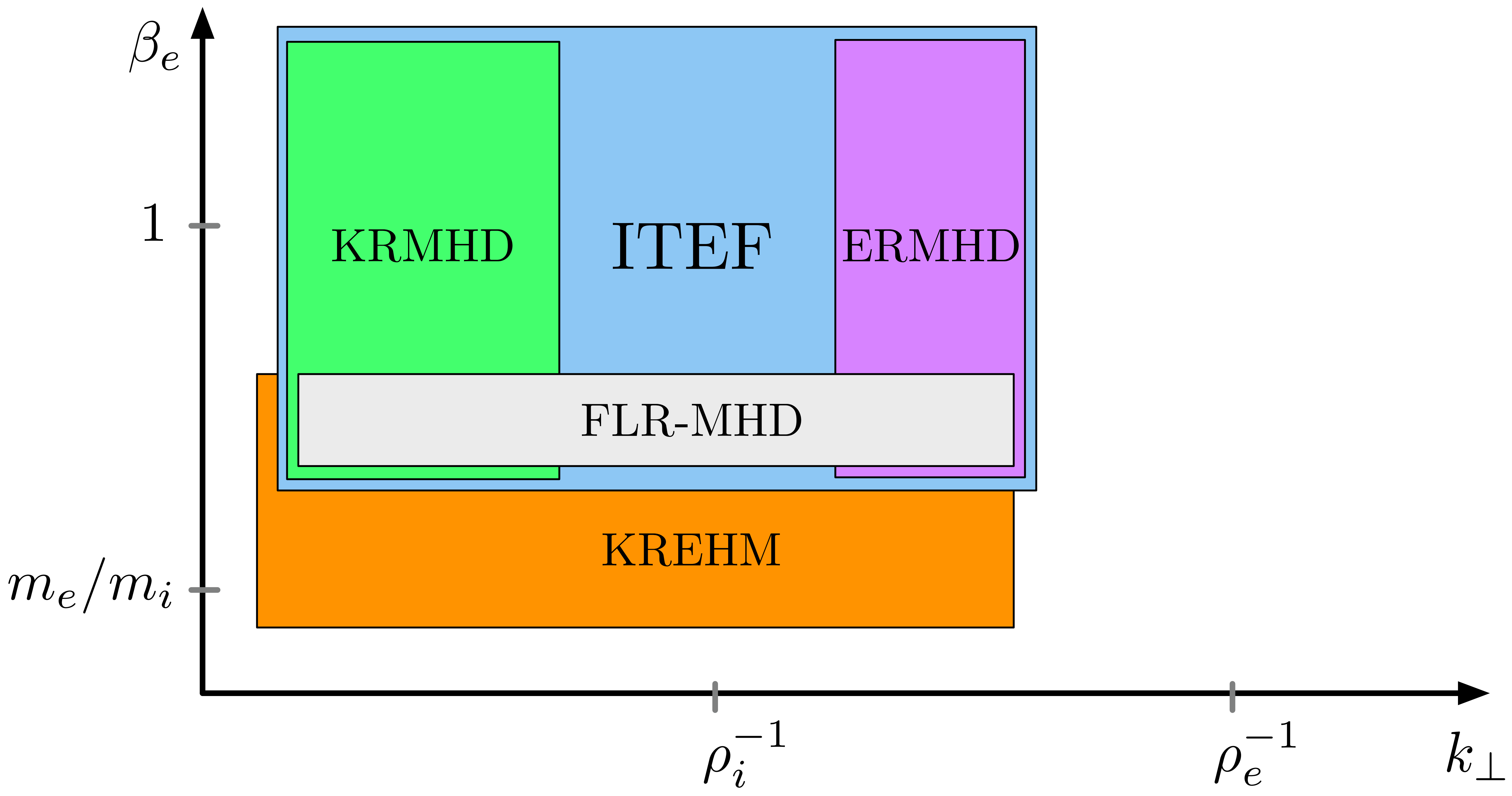
$$\psi_s \equiv h_s - \frac{q_s}{T_{0s}} \langle \chi \rangle_{\boldsymbol{R}_s} F_{0s}$$

$$\mathcal{H} = \sum_s \int \frac{\mathrm{d}^3\boldsymbol{R}_s}{V} \int_0^\infty \mathrm{d}v_\perp \pi v_\perp \left[\int_{-\infty}^{\infty} \frac{2T_{0s}\psi_{\mathrm{even}}\psi_{\mathrm{odd}}}{v_\parallel F_{0s}} \mathrm{d}v_\parallel + \mathrm{i}\pi \left(\frac{T_{0s}\psi_s^2}{F_{0s}} \right)_{v_\parallel = 0} \right]$$

$$\psi_{\mathrm{even}}(\boldsymbol{R}_s, v_\perp, v_\parallel, t) = \frac{1}{2}\left[\psi(\boldsymbol{R}_s, v_\perp, v_\parallel, t) + \psi(\boldsymbol{R}_s, v_\perp, -v_\parallel, t) \right]$$

$$\psi_{\mathrm{odd}}(\boldsymbol{R}_s, v_\perp, v_\parallel, t) = \frac{1}{2}\left[\psi(\boldsymbol{R}_s, v_\perp, v_\parallel, t) - \psi(\boldsymbol{R}_s, v_\perp, -v_\parallel, t) \right]$$

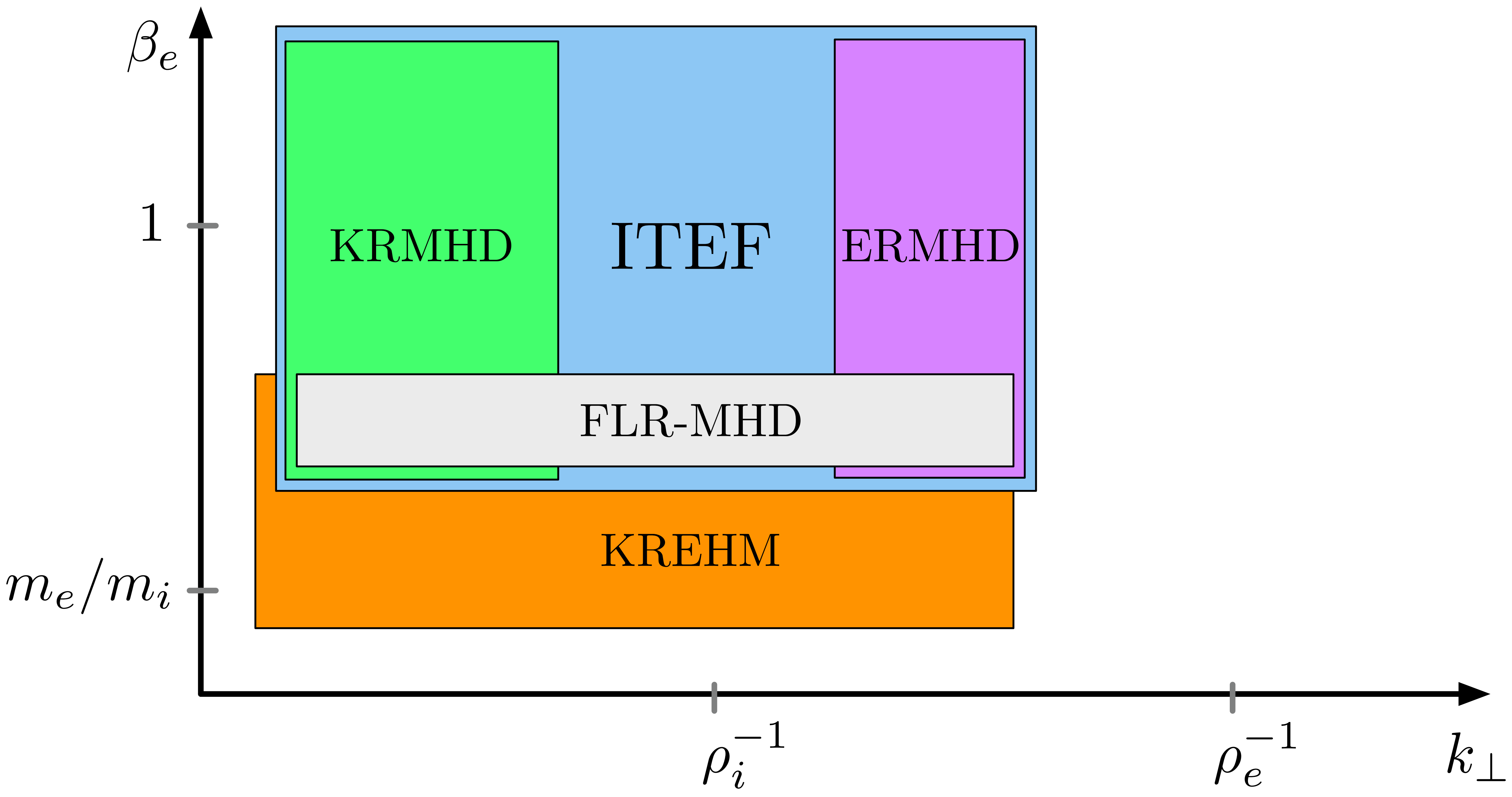
Subsidiary Regimes of Astrophysical Gyrokinetics



- ITEF: isothermal electron fluid
- KRMHD: kinetic reduced MHD
- ERMHD: electron MHD
- FLRMHD: finite-Larmor-radius MHD
- KREHM: kinetic reduced electron heating model

Chandran, Mallet, & Meyrand (submitted) work out $\mathscr{H}$ in each regime

Isothermal Electron Fluid (ITEF)



$$\mathrm{Re}\,\mathcal{H} = \mathcal{H}_{\mathrm{mf}} + \mathcal{H}_{\text{ph-sp}}$$

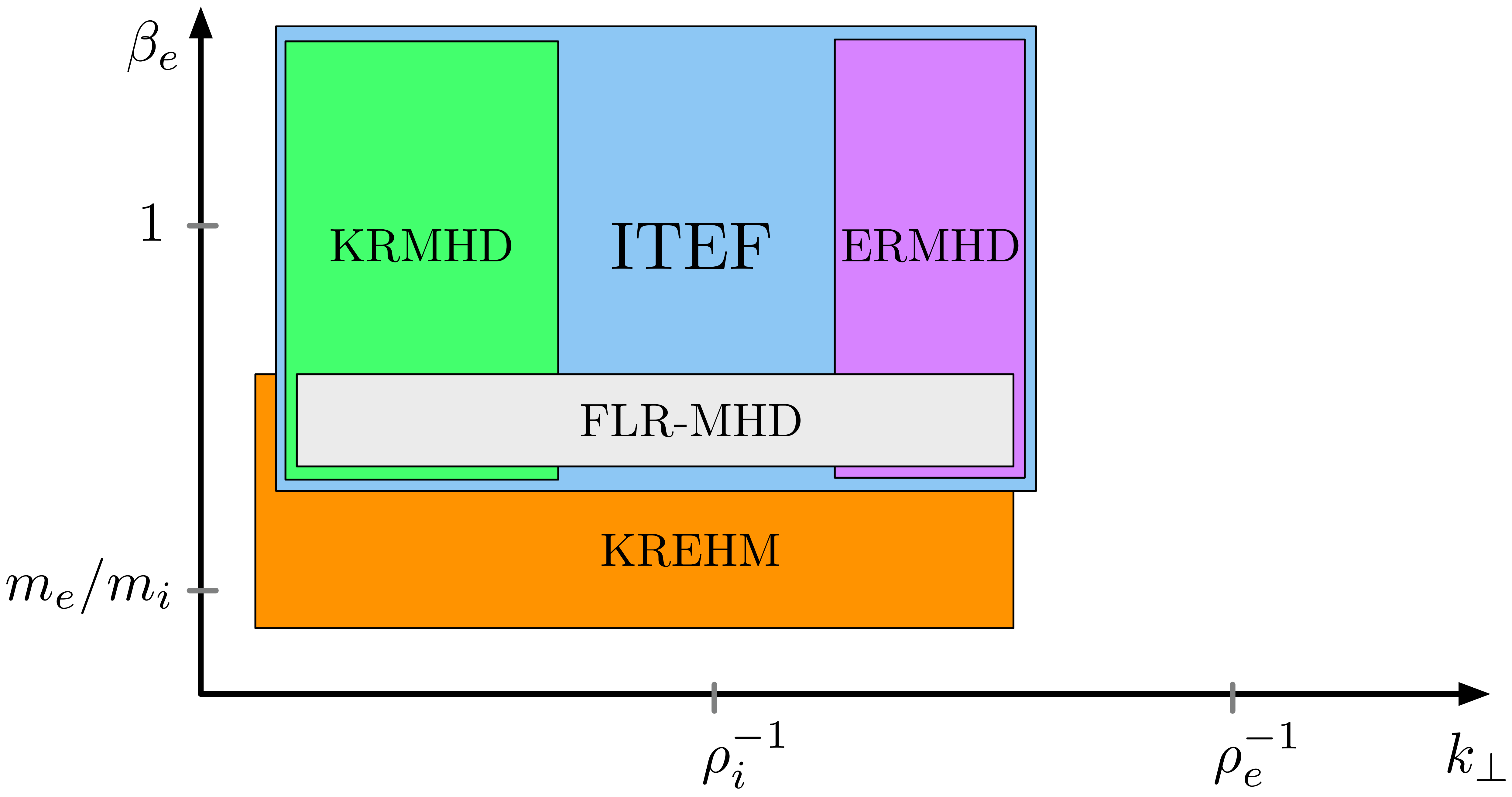
$$\mathcal{H}_{\mathrm{mf}} = m_i n_i v_{\mathrm{th}i} \int \frac{\mathrm{d}^3\boldsymbol{r}}{V} A \left[\left(1 - \hat{\Gamma}_0\right) \varphi + \left(1 - \hat{\Gamma}_1\right) \frac{\delta B}{B} \right]$$

(related to known helicities)

$$\mathcal{H}_{\text{ph-sp}} = T_i \int \frac{\mathrm{d}^3\boldsymbol{R}_i}{V} \int_0^\infty \mathrm{d}v_\perp \pi v_\perp \mathrm{P} \int_{-\infty}^\infty \mathrm{d}v_\parallel \frac{\left(g_i - \hat{v}_\perp^2 \hat{J}_1 \frac{\delta B}{B} F_{0i}\right)^2}{v_\parallel F_{0i}}$$

(new)

Isothermal Electron Fluid (ITEF)



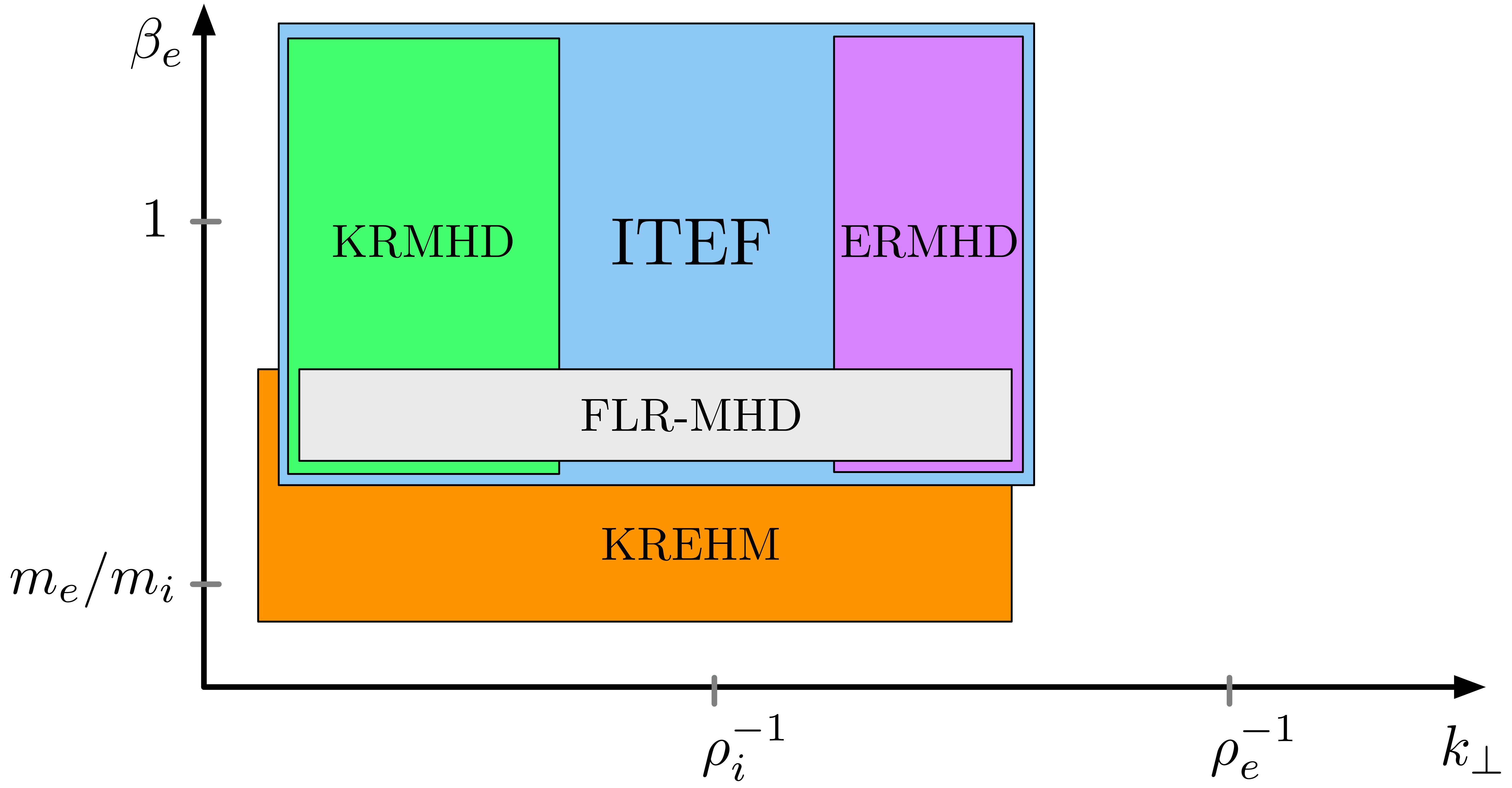
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\text{ph-sp}} = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\text{mf}}$$

(integrals dominated by $k_\perp \rho_i \sim 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\text{mf}} = -n_i T_i \int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\frac{Z}{\tau}\left(\hat{\Gamma}_1 \frac{\delta B}{B}\right)\nabla_\parallel \frac{\delta n}{n}$$
$$- n_i T_i \int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \frac{\mathrm{d}^3\boldsymbol{v}}{n_i}\Bigg\langle \left[\frac{Z}{\tau}\frac{\delta n(\boldsymbol{r},t)}{n} + \hat{v}_\perp^2 \hat{J}_1 \frac{\delta B(\boldsymbol{R}_i,t)}{B}\right]\nabla_\parallel g_i(\boldsymbol{R}_i, v_\perp, v_\parallel, t)$$
$$+ \rho_i \left[g_i(\boldsymbol{R}_i, v_\perp, v_\parallel, t) - \hat{v}_\perp^2 \hat{J}_1 \frac{\delta B(\boldsymbol{R}_i,t)}{B} F_{0i}\right]\left[\{\varphi(\boldsymbol{r},t), A(\boldsymbol{r},t)\} - \{\langle\varphi\rangle_{\boldsymbol{R}_i}, \langle A\rangle_{\boldsymbol{R}_i}\}\right]\Bigg\rangle_{\boldsymbol{r}}$$

(Chandran, Mallet, & Meyrand, submitted)

Magnetofluid Helicity $\mathscr{H}_{\mathrm{mf}}$ and Phase-Space Helicity $\mathscr{H}_{\text{ph-sp}}$



- $\mathscr{H}_{\mathrm{mf}}$ and $\mathscr{H}_{\text{ph-sp}}$ separately conserved in KRMHD and ERMHD
- KRMHD: $\mathscr{H}_{\mathrm{mf}} \to$ Alfvénic component of cross helicity. Three separately conserved components of $\mathscr{H}_{\text{ph-sp}}$, plus two additional helicity-like invariants.
- ERMHD: $\mathscr{H}_{\mathrm{mf}} \to$ magnetic helicity
- FLRMHD: $\mathscr{H}_{\mathrm{mf}} \to$ generalized helicity

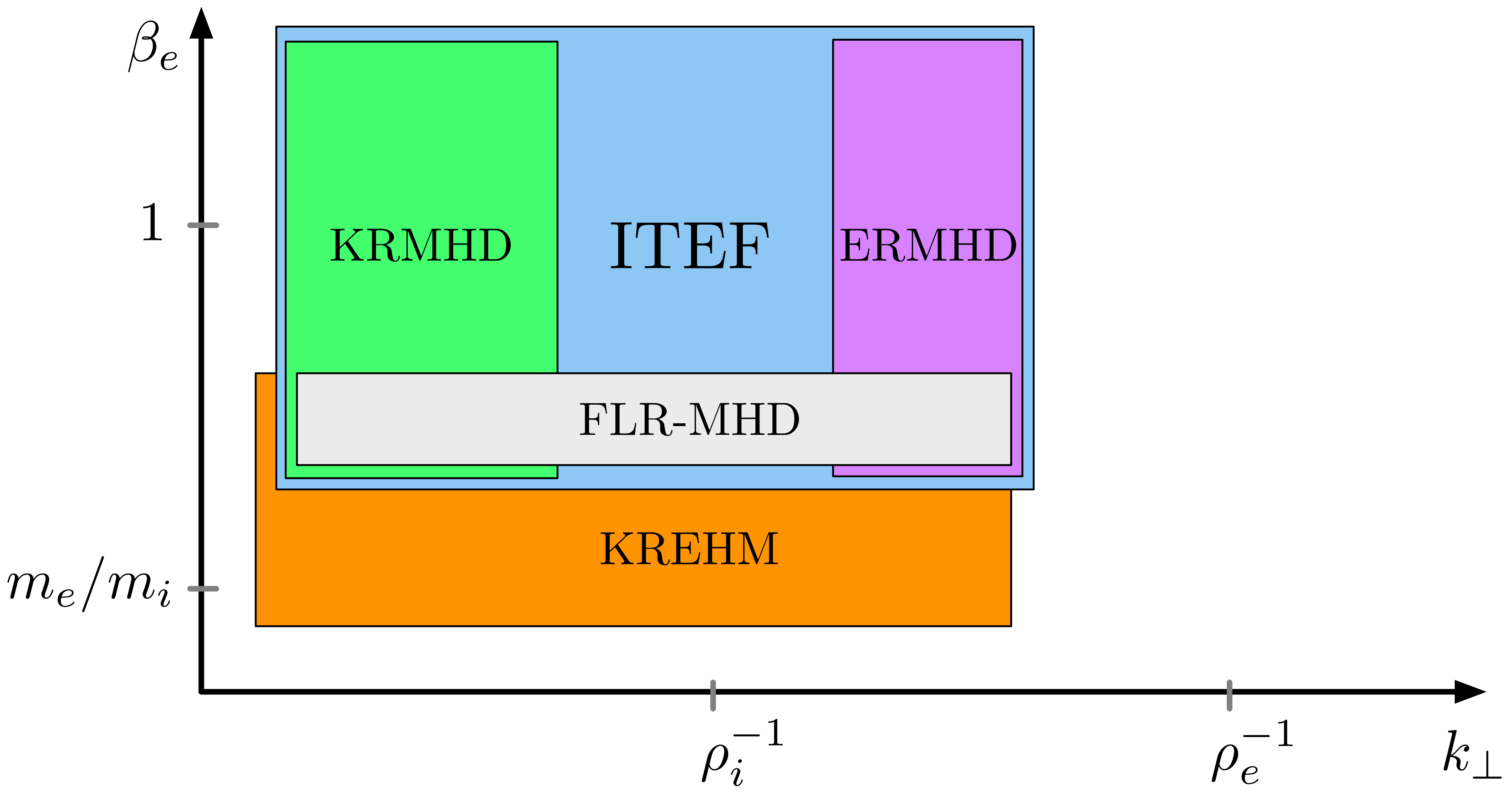
Kinetic Reduced Electron Heating Model (KREHM)

$$\mathcal{H}_{\mathrm{mf,KREHM}} = \frac{e}{c}\int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\delta n_e \left(d_e^2\nabla_\perp^2 A_\parallel - A_\parallel\right)$$

(also called the generalized helicity)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{mf,KREHM}} = -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\delta n_e \nabla_\parallel \delta T_{\parallel e}$$

(Adkins, Meyrand, & Squire 2025)



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$$\mathcal{H}_{\mathrm{ph\text{-}sp,KREHM}} = T_e\int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\int_0^\infty \mathrm{d}v_\perp \pi v_\perp \mathrm{P}\int_{-\infty}^{\infty}\mathrm{d}v_\parallel \frac{\left(g_e + \frac{\delta n}{n}F_{0e}\right)^2}{v_\parallel F_{0e}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{mf,KREHM}} = -\int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\delta n_e \nabla_\parallel \delta T_{\parallel e}$$

(Adkins, Meyrand, & Squire 2025)

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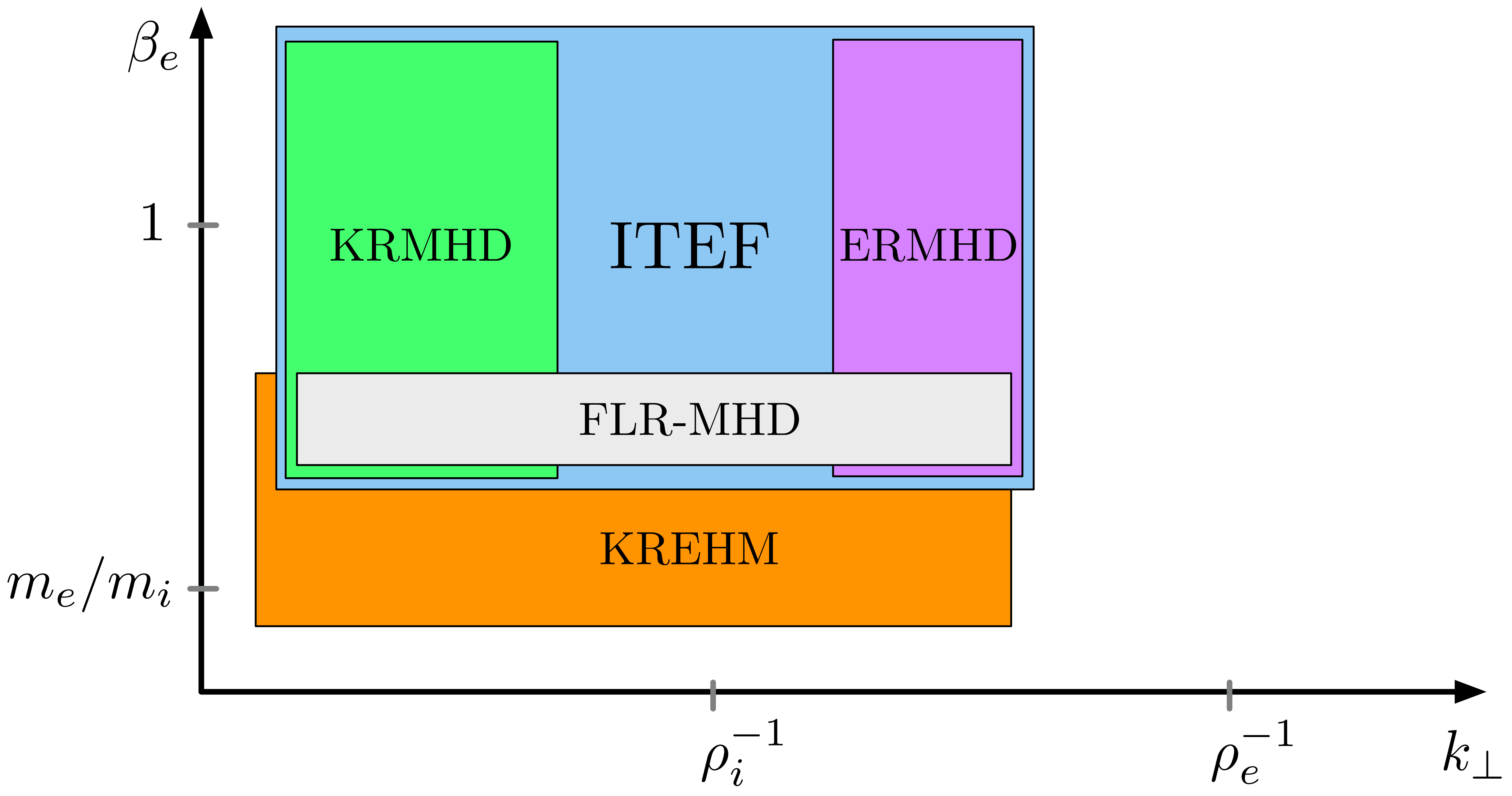
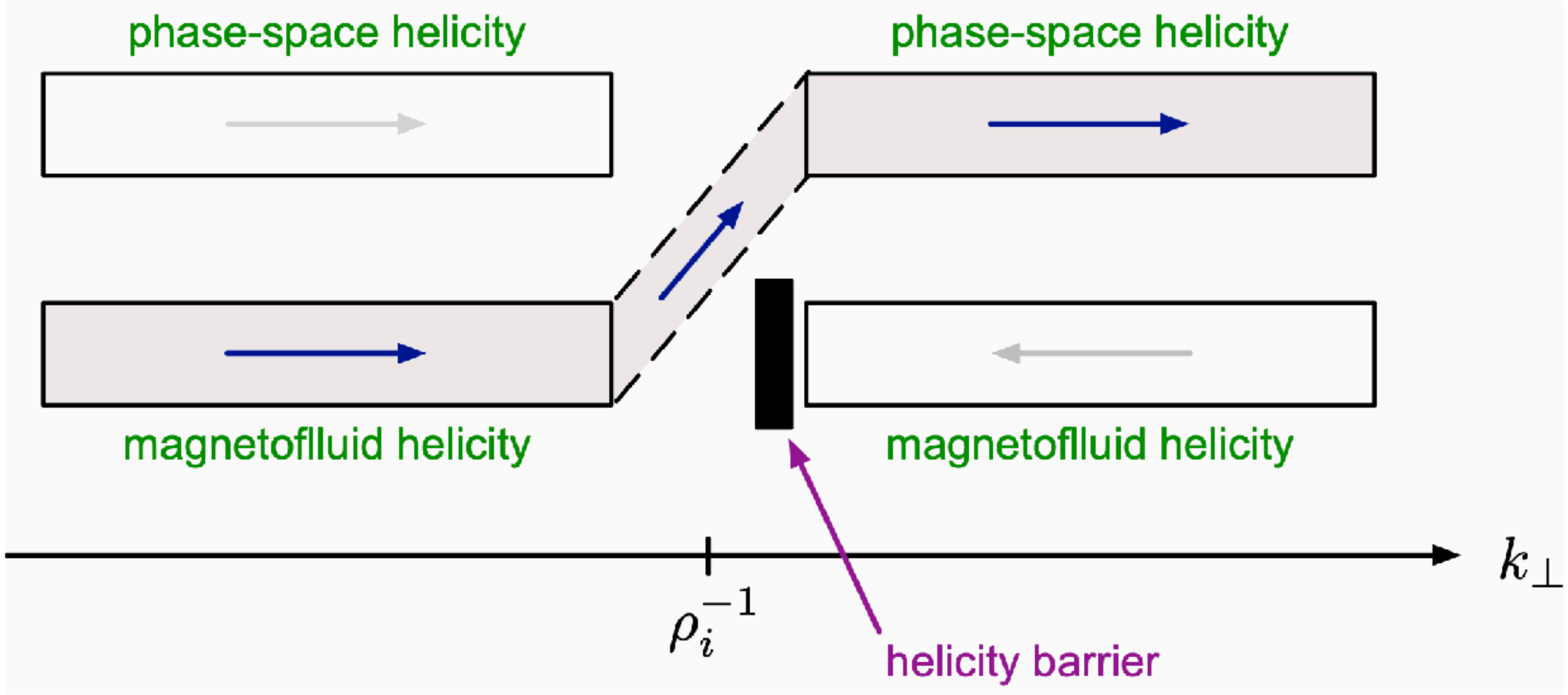


How Non-Alfvénic Fluctuations Can Bypass the HB to Fix SW Models



- $\mathscr{H}_{\mathrm{mf}}$ injected at $k_\perp \rho_i \ll 1$ cascades to $k_\perp \rho_i \sim 1$, transforms into $\mathscr{H}_{\text{ph-sp}}$, cascades to $k_\perp \rho_i \gg 1$, undergoes nonlinear perpendicular phase mixing, and dissipates via collisions.
- Forward cascade of $\mathscr{H}_{\mathrm{mf}}$ allows z^+ energy cascade rate to exceed z^- energy cascade rate, restoring the turbulent heating rate towards its RMHD value.

Last 3 slides: early work in progress

The Free Energy in Gyrokinetics in Rotating Tokamak Plasmas

(Abel et al 2013)

$$W(\psi) = \sum_s \left\langle \left\langle \int \mathrm{d}^3 \boldsymbol{w} \frac{T_s \langle h_s^2 \rangle_r}{2F_{0s}} \right\rangle_\perp \right\rangle_\psi - \sum_s \left\langle \left\langle \frac{Z_s^2 e^2 n_s \delta\varphi'^2}{2T_s} \right\rangle_\perp \right\rangle_\psi + \left\langle \left\langle \frac{\delta B^2}{8\pi} \right\rangle_\perp \right\rangle_\psi,$$

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right] h_s + \left(w_\parallel \boldsymbol{b} + \boldsymbol{V}_{\mathrm{D}s} + \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}}\right) \cdot \frac{\partial h_s}{\partial \boldsymbol{R}_s} - \langle C_L[h_s] \rangle_{\boldsymbol{R}}$$
$$= \frac{Z_s e F_{0s}}{T_s} \left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right] \langle \chi \rangle_{\boldsymbol{R}} - \left\{ \frac{\partial F_{0s}}{\partial \psi} + \frac{m_s F_{0s}}{T_s} \left[\frac{I(\psi) w_\parallel}{B} + \omega(\psi) R^2 \right] \frac{\mathrm{d}\omega}{\mathrm{d}\psi} \right\} \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}} \cdot \nabla\psi$$

$$\left.\frac{\partial}{\partial t}\right|_\psi W = \sum_s \left\langle \left\langle \int \mathrm{d}^3 \boldsymbol{w} \left\langle \frac{T_s h_s}{F_{0s}} C_L[h_s] \right\rangle_r \right\rangle_\perp \right\rangle_\psi - \sum_s \left\langle \left\langle \int \mathrm{d}^3 \boldsymbol{w} \langle h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi T_s \left(\frac{\mathrm{d} \ln N_s}{\mathrm{d}\psi} - \frac{3}{2} \frac{\mathrm{d} \ln T_s}{\mathrm{d}\psi} \right)$$
$$- \sum_s \left\langle \left\langle \int \mathrm{d}^3 \boldsymbol{w} \varepsilon_s \langle h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi \frac{\mathrm{d} \ln T_s}{\mathrm{d}\psi}$$
$$- \left\langle R^2 (\nabla\phi) \cdot \left[\sum_s \left\langle \int \mathrm{d}^3 \boldsymbol{w} \langle \boldsymbol{v} h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp - \left\langle \frac{1}{4\pi} \delta\boldsymbol{B} \delta\boldsymbol{B} + \frac{1}{c} \delta\boldsymbol{A} \delta\boldsymbol{j} \right\rangle_\perp \right] \cdot \nabla\psi \right\rangle_\psi \frac{\mathrm{d}\omega}{\mathrm{d}\psi}.$$

The Free Energy in Gyrokinetics in Rotating Tokamak Plasmas

(Abel et al 2013)

$$W(\psi) = \sum_s \left\langle\left\langle \int d^3w \frac{T_s \langle h_s^2 \rangle_r}{2F_{0s}} \right\rangle_\perp \right\rangle_\psi - \sum_s \left\langle\left\langle \frac{Z_s^2 e^2 n_s \delta\varphi'^2}{2T_s} \right\rangle_\perp \right\rangle_\psi + \left\langle\left\langle \frac{\delta B^2}{8\pi} \right\rangle_\perp \right\rangle_\psi,$$

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right] h_s + \left(w_\parallel \boldsymbol{b} + \boldsymbol{V}_{Ds} + \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}}\right) \cdot \frac{\partial h_s}{\partial \boldsymbol{R}_s} - \langle C_L[h_s] \rangle_{\boldsymbol{R}}$$

$$= \frac{Z_s e F_{0s}}{T_s} \left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s) \cdot \frac{\partial}{\partial \boldsymbol{R}_s}\right] \langle \chi \rangle_{\boldsymbol{R}} - \left\{ \frac{\partial F_{0s}}{\partial \psi} + \frac{m_s F_{0s}}{T_s} \left[\frac{I(\psi) w_\parallel}{B} + \omega(\psi) R^2\right] \frac{d\omega}{d\psi} \right\} \langle \boldsymbol{V}_\chi \rangle_{\boldsymbol{R}} \cdot \nabla\psi$$

$$\left.\frac{\partial}{\partial t}\right|_\psi W = \sum_s \left\langle\left\langle \int d^3w \left\langle \frac{T_s h_s}{F_{0s}} C_L[h_s] \right\rangle_r \right\rangle_\perp \right\rangle_\psi - \sum_s \left\langle\left\langle \int d^3w \langle h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi T_s \left(\frac{d \ln N_s}{d\psi} - \frac{3}{2} \frac{d \ln T_s}{d\psi} \right)$$

$$- \sum_s \left\langle\left\langle \int d^3w \, \varepsilon_s \langle h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp \cdot \nabla\psi \right\rangle_\psi \frac{d \ln T_s}{d\psi}$$

$$- \left\langle R^2 (\nabla\phi) \cdot \left[\sum_s \left\langle \int d^3w \langle \boldsymbol{v} h_s \boldsymbol{V}_\chi \rangle_r \right\rangle_\perp - \left\langle \frac{1}{4\pi} \delta\boldsymbol{B} \delta\boldsymbol{B} + \frac{1}{c} \delta\boldsymbol{A} \delta\boldsymbol{j} \right\rangle_\perp \right] \cdot \nabla\psi \right\rangle_\psi \frac{d\omega}{d\psi}.$$

Gyrokinetic Helicity in Rotating Tokamak Plasmas?

$$\mathscr{H}(\psi) = \lim_{\epsilon\to 0+} \left\langle \left\langle \sum_s T_{0s} \int \mathrm{d}^3\boldsymbol{w} \frac{\left(h_s - q_s T_{0s}^{-1} \langle\chi\rangle_{\boldsymbol{R}_s} F_{0s}\right)^2}{2(w_\| - i\epsilon v_{\mathrm{th},s})F_{0s}} \right\rangle_\perp \right\rangle_\psi$$

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{R}_s)\cdot\frac{\partial}{\partial \boldsymbol{R}_s}\right]\left(h_s - q_s T_{0s}^{-1}\langle\chi\rangle_{\boldsymbol{R}_s} F_{0s}\right) + \left(w_\|\boldsymbol{b} + \boldsymbol{V}_{\mathrm{D}s} + \left\langle \boldsymbol{V}_\chi \right\rangle_{\boldsymbol{R}_s}\right)\cdot\frac{\partial h_s}{\partial \boldsymbol{R}_s} - \left\langle C_L[h_s]\right\rangle_{\boldsymbol{R}}$$

$$= -\left\{\frac{\partial F_{0s}}{\partial \psi} + \frac{m_s F_{0s}}{T_s}\left[\frac{I(\psi)w_\|}{B} + \omega(\psi)R^2\right]\frac{\mathrm{d}\omega}{\mathrm{d}\psi}\right\}\left\langle \boldsymbol{V}_\chi \right\rangle_{\boldsymbol{R}_s}\cdot\nabla\psi$$

More work needed to see if this is useful concept for tokamaks.

Conclusion

- Astrophysical gyrokinetics possesses a second 3D quadratic invariant, the gyrokinetic helicity, which generalizes and unifies several previously identified helicities in different sub-regimes of gyrokinetics

- The transformation of magnetofluid helicity to phase-space helicity via non-Alfvénic ion fluctuations enables the forward cascade of helicity, circumventing the helicity barrier and enhancing turbulent heating in coronal holes.

- It remains unclear whether there is a useful generalization of the gyrokinetic helicity for more general magnetic geometry, including toroidal fusion plasmas.