The reduced MHD limit of the compressible MHD for a strongly magnetised and anisotropic plasma: a rigorous justification

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joint work with

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17th Plasma Kinetics Working Meeting

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  - $v \in \mathbb{R}^3$ is the fluid velocity.
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  - Barotropic pressure : $p = \mathfrak{a}\rho^\gamma$, with $\mathfrak{a} > 0$, and $\gamma > 1$.
  - $e_\parallel := {}^t(0, 0, 1)$ is the parallel direction (without loss of generality).
  - $B = {}^t({}^tB_\perp, B_\parallel)$, with $B_\parallel = B_3 := B \cdot e_\parallel \in \mathbb{R}$, and $B_\perp := {}^t(B_1, B_2) \in \mathbb{R}^2$.
  - $x := (x_\perp, x_\parallel)$, with $x_\perp := (x_1, x_2)$, and $x_\parallel := x_3$.
  - $\nabla_\perp := {}^t(\partial_1, \partial_2)\,, \quad \nabla_\parallel := e_\parallel\, \partial_\parallel\,, \quad \partial_\parallel \equiv \partial_3\,.$
  - $\Delta_\perp := \partial^2_{11} + \partial^2_{22}, \quad \Delta_\parallel := \partial^2_{33}.$

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- On $\mathbb{R}_+ \times \Omega \ni (t,x)$ the system of compressible MHD equations reads:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0\,, \\ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p + B \times (\nabla \times B) - \mu_\perp \Delta_\perp v - \mu_\parallel \Delta_\parallel v - \lambda \nabla(\nabla \cdot v) = 0\,, \\ \partial_t B + \nabla \times (B \times v) - \eta_\perp \Delta_\perp B - \eta_\parallel \Delta_\parallel B = 0\,, \qquad \nabla \cdot B = 0. \end{cases} \tag{1}$$

Non-dimensionalization and scaling: the large aspect ratio framework

• Putting dimensions into the "bar" quantities, we define the dimensionless unknowns and variables as :

$\mathrm{t} = \bar{\mathrm{t}}\, t$, $\mathrm{x}_\perp = \bar{\mathrm{x}}_\perp\, x_\perp$, $\mathrm{x}_\parallel = \bar{\mathrm{x}}_\parallel\, x_\parallel$, $\upmu_\perp = \bar{\upmu}_\perp\, \mu_\perp$, $\upmu_\parallel = \bar{\upmu}_\parallel\, \mu_\parallel$, $\uplambda = \bar{\uplambda}\, \lambda$, $\upeta_\perp = \bar{\upeta}_\perp\, \eta_\perp$, $\upeta_\parallel = \bar{\upeta}_\parallel\, \eta_\parallel$, $\mathfrak{a} = \bar{\mathfrak{a}}\, a$, $\uprho = \bar{\uprho}\, \rho$, $\mathrm{v} = \bar{\mathrm{v}}\, v$, $\mathrm{B} = \bar{\mathrm{B}}\, B$, and $\mathrm{p} = \bar{\mathrm{p}}\, p$, with $\bar{\mathrm{p}} = \bar{\mathfrak{a}}\bar{\uprho}^\gamma$ and $p = a\rho^\gamma$.

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- Some definitions. The Alfvén velocity : $v_A := \bar{\mathrm{B}}/\sqrt{\bar{\uprho}}$; the sound velocity : $v_s := \sqrt{\gamma\bar{\mathrm{p}}/\bar{\uprho}}$; the parameter $\beta := \bar{\mathrm{p}}/|\bar{\mathrm{B}}|^2 = v_s^2/(\gamma v_A^2)$; the Alfvén number : $\varepsilon_A := \bar{\mathrm{v}}/v_A$; the Mach number : $\varepsilon_M := \bar{\mathrm{v}}/v_s = \varepsilon_A/(\sqrt{\gamma\beta})$. Here we choose $\beta = 1$, thus we have $v_s = \sqrt{\gamma} v_A \simeq v_A$.

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Since we want to describe the dynamics on a time scale longer that $\tau_\perp$, we set $\bar{\mathrm{t}} = \tau_\perp/\varepsilon$, with $\varepsilon \ll 1$. This is equivalent to describe the dynamics of waves which propagate with velocity slower than $\mathcal{C}_F$ (or v_A). Hence, we have $\bar{\mathrm{v}} = \varepsilon v_A$, $\varepsilon_A = \varepsilon$, , and $\varepsilon_M = \varepsilon/(\sqrt{\gamma\beta}) \simeq \varepsilon$.

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- Moreover, we suppose a strong anisotropy between the perpendicular and parallel direction, that is $\bar{\mathrm{x}}_\perp/\bar{\mathrm{x}}_\parallel = \varepsilon$. In other words the dimensional gradient ∇_{x} becomes the anisotropic dimensionless gradient $\nabla_\varepsilon := {}^t({}^t\nabla_\perp, 0) + \varepsilon\nabla_\parallel$.

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- In addition, we suppose the presence of a strong constant background magnetic field in the parallel direction: $B = e_\parallel + \varepsilon B^\varepsilon$.

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- Finally, it remains to choose some scalings with respect to the small parameter ε for the dimensionless viscosities and resistivities:

- We choose $\{\mu_\perp = \varepsilon\mu_\perp^\varepsilon,\, \mu_\parallel = \mu_\parallel^\varepsilon/\varepsilon,\, \lambda = \varepsilon\lambda^\varepsilon\}$, where viscosities $\{\mu_\perp^\varepsilon,\, \mu_\parallel^\varepsilon,\, \lambda^\varepsilon\}$ satisfy

$$\mu_\perp^\varepsilon \longrightarrow \mu_\perp > 0\,, \quad \mu_\parallel^\varepsilon \longrightarrow \mu_\parallel > 0\,, \quad \lambda^\varepsilon \longrightarrow \lambda > 0\,, \quad \text{as} \quad \varepsilon \longrightarrow 0_+\,. \tag{2}$$

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- The system (1) is then equivalent to the following system written in the dimensionless unknowns $(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)(t,x)$:

$$\begin{cases} \partial_t\rho^\varepsilon + \nabla_\varepsilon\cdot(\rho^\varepsilon v^\varepsilon) = 0\,, \\ \partial_t(\rho^\varepsilon v^\varepsilon) + \nabla_\varepsilon\cdot(\rho^\varepsilon v^\varepsilon\otimes v^\varepsilon) + \dfrac{1}{\varepsilon^2}\nabla_\varepsilon p^\varepsilon + \dfrac{1}{\varepsilon}(e_\parallel + \varepsilon B^\varepsilon)\times(\nabla_\varepsilon\times B^\varepsilon) \\ \qquad\qquad - \mu_\perp^\varepsilon\Delta_\perp v^\varepsilon - \mu_\parallel^\varepsilon\Delta_\parallel v^\varepsilon - \lambda^\varepsilon\nabla_\varepsilon(\nabla_\varepsilon\cdot v^\varepsilon) = 0\,, \\ \partial_t B^\varepsilon + \dfrac{1}{\varepsilon}\nabla_\varepsilon\times\big((e_\parallel+\varepsilon B^\varepsilon)\times v^\varepsilon\big) - \eta_\perp^\varepsilon\Delta_\perp B^\varepsilon - \eta_\parallel^\varepsilon\Delta_\parallel B^\varepsilon = 0\,, \quad \nabla_\varepsilon\cdot B^\varepsilon = 0\,. \end{cases} \tag{4}$$

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- We then look for solution like

$$\begin{pmatrix} \rho/\bar{\rho} \\ v/\bar{v} \\ B/\bar{B} \end{pmatrix}(t, x_\perp, x_\parallel) = \begin{pmatrix} \rho^\varepsilon(t, \varepsilon^{-1} x_\perp, x_\parallel) \\ \varepsilon\, v^\varepsilon(t, \varepsilon^{-1} x_\perp, x_\parallel) \\ e_\parallel + \varepsilon B^\varepsilon(t, \varepsilon^{-1} x_\perp, x_\parallel) \end{pmatrix} = \begin{pmatrix} \overline{\rho^\varepsilon} \\ 0 \\ e_\parallel \end{pmatrix} + \varepsilon \begin{pmatrix} \varrho^\varepsilon \\ v^\varepsilon \\ B^\varepsilon \end{pmatrix}(t, \varepsilon^{-1} x_\perp, x_\parallel)$$

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- Here, we choose viscosities and resistivities which accommodate the propagation of oscillating waves with wavelengths approximately ε. For this, we impose viscosities $\{\mu_\perp = \varepsilon^2 \mu^\varepsilon_\perp,$ $\mu_\parallel = \mu^\varepsilon_\parallel, \lambda = \varepsilon^2 \lambda^\varepsilon\}$, where dimensionless viscosities $\{\mu^\varepsilon_\perp, \mu^\varepsilon_\parallel, \lambda^\varepsilon\}$ satisfy (2), and resistivities $\{\eta_\perp = \varepsilon^2 \eta^\varepsilon_\perp, \eta_\parallel = \eta^\varepsilon_\parallel\}$, where dimensionless resistivities $\{\eta^\varepsilon_\perp, \eta^\varepsilon_\parallel\}$ satisfy (3).

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- Therefore, we investigate the dynamics of a magnetized plasma near a fixed large constant magnetic field where anisotropic oscillations in space can develop.

- The first term of the right-hand side of (5), which is of order of unity, is a stationary solution of (1).

- The second term of the right-hand side of (5) is the perturbation, which is of small amplitude ($\varepsilon \ll 1$) and of high frequency ($\varepsilon^{-1} \gg 1$).

- Such a framework belongs to the so called weakly nonlinear geometric optics regime.

Slow versus fast regime

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- The **fast** regime when it is not slow. In this case, rapid time oscillations with non-vanishing amplitudes can persist on a long time scale, preventing the convergence in a usual strong sense.

With which waves is the fast regime associated ?

- Since the singular part involves the sole action of the operator $\nabla_\perp$, it induces a propagation which can only be achieved with respect to the perpendicular direction. Then, because Alfvén waves do not propagate in the directions orthogonal to the ambiant magnetic field (here $e_\parallel$), we are necessarily concerned with transverse fast magnetosonic waves as justified below.

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- Since here $\mathbb{b} := e_\|$, Alfvén waves cannot propagate in the perpendicular direction to $e_\|$. Indeed, it is well-known that Alfvén waves propagate mainly along the direction ($\mathbb{b} := e_\|$) of the ambiant magnetic field. For a wave propagating in the perpendicular direction to $e_\|$, we obtain:

$\lambda_F^\pm = \pm (V_s^2 + V_A^2)^{1/2} \simeq \pm V_A \simeq \pm V_s$, whereas $\lambda_S^\pm = 0$.

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- In dimensionless variables we have $\lambda_F^{\pm} = \pm\sqrt{b+1}$, with $b = a\gamma$.

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- In order to understand now the nature of the waves that are filtered out from the singular linear part of the system (7), we rewrite it in the fast time variable t to obtain

$$\partial_{\mathrm{t}}\varrho + \nabla_\perp \cdot v_\perp = 0\,, \qquad \partial_{\mathrm{t}} v_\perp + \nabla_\perp(b\varrho + B_\parallel) = 0\,, \qquad \partial_{\mathrm{t}} B_\parallel + \nabla_\perp \cdot v_\perp = 0\,. \tag{9}$$

- With $U := {}^t(\varrho, v_1, v_2, B_\parallel)$, the previous system can be recast as

$$\partial_{\mathrm{t}} U + (A_1\partial_{x_1} + A_1\partial_{x_2})U = 0,$$

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- With the linear system (9), we can associated a **unitary group** $\mathcal{S}_F(\mathrm{t})$, which takes into account of propagation of transverse fast magnetosonic waves.

Prepared versus unprepared (or general) initial data

- At time $t = 0$, we start with

$$\rho^{\varepsilon}_{|t=0} = \rho^{\varepsilon}_0\,, \qquad v^{\varepsilon}_{|t=0} = v^{\varepsilon}_0\,, \qquad B^{\varepsilon}_{|t=0} = B^{\varepsilon}_0\,, \quad \nabla_{\varepsilon} \cdot B^{\varepsilon}_0 = 0\,. \tag{10}$$

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- The next stage is to identify the limit (ϱ, v, B).

The limit model RMHD : perpendicular components

- The perpendicular component $B_\perp := {}^t(B_1, B_2) \in \mathbb{R}^2$ and the perpendicular component $v_\perp := {}^t(v_1, v_2) \in \mathbb{R}^2$ can be identified independently by solving the following nonlinear closed system:

$$\begin{cases} \partial_t B_\perp - \partial_\parallel v_\perp + \nabla_\perp \cdot (B_\perp \otimes v_\perp - v_\perp \otimes B_\perp) - \eta_\perp \Delta_\perp B_\perp - \eta_\parallel \Delta_\parallel B_\perp = 0\,, \\ \partial_t v_\perp - \partial_\parallel B_\perp + \nabla_\perp \cdot (v_\perp \otimes v_\perp - B_\perp \otimes B_\perp) + \nabla_\perp \pi - \mu_\perp \Delta_\perp v_\perp - \mu_\parallel \Delta_\parallel v_\perp = 0\,, \end{cases} \tag{12}$$

together with the (transverse velocity) divergence-free condition: $\nabla_\perp \cdot v_\perp = 0\,,$ (13)

and the initial data: $(B_\perp, v_\perp)_{|t=0} = (\mathbb{P}_\perp B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\Omega; \mathbb{R}^4)\,,$ (14)

where the projection $\mathbb{P}_\perp := \mathbb{I}_\perp - \nabla_\perp \Delta_\perp^{-1} \nabla_\perp$ denotes the 2D transverse Leray operator:

$$v_\perp = \mathbb{P}_\perp v_\perp + \mathbb{Q}_\perp v_\perp\,, \quad \nabla_\perp \cdot (\mathbb{P}_\perp v_\perp) = 0\,, \quad \nabla_\perp \times (\mathbb{Q}_\perp v_\perp) = 0\,, \quad \nabla_\perp \times v_\perp := \partial_1 v_2 - \partial_2 v_1\,.$$

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- In the second equation of (12), the pressure π plays the role of a Lagrange multiplier to ensure the transverse incompressibility of the flow. Using (13), we have

$$\partial_t (\nabla_\perp \cdot B_\perp) - \eta_\perp \Delta_\perp (\nabla_\perp \cdot B_\perp) - \eta_\parallel \Delta_\parallel (\nabla_\perp \cdot B_\perp) = 0 \,.$$

It follows that the divergence-free condition on $B_{0\perp}$ is propagated, i.e. $\nabla_\perp \cdot B_\perp = 0 \,.$

The limit model RMHD : parallel components

- The weak limit $(B_\parallel, v_\parallel)$ is decoupled from the one of $(B_\perp, v_\perp)$. In fact, knowing the content of $(B_\perp, v_\perp)$, with the constant $\mathbb{c} := 1 + (1/b) > 1$, we have access to $(B_\parallel, v_\parallel)$ through:

$$\begin{cases} \mathbb{c}\big(\partial_t B_\parallel + (v_\perp \cdot \nabla_\perp) B_\parallel\big) - \partial_\parallel v_\parallel - (B_\perp \cdot \nabla_\perp) v_\parallel - \eta_\perp \Delta_\perp B_\parallel - \eta_\parallel \Delta_\parallel B_\parallel = 0\,, \\ \partial_t v_\parallel + (v_\perp \cdot \nabla_\perp) v_\parallel - \partial_\parallel B_\parallel - (B_\perp \cdot \nabla_\perp) B_\parallel - \mu_\perp \Delta_\perp v_\parallel - \mu_\parallel \Delta_\parallel v_\parallel = 0\,, \end{cases} \tag{15}$$

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$$(B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel})\,, \qquad \mathbb{B}_{0\parallel} := (B_{0\parallel} - \varrho_0)/\mathbb{c}\,. \tag{16}$$

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- Given $\varepsilon > 0$, the fluid is slightly compressible since $\rho^\varepsilon = 1 + \varepsilon\, \varrho^\varepsilon$ with $\varrho^\varepsilon \sim \varrho$.

- The expression ϱ^ε (and its weak limit ϱ) plays at the level of (4) the part of a one-order corrector which keeps track of the original compressibility.

- Now, looking at (7), ϱ^ε acts in the equations with the same order as the components $v_\perp^\varepsilon$ and $B_\parallel^\varepsilon$. It is therefore reasonable to find a link between ϱ, $v_\perp$ and $B_\parallel$.

- In view of the second relation inside (8), we can already infer that $\mathbb{p} + B_\parallel = 0$, where $\mathbb{p} := b\,\varrho$ is the weak limit of $\mathbb{p}^\varepsilon$. By this way, $B_\parallel$ acquires asymptotically the status of a pressure which can serve to measure some compressibility in the parallel direction.

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- For prepared data, that is when $B_{0\parallel} + b\varrho_0 = 0$, we start with $\mathbb{B}_{0\parallel} = B_{0\parallel}$.

- Otherwise, for general unprepared data (which is our framework), we find that $\mathbb{B}_{0\parallel} \not\equiv B_{0\parallel}$. Again, this is the hallmark of a boundary layer occuring at time $t = 0$ concerning the component $B_\parallel^\varepsilon$.

The periodic case: $\Omega = \mathbb{T}^3$

- The functional framework (existence of weak solutions) is based on Lions's 1998 book (L98), Lions–Masmoudi-JMPA-98 (LM98), Feireisl-Novotny-Petzeltova-JMFM-2000 (FNP00), and Hu–Wang-ARMA-2010 (HW10).

- Select initial data satisfying
$$\begin{cases} \rho^\varepsilon_{|_{t=0}} = \rho^\varepsilon_0 \in L^\gamma(\mathbb{T}^3)\,, \qquad \rho^\varepsilon_0 \geq 0\,, \qquad \sqrt{\rho^\varepsilon_0}\, v^\varepsilon_0 \in L^2(\mathbb{T}^3)\,, \\ (\rho^\varepsilon v^\varepsilon)_{|_{t=0}} = m^\varepsilon_0 = \begin{Bmatrix} \rho^\varepsilon_0 v^\varepsilon_0 & \text{if } \rho^\varepsilon_0 \neq 0 \\ 0 & \text{if } \rho^\varepsilon_0 = 0 \end{Bmatrix} \in L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)\,, \\ B^\varepsilon_{|_{t=0}} = B^\varepsilon_0 \in L^2(\mathbb{T}^3)\,, \qquad \nabla_\varepsilon \cdot B^\varepsilon_0 = 0\,, \qquad \int_{\mathbb{T}^3} dx\, B^\varepsilon_0 = 0\,. \end{cases} \tag{17}$$

We assume that these regularity assumptions are uniform with respect to ε.

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- Furthermore, given a constant C_0 (not depending on ε), we impose
$$\frac{1}{2}\int_{\mathbb{T}^3} dx\,\left(\rho^\varepsilon_0 |v^\varepsilon_0|^2 + |B^\varepsilon_0|^2\right) + \frac{a}{\varepsilon^2(\gamma-1)}\int_{\mathbb{T}^3} dx\,\left((\rho^\varepsilon_0)^\gamma - \gamma\rho^\varepsilon_0(\overline{\rho^\varepsilon_0})^{\gamma-1} + (\gamma-1)(\overline{\rho^\varepsilon_0})^\gamma\right) \leq C_0\,. \tag{18}$$

This is completed by:
$$\overline{\rho^\varepsilon_0} := \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} dx\, \rho^\varepsilon_0 \longrightarrow 1\,, \quad \text{as } \varepsilon \longrightarrow 0_+\,. \tag{19}$$

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- As soon as $\gamma > 3/2$, works of [L98, LM98, FNP00, HW10] furnish a weak solution to (4) with

$$\begin{cases} \rho^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3))\,, \quad v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,, \\ \sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))\,, \quad \rho^\varepsilon v^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)) \cap \mathscr{C}_{\text{loc}}(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\text{weak}}(\mathbb{T}^3))\,, \\ B^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap \mathscr{C}_{\text{loc}}(\mathbb{R}_+; L^2_{\text{weak}}(\mathbb{T}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,, \qquad \int_{\mathbb{T}^3} dx\, B^\varepsilon = 0\,. \end{cases} \tag{20}$$

The periodic case: $\Omega = \mathbb{T}^3$

- The mass is conserved

$$\overline{\rho^\varepsilon} := \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} dx \, \rho^\varepsilon = \overline{\rho_0^\varepsilon} \,,$$

and thus, using (19), we deduce that $\overline{\rho^\varepsilon} \to 1$, as $\varepsilon \to 0_+$.

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- Moreover, we have two energy inequalities

$$\mathbb{E}^\varepsilon(t) + \int_0^t ds\, \mathbb{D}^\varepsilon(s) \le \mathbb{E}_0^\varepsilon\,, \quad \text{a.e. } t \in [0, +\infty)\,, \tag{21}$$

with $\mathbb{E} \in \{\mathcal{E}_1, \mathcal{E}_2\}$, where for $i = 1, 2$,

$$\mathcal{E}_i^\varepsilon(t) = \int_\Omega dx \left(\tfrac{1}{2}\rho^\varepsilon |v^\varepsilon|^2 + \tfrac{1}{2}|B^\varepsilon|^2 + \Pi_i(\rho^\varepsilon)\right), \quad \mathcal{E}_{i0}^\varepsilon = \int_\Omega dx \left(\tfrac{1}{2}\rho_0^\varepsilon |v_0^\varepsilon|^2 + \tfrac{1}{2}|B_0^\varepsilon|^2 + \Pi_i(\rho_0^\varepsilon)\right), \tag{22}$$

$$\mathbb{D}^\varepsilon = \int_\Omega dx \left(\mu_\perp^\varepsilon |\nabla_\perp v^\varepsilon|^2 + \mu_\parallel^\varepsilon |\partial_\parallel v^\varepsilon|^2 + \lambda^\varepsilon |\nabla_\varepsilon \cdot v^\varepsilon|^2 + \eta_\perp^\varepsilon |\nabla_\perp B^\varepsilon|^2 + \eta_\parallel^\varepsilon |\partial_\parallel B^\varepsilon|^2\right), \tag{23}$$

and

$$\Pi_1(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma - 1)}(\rho^\varepsilon)^\gamma\,, \quad \Pi_2(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma - 1)}\left((\rho^\varepsilon)^\gamma - \gamma\rho^\varepsilon(\overline{\rho^\varepsilon})^{\gamma-1} + (\gamma - 1)(\overline{\rho^\varepsilon})^\gamma\right). \tag{24}$$

- For $i = 1$, the inequality (21) is a consequence of straightforward calculation involving (4).

- Using the mass conservation, we can check that the inequality (21) for $i = 2$ is equivalent to (21) for $i = 1$. The case $i = 2$ is introduced because it allows a better comparison of ρ^ε with $\overline{\rho^\varepsilon}$.

The periodic case: $\Omega = \mathbb{T}^3$

Theorem (Convergence of MHD to RMHD on a periodic domain)

Assume $\Omega = \mathbb{T}^3$ *and* $\gamma > 3/2$. *Consider a sequence* $\{(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ *of weak solutions to the compressible MHD system* (4) *with initial data* $\{(\rho_0^\varepsilon, v_0^\varepsilon, B_0^\varepsilon)\}_{\varepsilon>0}$ *as in* (18). *Let us set* $\varrho^\varepsilon := (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$. *Then, up to a subsequence, the family* $\{(\rho^\varepsilon, \varrho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ *converges to* $(1, \varrho, v, B)$ *as:*

$$
\begin{aligned}
&\rho^\varepsilon \longrightarrow 1 \ \text{ in } \ L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3)) - \text{strong}\,,\\
&\varrho^\varepsilon \rightharpoonup \varrho \ \text{ in } \ L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3)) - \text{weak}-*\,, \quad \kappa = \min\{2,\gamma\}\,,\\
&\mathbb{P}_\perp v_\perp^\varepsilon \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \ \text{ in } \ L^2_{\mathrm{loc}}(\mathbb{R}_+; L^p \cap H^s(\mathbb{T}^3)) - \text{strong}\,, \quad 1 \le p < 6\,, \quad 0 \le s < 1\,,\\
&\mathbb{Q}_\perp v_\perp^\varepsilon \rightharpoonup 0 \ \text{ in } \ L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)) - \text{weak}\,,\\
&v_\parallel^\varepsilon \rightharpoonup v_\parallel \ \text{ in } \ L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)) - \text{weak}\,,\\
&B_\perp^\varepsilon \longrightarrow B_\perp \ \text{ in } \ L^r_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) - \text{strong}\,, \quad 1 \le r < \infty\,,\\
&B_\parallel^\varepsilon \rightharpoonup B_\parallel \ \text{ in } \ L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)) - \text{weak} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) - \text{weak}-*\ .
\end{aligned}
$$

The limit point (v, B) *is a weak solution to the RMHD equations* (12)-(13)-(15) *with initial data*

$$
(B_\perp, v_\perp)_{|t=0} = (B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\mathbb{T}^3)\,, \qquad (B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel}) \in L^2(\mathbb{T}^3)\,,
$$

where $\mathbb{B}_{0\parallel}$ *is as in* (16), *and it satisfies the following regularity properties*

$$
B \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,, \qquad v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,.
$$

Moreover, ϱ *and* $B_\parallel$ *satisfy the relation* $b\,\varrho + B_\parallel = 0$, *for a.e.* $(t,x) \in]0,+\infty[\times\mathbb{T}^3$.

The whole space case: $\Omega = \mathbb{R}^3$

• In order to define weak solutions in the whole space, we need to introduce the following special type of Orlicz spaces $L^p_q(\Omega)$,

$$L^p_q(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; f\mathbb{1}_{\{|f|\leq\delta\}} \in L^q(\Omega), \;\; f\mathbb{1}_{\{|f|>\delta\}} \in L^p(\Omega), \; \delta > 0 \right\},$$

where the function $\mathbb{1}_S$ denotes the indicator function of the set S. Obviously $L^p_p(\Omega) \equiv L^p(\Omega)$.

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where the function $\mathbb{1}_S$ denotes the indicator function of the set S. Obviously $L^p_p(\Omega) \equiv L^p(\Omega)$.

- The functional framework is based on [L98, LM98, FNP00, HW10]. Select initial data satisfying

$$\begin{cases} \rho^\varepsilon_{|_{t=0}} = \rho^\varepsilon_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)\,, \quad \rho^\varepsilon_0 - 1 \in L^\gamma_2(\mathbb{R}^3)\,, \quad \gamma > 3/2\,, \quad \rho^\varepsilon_0 \geq 0\,, \quad \sqrt{\rho^\varepsilon_0}v^\varepsilon_0 \in L^2(\mathbb{R}^3)\,, \\ (\rho^\varepsilon v^\varepsilon)_{|_{t=0}} = m^\varepsilon_0 = \begin{Bmatrix} \rho^\varepsilon_0 v^\varepsilon_0 & \text{if } \rho^\varepsilon_0 \neq 0 \\ 0 & \text{if } \rho^\varepsilon_0 = 0 \end{Bmatrix} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)\,, \\ B^\varepsilon_{|_{t=0}} = B^\varepsilon_0 \in L^2(\mathbb{R}^3)\,, \qquad \nabla_\varepsilon \cdot B^\varepsilon_0 = 0\,, \qquad \displaystyle\int_{\mathbb{R}^3} dx\, B^\varepsilon_0 = 0\,, \\ \rho^\varepsilon_0 \longrightarrow 1\,, \quad v^\varepsilon_0 \longrightarrow 0\,, \quad B^\varepsilon_0 \longrightarrow 0\,, \quad \text{as } |x| \longrightarrow \infty\,. \end{cases} \tag{25}$$

We assume that these regularity assumptions are uniform with respect to ε.

- Furthermore, given a constant C_0 (not depending on ε), we impose

$$\frac{1}{2}\int_{\mathbb{R}^3} dx\,\left(\rho^\varepsilon_0|v^\varepsilon_0|^2 + |B^\varepsilon_0|^2\right) + \frac{a}{\varepsilon^2(\gamma-1)}\int_{\mathbb{R}^3} dx\,\left((\rho^\varepsilon_0)^\gamma - \gamma\rho^\varepsilon_0 + \gamma - 1\right) \leq C_0\,. \tag{26}$$

The whole space case: $\Omega = \mathbb{R}^3$

- As soon as $\gamma > 3/2$, the contributions [L98, LM98, FNP00, HW10] furnish a weak solution to (4) with

$$
\begin{cases}
\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_{\mathrm{loc}}(\mathbb{R}^3))\,, \quad \rho^\varepsilon - 1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_2(\mathbb{R}^3))\,, \quad \nabla v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))\,, \\
\sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))\,, \quad \rho^\varepsilon v^\varepsilon \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\mathrm{loc}}(\mathbb{R}^3)\big) \cap \mathscr{C}_{\mathrm{loc}}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\mathrm{loc\ weak}}(\mathbb{R}^3)\big)\,, \\
B^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap \mathscr{C}_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{weak}}(\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,, \quad \displaystyle\int_{\mathbb{T}^3} dx\, B^\varepsilon = 0\,, \\
\rho^\varepsilon \longrightarrow 1\,, \quad v^\varepsilon \longrightarrow 0\,, \quad B^\varepsilon \longrightarrow 0\,, \quad \text{as } |x| \longrightarrow \infty\,.
\end{cases}
\tag{27}
$$

The whole space case: $\Omega = \mathbb{R}^3$

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$$\begin{cases} \rho^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma_{\text{loc}}(\mathbb{R}^3)), \quad \rho^\varepsilon - 1 \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma_2(\mathbb{R}^3)), \quad \nabla v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)), \\ \sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)), \quad \rho^\varepsilon v^\varepsilon \in L^\infty_{\text{loc}}\left(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\text{loc}}(\mathbb{R}^3)\right) \cap \mathscr{C}_{\text{loc}}\left(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\text{loc weak}}(\mathbb{R}^3)\right), \\ B^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap \mathscr{C}_{\text{loc}}(\mathbb{R}_+; L^2_{\text{weak}}(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)), \quad \displaystyle\int_{\mathbb{T}^3} dx\, B^\varepsilon = 0, \\ \rho^\varepsilon \longrightarrow 1, \quad v^\varepsilon \longrightarrow 0, \quad B^\varepsilon \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty. \end{cases} \tag{27}$$

- Moreover, we have the energy inequality (21) with $\mathbb{E}^\varepsilon = \mathcal{E}^\varepsilon_3$ given by the formula (22) and $\Pi_i = \Pi_3$, where Π_3 is defined by

$$\Pi_3(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma - 1)}\left((\rho^\varepsilon)^\gamma - \gamma\rho^\varepsilon + \gamma - 1\right). \tag{28}$$

This energy inequality is the consequence of straightforward calculation involving (4).

The whole space case: $\Omega = \mathbb{R}^3$

Theorem (Convergence of MHD to RMHD on the whole space)

Assume $\Omega = \mathbb{R}^3$ *and* $\gamma > 3/2$. *Consider* $\{(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ *a sequence of weak solutions to the compressible MHD system* (4) *with initial data* $\{(\rho_0^\varepsilon, v_0^\varepsilon, B_0^\varepsilon)\}_{\varepsilon>0}$ *as in* (26). *Let us set* $\varrho^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$. *Then, up to a subsequence, the family* $\{(\rho^\varepsilon, \varrho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ *converge to* $(1, \varrho, v, B)$ *as:*

$$\begin{aligned}
&\rho^\varepsilon \longrightarrow 1 \ \text{ in } \ L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma_2 \cap L^\gamma_{\text{loc}} \cap H^{-\alpha}(\mathbb{R}^3)) - \text{strong}\,, \quad \alpha \geq 1/2\,,\\
&\varrho^\varepsilon \rightharpoonup \varrho \ \text{ in } \ L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa_{\text{loc}} \cap H^{-\alpha}(\mathbb{R}^3)) - \text{weak}-*\,, \quad \kappa = \min\{2,\gamma\}\,, \quad \alpha \geq 1/2\,,\\
&\mathbb{P}_\perp v_\perp^\varepsilon \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; L^p_{\text{loc}} \cap H^s_{\text{loc}}(\mathbb{R}^3)) - \text{strong}\,, \quad 1 \leq p < 6\,, \quad 0 \leq s < 1\,,\\
&\mathbb{Q}_\perp v_\perp^\varepsilon \rightharpoonup 0 \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3)) - \text{weak}\,,\\
&v_\parallel^\varepsilon \rightharpoonup v_\parallel \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3)) - \text{weak}\,,\\
&B_\perp^\varepsilon \longrightarrow B_\perp \ \text{ in } \ L^r_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}}(\mathbb{R}^3)) - \text{strong}\,, \quad 1 \leq r < \infty\,,\\
&B_\parallel^\varepsilon \rightharpoonup B_\parallel \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3)) - \text{weak} \, \cap \, L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) - \text{weak}-*\,.
\end{aligned}$$

The limit point (v, B) *is a weak solution to the RMHD equations* (12)-(13)-(15) *with initial data*

$$(B_\perp, v_\perp)_{|t=0} = (\mathbb{P}_\perp B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\mathbb{R}^3)\,, \qquad (B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel}) \in L^2(\mathbb{R}^3)\,,$$

where $\mathbb{B}_{0\parallel}$ *is as in* (16), *and it satisfies the following regularity properties*

$$B \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))\,, \qquad v \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))\,.$$

Moreover, ϱ *and* $B_\parallel$ *satisfy the relation* $b\,\varrho + B_\parallel = 0$, *for a.e.* $(t,x) \in]0,+\infty[\times\mathbb{R}^3$.

The periodic case : $\Omega = \mathbb{T}^3$ (Global-in-time, No prepared initial data)

- Compactness (weak or strong) for the sequences :

$$\{(\rho^\varepsilon,\ \varrho^\varepsilon,\ v^\varepsilon,\ \mathbb{P}_\perp v_\perp^\varepsilon,\ \mathbb{Q}_\perp v_\perp^\varepsilon,\ \rho^\varepsilon v^\varepsilon,\ \mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon),\ B^\varepsilon,\ B_\perp^\varepsilon)\}_{\varepsilon>0}\ :$$

  - Physical a priori estimates : energy inequality, convexity of pressure law.
  - Mathematical a priori estimates : Hölder inequality, Poincaré–Wirtinger inequality, "infinitely" iterated Gagliardo–Nirenberg interpolation inequality.
  - Standard compactness tools : Sobolev embeddings (spatial compactness), Aubin–Lions–Simon theorems (time compactness).

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  - Standard compactness tools : Sobolev embeddings (spatial compactness), Aubin–Lions–Simon theorems (time compactness).

- Passage to the limit in the perpendicular direction.

  - We use the unitary group method (Schochet-JDE-1994, LM98).
  - Recall the unitary group $\mathcal{S}_F(t)$, which is associated with the perpendicular wave operator related to the propagation of transverse fast magnetosonic waves.
  - Filtering the unknows $U^\varepsilon := {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon)$, with $\phi^\varepsilon := b^\varepsilon \varrho^\varepsilon + B_\parallel^\varepsilon$, and $\Phi^\varepsilon := \mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon)$, by using the group $\mathcal{S}_F(t/\varepsilon)$, we get existence of the limit $\mathcal{U} = {}^t(\psi, {}^t\Psi) \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3; \mathbb{R}^3))$.

    To pass to the limit in both singular terms and nonlinear terms, we construct an approximate solution given by

$$U^\varepsilon = \mathcal{S}_F(t/\varepsilon)\mathcal{U} + \mathcal{R}^\varepsilon, \quad \text{with } \mathcal{R}^\varepsilon \longrightarrow 0 \ \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3)) - \mathrm{strong}. \tag{29}$$

The periodic case : $\Omega = \mathbb{T}^3$ (Global-in-time, No prepared initial data)

- The pressure term π, which can be seen as a Lagrange multiplier ensuring the constraint $\nabla_\perp \cdot v_\perp = 0$, comes from the following three contributions π_1, π_2 and π_3.
  - The pressure term π_1 comes from the singular (compressible) fluid pressure term and the singular magnetic term.
  - The pressure π_2 comes from the non-singular fluid and magnetic pressure terms.
  - The pressure term π_3, which comes from the Reynolds stress tensor, results from taking into account the resonant interactions of the compressible modes on the incompressible mode, while the non-resonant interaction terms vanish in the limit by using Riemann–Lebesgue or stationary phase arguments.

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- <u>Passage to limit in the parallel direction.</u>
  - We use special cancellations due to the particular structure of the equations. In particular we use the combination of the equations of ρ^ε and $B_\parallel^\varepsilon$.
  - To pass to the limit in weak products, where each term of the product converges only weakly, we use a space-time compactness argument due to P.-L. Lions (in “Mathematical Topics in Fluid Mechanics. Compressible models”, 1998) for showing the existence of weak solutions for the compressible Navier–Stokes equations.
    This is link to the resurgence of compressibility in the parallel direction.

The wholde space case : $\Omega = \mathbb{R}^3$ (Global-in-time, No prepared initial data)

- Compactness (weak or strong) for the sequences :

  $\{(\rho^\varepsilon,\ \varrho^\varepsilon,\ v^\varepsilon,\ \mathbb{P}_\perp v^\varepsilon_\perp,\ \mathbb{Q}_\perp v^\varepsilon_\perp,\ \rho^\varepsilon v^\varepsilon,\ \mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp),\ B^\varepsilon,\ B^\varepsilon_\perp)\}_{\varepsilon>0}$:

  - The proof of a priori estimates follows more or less the same lines than the periodic case.
  - Since in the whole case the density is only locally integrable in space, we have to deal with estimates in Orlicz spaces, particularly for ρ^ε, but not only that.
  - compactness properties: local-in-space version for strong compactness results.

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  - The proof of a priori estimates follows more or less the same lines than the periodic case.
  - Since in the whole case the density is only locally integrable in space, we have to deal with estimates in Orlicz spaces, particularly for ρ^ε, but not only that.
  - compactness properties: local-in-space version for strong compactness results.
- Passage to the limit. It follows the same strategy as the periodic case but it is more technical :
  - Contrary to the periodic case, we are not able to get the strong convergence (29) for U^ε. Indeed, we can still prove locally strong compactness in space-time for $\mathcal{U}^\varepsilon := \mathcal{S}_F(-t/\varepsilon)U^\varepsilon$ (using an Aubin–Lions theorem and local a priori estimates), but returning to U^ε, via the group of isometry $\mathcal{S}_F(t/\varepsilon)$, such local strong compactness seems unaccessible since $\mathcal{S}_F$ is non local. To recover some strong compactness for U^ε we consider a truncated version of U^ε and use the uniform integrability in space of such truncated sequences as well as the energy inequality and the strong convergence of ρ^ε.
  - By contrast with the zero-Mach-number limit of compressible Navier–Stokes equations, where in Desjardins-Grenier-PRSLA-1999 the dispersive effects (and Strichartz estimates) of the wave equation are used to show strong convergence to zero of the irrotational part the velocity, $\mathbb{Q}v^\varepsilon$, (fast acoustic waves), here, it seems not possible to use dispersive effects of the perpendicular wave operator (associated with the transverse fast magnetosonic waves) to show that $\mathbb{Q}_\perp v_\perp^\varepsilon$ converges strongly to zero. This is an extra difficulty due to anisotropy.
- Open issue : Can we get extra strong compactness for $\mathbb{Q}_\perp v_\perp^\varepsilon$ (in all variables) using some dispersive effects of the perpendicular wave operator ?

THANK YOU FOR YOUR ATTENTION