A Weakly Nonlinear Theory of Zonal-Flow Forcing in Gyrokinetic Turbulence

G. Acton, E. Rodríguez, G. Roberg-Clark, A. Zocco

Max Planck Institute for Plasma Physics, Greifswald

Thursday 30$^{\text{th}}$ July 2026
WPI workshop

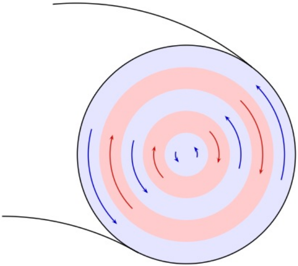
MAX-PLANCK-INSTITUT
FÜR PLASMAPHYSIK

Table of Contents

Table of Contents

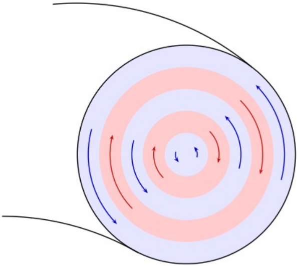
Motivation

- **Zonal flows** (turbulence-generated): Self-organised, large-scale shear flows that regulate turbulent transport in magnetised plasmas



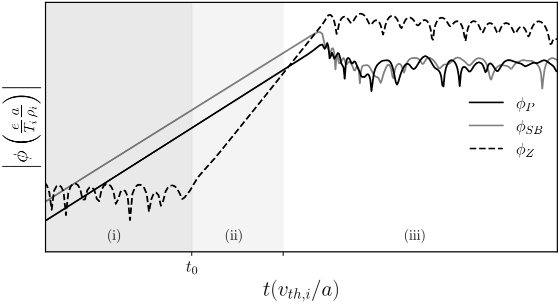
Motivation

- **Zonal flows** (turbulence-generated): Self-organised, large-scale shear flows that regulate turbulent transport in magnetised plasmas
- Ion gyroradius scale instabilities tend to generate zonal flows
- These flows are defined as $k_\alpha = 0$
- Can shear apart turbulent eddies and reduce saturation levels
- Many existing theories describe zonal flows in terms of secondary instabilities



Motivation

- **Zonal flows** (turbulence-generated): Self-organised, large-scale shear flows that regulate turbulent transport in magnetised plasmas
- Ion gyroradius scale instabilities tend to generate zonal flows
- These flows are defined as $k_\alpha = 0$
- Can shear apart turbulent eddies and reduce saturation levels
- Many existing theories describe zonal flows in terms of secondary instabilities
- Saturation region important for confinement $\rightarrow$ early times may be illuminating



Gyrokinetic Equation

$$\underbrace{\left(\partial_t + v_\parallel \nabla_\parallel + i\left(k_\psi \nabla\psi + k_\alpha \nabla\alpha\right)\cdot \mathbf{v}_d\right)}_{\mathcal{L}_{\boldsymbol{k}}} h_{s,\boldsymbol{k}} =$$

$$\underbrace{\left(\partial_t + ic\frac{k_\alpha T_{0s}}{q_s}\frac{F'_{0s}}{F_{0s}}\right)}_{\mathcal{R}_{\boldsymbol{k}}} \frac{q_s F_{0s}}{T_{0s}} J_{0s,\boldsymbol{k}}\phi_{\boldsymbol{k}} - \underbrace{c\sum_{\boldsymbol{k}'}\left(k_\psi k'_\alpha - k_\alpha k'_\psi\right) J_{0s,\boldsymbol{k}}\phi_{\boldsymbol{k}-\boldsymbol{k}'} h_{s,\boldsymbol{k}'}}_{\mathcal{N}_{\boldsymbol{k}}}$$

- $\mathcal{L}_{\boldsymbol{k}}$, $\mathcal{R}_{\boldsymbol{k}}$: **Linear** dynamics
- $\mathcal{N}_{\boldsymbol{k}}$: **Nonlinear** mode coupling

Gyrokinetic Equation

$$\underbrace{\left(\partial_t + v_\parallel \nabla_\parallel + i\left(k_\psi \nabla\psi + k_\alpha \nabla\alpha\right)\cdot \mathbf{v}_d\right)}_{\mathcal{L}_{\boldsymbol{k}}} h_{s,\boldsymbol{k}} =$$

$$\underbrace{\left(\partial_t + ic\frac{k_\alpha T_{0s}}{q_s}\frac{F_{0s}'}{F_{0s}}\right)}_{\mathcal{R}_{\boldsymbol{k}}} \frac{q_s F_{0s}}{T_{0s}} J_{0s,\boldsymbol{k}}\phi_{\boldsymbol{k}} - \underbrace{c\sum_{\boldsymbol{k}'}\left(k_\psi k_\alpha' - k_\alpha k_\psi'\right) J_{0s,\boldsymbol{k}}\phi_{\boldsymbol{k}-\boldsymbol{k}'} h_{s,\boldsymbol{k}'}}_{\mathcal{N}_{\boldsymbol{k}}}$$

- $\mathcal{L}_{\boldsymbol{k}}$, $\mathcal{R}_{\boldsymbol{k}}$: **Linear** dynamics
- $\mathcal{N}_{\boldsymbol{k}}$: **Nonlinear** mode coupling

$$\frac{e^2 n}{T_{0,i}}(1+\tau)\phi_{\boldsymbol{k}} = \sum_s \frac{e}{n_{0,s}} \int \mathrm{d}^3\boldsymbol{v}\; h_{s,\boldsymbol{k}} J_{0,\boldsymbol{k}}$$

- Quasineutrality to close the system

Gyrokinetic Equation

$$\mathcal{L}_{\boldsymbol{k}} h_{s,\boldsymbol{k}} = \mathcal{R}_{\boldsymbol{k}} \frac{q_s F_{0s}}{T_{0s}} J_{0s,\boldsymbol{k}} \phi_{\boldsymbol{k}} - \mathcal{N}_{\boldsymbol{k}}$$

- $\mathcal{L}_{\boldsymbol{k}}$, $\mathcal{R}_{\boldsymbol{k}}$: **Linear** dynamics
- $\mathcal{N}_{\boldsymbol{k}}$: **Nonlinear** mode coupling

$$\phi_{\boldsymbol{k}} = \hat{Q}[h_{s,\boldsymbol{k}}]$$

- Quasineutrality to close the system

Gyrokinetic Equation

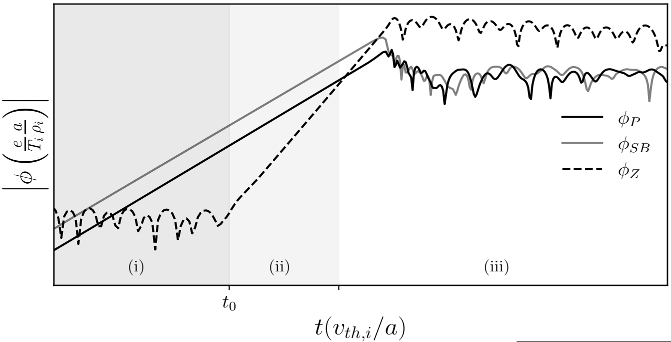
$$\mathcal{L}_{\boldsymbol{k}} h_{s,\boldsymbol{k}} = \mathcal{R}_{\boldsymbol{k}} \frac{q_s F_{0s}}{T_{0s}} J_{0s,\boldsymbol{k}} \phi_{\boldsymbol{k}} - \mathcal{N}_{\boldsymbol{k}}$$

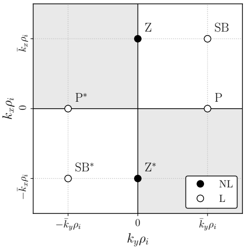
- $\mathcal{L}_{\boldsymbol{k}}$, $\mathcal{R}_{\boldsymbol{k}}$: **Linear** dynamics
- $\mathcal{N}_{\boldsymbol{k}}$: **Nonlinear** mode coupling

$$\phi_{\boldsymbol{k}} = \hat{Q}[h_{s,\boldsymbol{k}}]$$

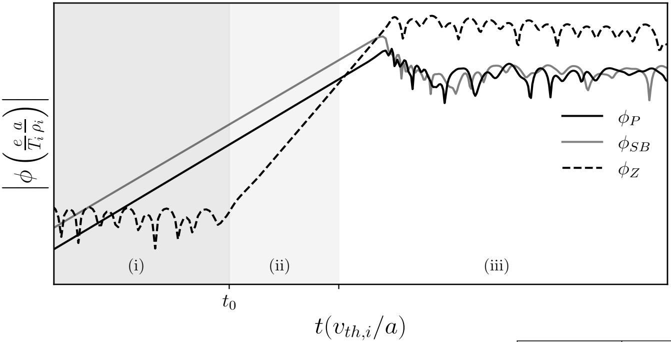
- Quasineutrality to close the system
- Modified Boltzmann response
- Flux-tube CBC geometry

Regime of Interest





Regime of Interest

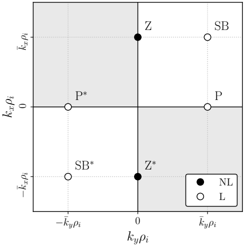


Three distinct dynamical regimes:

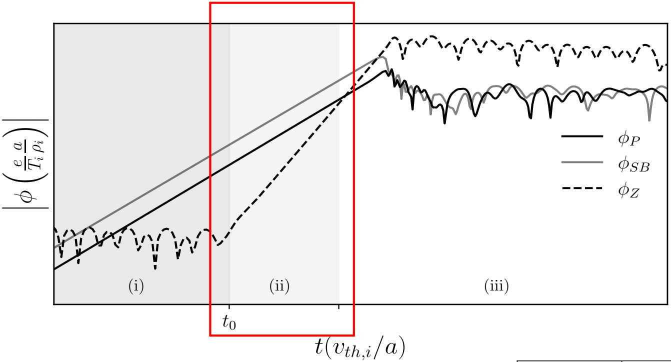
i) **Linear:** Independent mode growth

ii) **Weakly nonlinear:** Selective mode coupling

iii) **Strongly/Fully nonlinear**



Regime of Interest

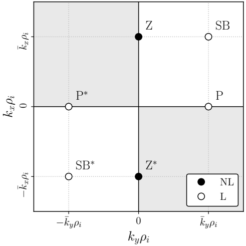


Three distinct dynamical regimes:

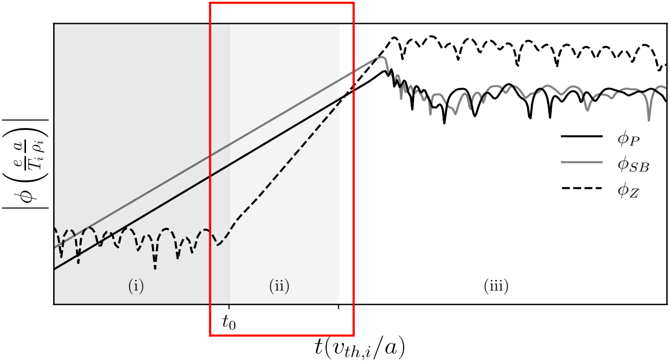
i) **Linear:** Independent mode growth

ii) **Weakly nonlinear:** Selective mode coupling

iii) **Strongly/Fully nonlinear**



Regime of Interest



$$\omega_Z^{\mathrm{lin}}\phi_Z \ll \mathcal{N}\phi_P\phi_{SB}, \quad \omega_P^{\mathrm{lin}}\phi_P \gg \mathcal{N}\phi_{SB}\phi_Z, \quad \omega_{SB}^{\mathrm{lin}}\phi_{SB} \gg \mathcal{N}\phi_P\phi_Z$$

Goal of This Work

- Zonal modes can be born via **nonlinear driving** by linearly unstable primary modes, not only through a secondary instability
- **Frequency-matching** selection rule of unstable non-zonal modes governs the growth rate of zonal modes
- Phase-space structure of driven zonal flow, using “natural” initial condition $\Rightarrow$ **residual significantly higher** than classical Rosenbluth–Hinton prediction

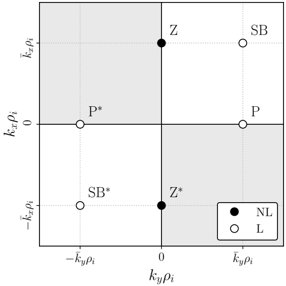
Table of Contents

Triadic Model

- Retain only the modes needed to capture zonal-mode dynamics:

$$\mathcal{L}_{k_\psi k_\alpha} h_{k_\psi k_\alpha} = \mathcal{R}_{k_\psi k_\alpha} \frac{qF_0}{T_0} J_{0,\boldsymbol{k}} \phi_{k_\psi k_\alpha} \qquad \text{for} \quad (k_\psi, k_\alpha) = \begin{cases} (0, \bar{k}_\alpha) \\ (\bar{k}_\psi, \bar{k}_\alpha) \end{cases}$$

$$\mathcal{L}_{k_\psi k_\alpha} h_{k_\psi k_\alpha} = \mathcal{R}_{k_\psi k_\alpha} \frac{qF_0}{T_0} J_{0,\boldsymbol{k}} \phi_{k_\psi k_\alpha} - \mathcal{N}_{k_\psi k_\alpha} \qquad \text{for} \quad (k_\psi, k_\alpha) = (\bar{k}_\psi, 0)$$



- Number of modes is regime-dependent
  → Triadic model suffices in weakly nonlinear phase

Weakly Nonlinear Regime

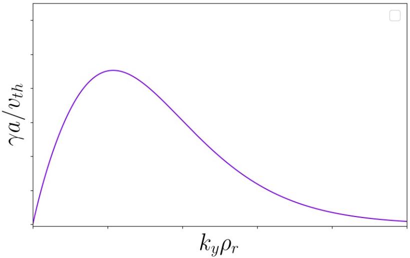
- Laplace/Fourier analysis in time
  $\rightarrow$ Non-zonal dynamics given by their linear equations
  $\rightarrow$ Zonal mode driven by its nonlinearity
  $\rightarrow$ Zonal response determined and can be analytically calculated

Weakly Nonlinear Regime

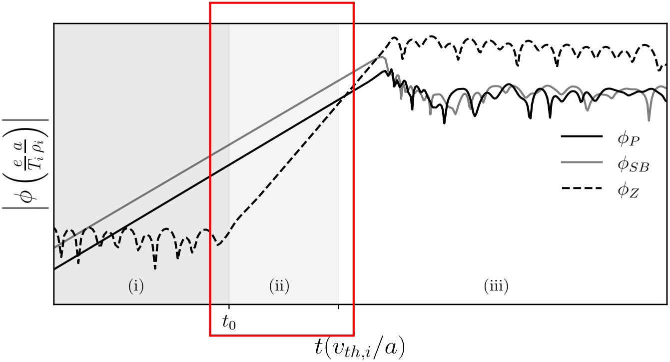
- Laplace/Fourier analysis in time
  $\rightarrow$ Non-zonal dynamics given by their linear equations
  $\rightarrow$ Zonal mode driven by its nonlinearity
  $\rightarrow$ Zonal response determined and can be analytically calculated
- Result yields zonal amplitude and enforces frequency matching
- No dispersion relation $\Rightarrow$ no marginal point
- Always there, even if subdominant

Weakly Nonlinear Regime

- Laplace/Fourier analysis in time
  $\rightarrow$ Non-zonal dynamics given by their linear equations
  $\rightarrow$ Zonal mode driven by its nonlinearity
  $\rightarrow$ Zonal response determined and can be analytically calculated
- Result yields zonal amplitude and enforces frequency matching
- No dispersion relation $\Rightarrow$ no marginal point
- Always there, even if subdominant
- To carry out numerical experiments $\Rightarrow$ need systematic process to find triad
  $\rightarrow$ For any given k_x plot k_y spectrum
  $\rightarrow$ Identify triad using peak in k_y

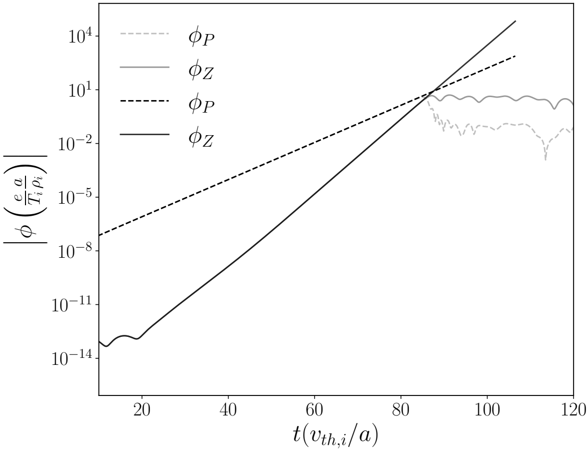


Validation of Triadic Model in Weakly Nonlinear Regime



- The reduced model retains only the nonlinearity on the zonal modes
- How reasonable is our model choice?

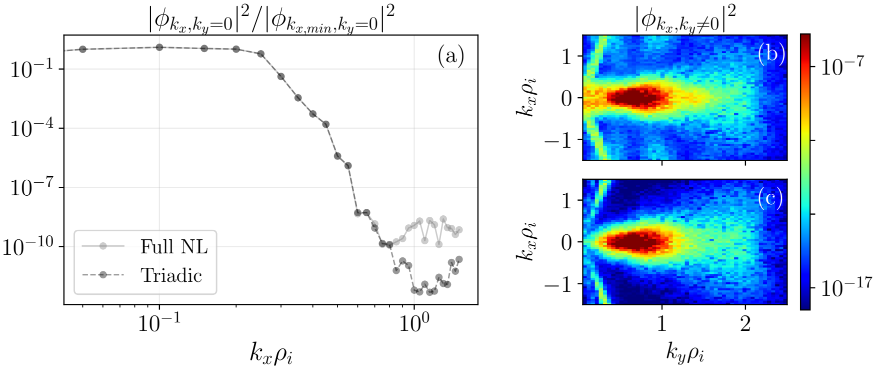
Validation of Triadic Model in Weakly Nonlinear Regime



- Light grey: Full GK → All modes nonlinear
- Black: Reduced Model → Zonal = Nonlinear; Non-zonal = Linear

Validation of Triadic Model in Weakly Nonlinear Regime

- Reproduces the ϕ^2 spectra of the full system



Left: Zonal spectra

Right: Non-zonal spectra
b) Full GK c) Model

Validation of Triadic Model in Weakly Nonlinear Regime

- Reproduces the ϕ^2 spectra of the full system

Left: Zonal spectra

Right: Non-zonal spectra
b) Full GK c) Model

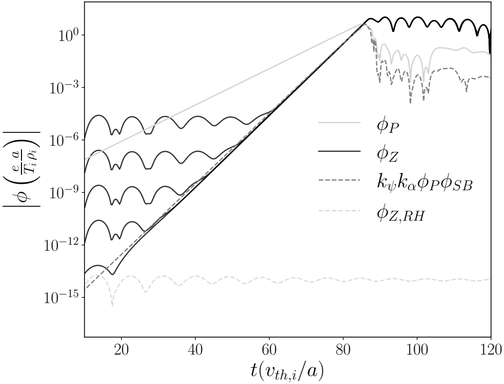
- How well does this “driven” concept hold?

Weakly Nonlinear Regime: Numerical Experiment 1

- Evolution of the zonal potential for varying initial amplitudes
- Non-zonal modes are initialised identically; only zonal amplitude is varied

Weakly Nonlinear Regime: Numerical Experiment 1

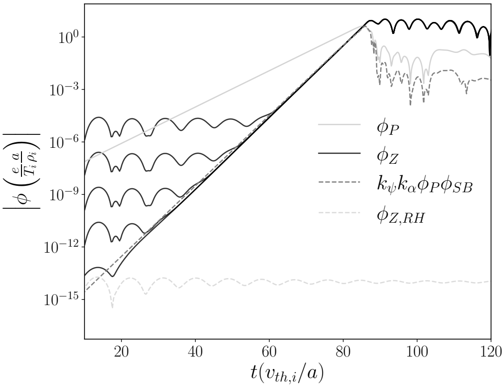
- Evolution of the zonal potential for varying initial amplitudes
- Non-zonal modes are initialised identically; only zonal amplitude is varied



- The dashed curve is a proxy for the nonlinear drive $\propto k_\psi k_\alpha \, \phi_{k_\psi, k_\alpha} \phi_{k_\alpha}$

Weakly Nonlinear Regime: Numerical Experiment 1

- Evolution of the zonal potential for varying initial amplitudes
- Non-zonal modes are initialised identically; only zonal amplitude is varied



- The dashed curve is a proxy for the nonlinear drive $\propto k_\psi k_\alpha \, \phi_{k_\psi, k_\alpha} \phi_{k_\alpha}$
- Zonal growth “begins” when the nonlinearly driven mode becomes dominant

Weakly Nonlinear Regime: Numerical Experiment 2

- Analysis predicts frequency matching in the weakly nonlinear regime:

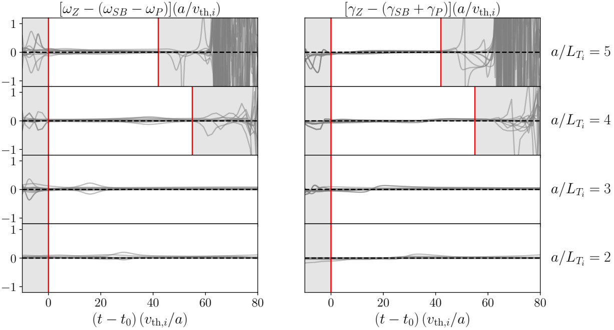
$$\boxed{\omega = \omega_{k_\psi, k_\alpha} - \omega^*_{k_\alpha}}$$

NB: Zonal growth is largest for small k_ψ
Also as $k_\psi \to 0$, then $\gamma_{k_\psi} \to 2\gamma_{k_\alpha}$

Weakly Nonlinear Regime: Numerical Experiment 2

- Analysis predicts frequency matching in the weakly nonlinear regime:

$$\boxed{\omega = \omega_{k_\psi,k_\alpha} - \omega^*_{k_\alpha}}$$



Left: Frequency $\omega_{k_\psi} - (\omega_{k_\psi k_\alpha} - \omega_{k_\alpha})$ *Right:* Growth rate $\gamma_{k_\psi} - (\gamma_{k_\psi k_\alpha} + \gamma_{k_\alpha})$.

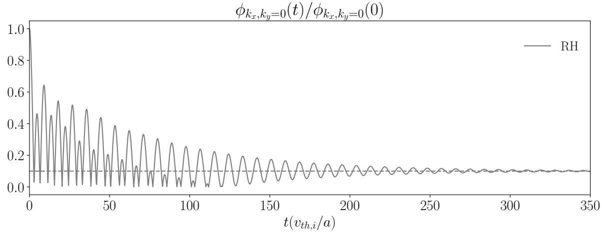
- Both the zonal frequency and growth rate are captured by the convolution model across temperature gradients for multiple k_ψ-modes

Table of Contents

Introduction to the Residual

Based on linear physics, Rosenbluth & Hinton (1998):

$$\mathcal{R} = \frac{\langle \phi_\infty \rangle_\psi}{\langle \phi_0 \rangle_\psi},$$

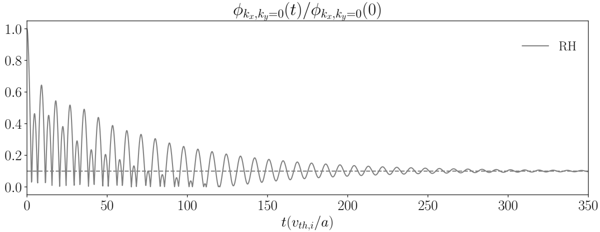


A measure of how **resilient** the zonal mode is in saturation

Introduction to the Residual

Based on linear physics, Rosenbluth & Hinton (1998):

$$\mathcal{R} = \frac{\langle \phi_\infty \rangle_\psi}{\langle \phi_0 \rangle_\psi},$$

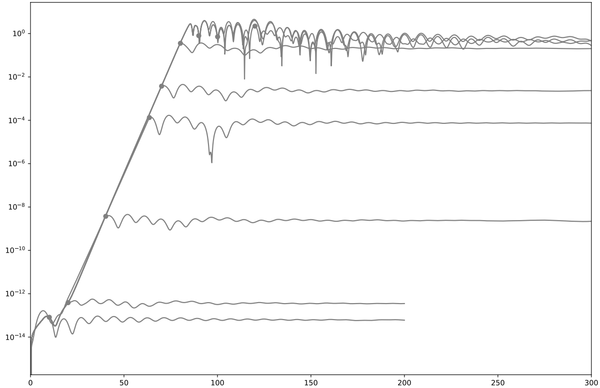


$$\langle \phi_\infty \rangle_\psi = \frac{\frac{1}{n}\left\langle \int \mathrm{d}^3\mathbf{v}\, J_{0,k_\psi}\, e^{-iQ}\overline{e^{iQ}\langle \delta f_{k_\psi,t=0}\rangle_{\mathbf{R}}}\right\rangle_\psi}{1 - \frac{1}{n}\left\langle \int \mathrm{d}^3\mathbf{v}\, J_{0,k_\psi}\, e^{-iQ}\overline{J_{0,k_\psi} e^{iQ}}\, F_0\right\rangle_\psi}$$

$$\langle \phi_0 \rangle_\psi = \frac{1}{1-\Gamma_0}\frac{1}{n}\left\langle \int \mathrm{d}^3\mathbf{v}\, J_{0,k_\psi}\langle \delta f_{k_\psi,t=0}\rangle_{\mathbf{R}}\right\rangle_\psi$$

Introduction to the Residual

- Residual is built up in the "2γ-phase" $\rightarrow$ Ratio of final to initial ϕ is $\sim$ constant (see Richard's talk from Monday)

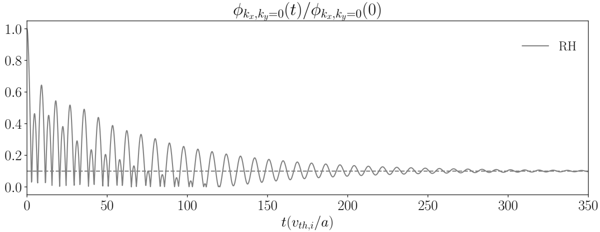


- Natural "reservoir" of energy that is fed in this region and not damped away
- Persistent zonal component $\rightarrow$ may play a role in saturation?

Introduction to the Residual

Based on linear physics, Rosenbluth & Hinton (1998):

$$\mathcal{R} = \frac{\langle \phi_\infty \rangle_\psi}{\langle \phi_0 \rangle_\psi},$$



Importantly this is usually done with a pure-density initial condition:

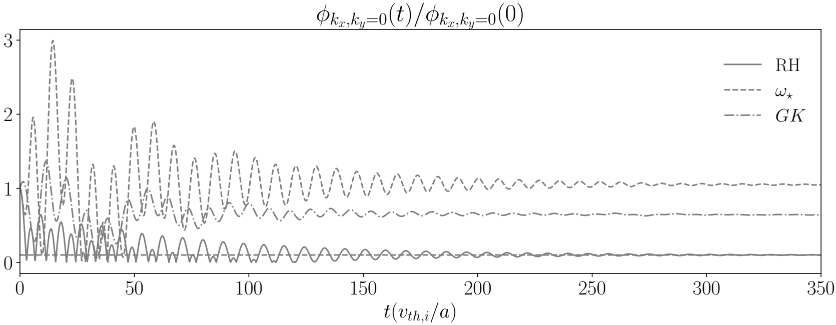
$$\langle \delta f_{k_\psi} \rangle_{\mathbf{R}} \sim k_\perp^2 F_0$$

Weakly Nonlinear Residual: Phase-space Structure

- Find that the residual is affected by the phase-space structure of the distribution function at $t = 0$

Weakly Nonlinear Residual: Phase-space Structure

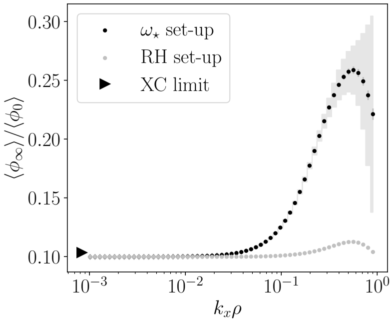
- Find that the residual is affected by the phase-space structure of the distribution function at $t = 0$

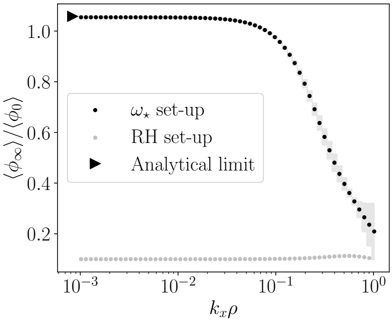


- RH initialisation: $\langle \delta f_{k_\psi} \rangle_{\mathbf{R}} = k_\perp^2 F_0$
- ω_* initialisation: $\langle \delta f_{k_\psi} \rangle_{\mathbf{R}} = \omega_{*,k_\alpha} \left(\hat{v}^2 - 3/2 \right) e^{z^2} F_0$
- GK initialisation: Restart using nonlinear GK simulation from 2γ-region

Weakly Nonlinear Residual: Phase-space Structure

- Find that the residual is affected by the phase-space structure of the distribution function at $t = 0$





$$\langle \delta f_{k_\psi} \rangle_{\mathbf{R}} = J_{0,k_\alpha} J_{0,k_\psi k_\alpha} \omega_{*,k_\alpha} \left(\hat{v}^2 - 3/2\right) e^{z^2} F_0$$

$$\langle \delta f_{k_\psi} \rangle_{\mathbf{R}} = \omega_{*,k_\alpha} \left(\hat{v}^2 - 3/2\right) e^{z^2} F_0$$

- RH acts as lower bound, and $\omega_{*,k_\alpha} \left(\hat{v}^2 - 3/2\right) e^{z^2} F_0$ as upper bound
- Different phase-space initialisations yield different residual values and k_ψ-spectra

Weakly Nonlinear Residual: Spectrum

- Non-zonal modes obey linear dynamics:

$$h_{\boldsymbol{k}} \approx J_{0,\boldsymbol{k}} \, \frac{q\phi_{\boldsymbol{k}}}{T} \, \frac{\omega_{\boldsymbol{k}} - \hat{\omega}_{*,\boldsymbol{k}}}{\omega_{\boldsymbol{k}} - \hat{\omega}_{d,\boldsymbol{k}}} \, F_0,$$

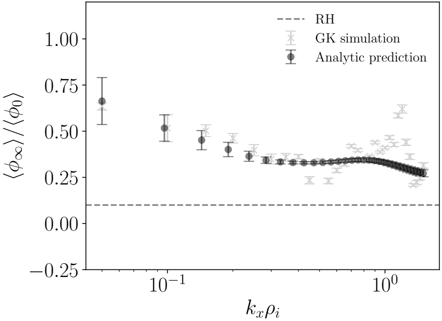
  and the zonal mode is **driven** by them
- This fixes the **phase-space structure** of the zonal flow, not just the amplitude

Weakly Nonlinear Residual: Spectrum

- Non-zonal modes obey linear dynamics:

$$h_{\boldsymbol{k}} \approx J_{0,\boldsymbol{k}} \frac{q\phi_{\boldsymbol{k}}}{T} \frac{\omega_{\boldsymbol{k}} - \hat{\omega}_{*,\boldsymbol{k}}}{\omega_{\boldsymbol{k}} - \hat{\omega}_{d,\boldsymbol{k}}} F_0,$$

  and the zonal mode is **driven** by them
- This fixes the **phase-space structure** of the zonal flow, not just the amplitude



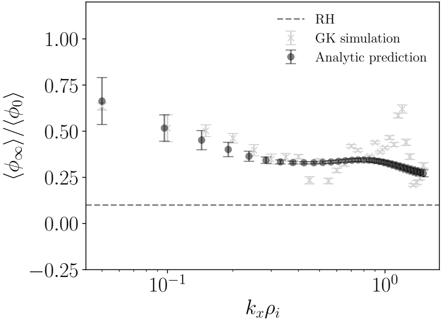
Weakly Nonlinear Residual: Spectrum

- Non-zonal modes obey linear dynamics:

$$h_{\boldsymbol{k}} \approx J_{0,\boldsymbol{k}} \frac{q\phi_{\boldsymbol{k}}}{T} \frac{\omega_{\boldsymbol{k}} - \hat{\omega}_{*,\boldsymbol{k}}}{\omega_{\boldsymbol{k}} - \hat{\omega}_{d,\boldsymbol{k}}} F_0,$$

  and the zonal mode is **driven** by them

- This fixes the **phase-space structure** of the zonal flow, not just the amplitude



- Spectrum is very sensitive to initial condition → is this useful?

Summary and Future Work

Summary

- Zonal-modes can be **driven by nonlinear forcing** from unstable primary modes (always present, even if subdominant)
- A **minimal three-wave model** captures the weakly nonlinear dynamics
- The “natural” phase-space initial condition yields a **residual that exceeds** the classical Rosenbluth–Hinton level
- Can **predict k_ψ spectrum** of residual

Summary and Future Work

Summary

- Zonal-modes can be **driven by nonlinear forcing** from unstable primary modes (always present, even if subdominant)
- A **minimal three-wave model** captures the weakly nonlinear dynamics
- The “natural” phase-space initial condition yields a **residual that exceeds** the classical Rosenbluth–Hinton level
- Can **predict k_ψ spectrum** of residual

Future Work

- Investigate how/if the residual is important in fully developed turbulence
- Investigate stellarator configurations → Importance of $k_\parallel$
- Relax the modified-Boltzmann electron response
- Extend minimal model to fully developed turbulence

Table of Contents

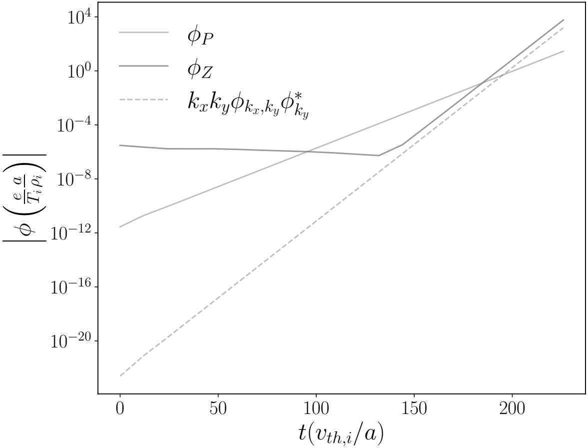
Ongoing Work

"This is work in progress. Not really much work, and definitely not much progress."

— Ben Chandran
(yesterday)

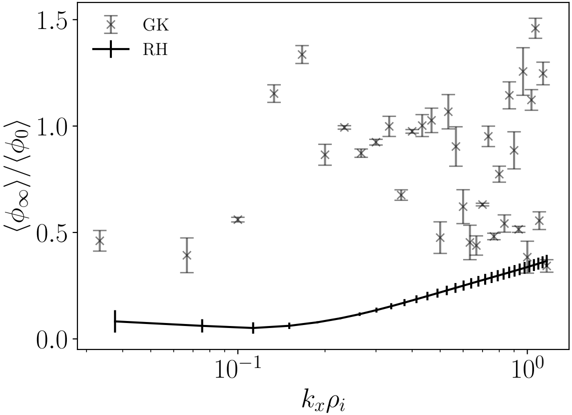
Ongoing Work – 1. Residuals in W7-X

- Similar story in W7-X geometry → parallel dependence has little impact on triadic model



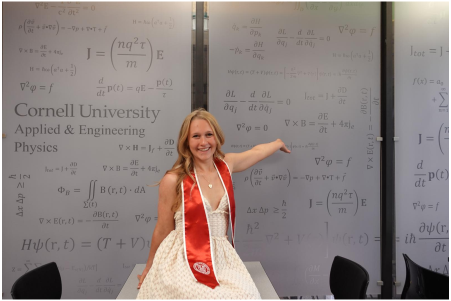
Ongoing Work – 1. Residuals in W7-X

- Similar story in W7-X geometry → parallel dependence has little impact on triadic model
- Residual spectrum is similar and we see large $\delta u_\parallel$ and δT moments



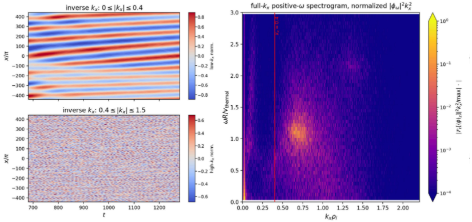
Ongoing Work – 2. TSM in W7-X

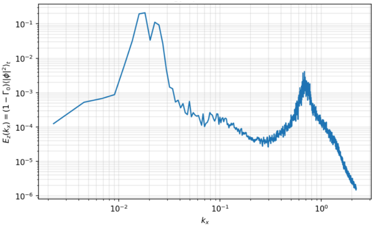
- Work with Sophia Arnold: Master's (soon Ph.D.) student at Greifswald



Ongoing Work – 2. TSM in W7-X

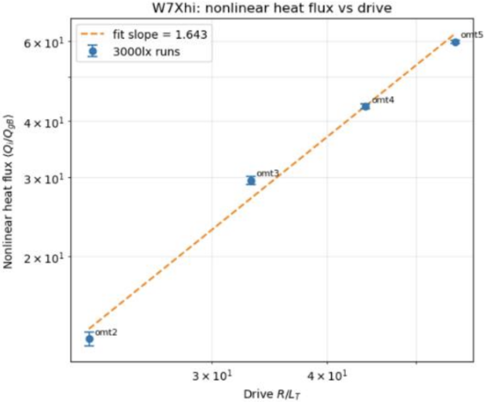
- TSM has been observed in W7-X when strongly driven
  (see work by Richard: Saturation of magnetized plasma turbulence by propagating zonal flows)





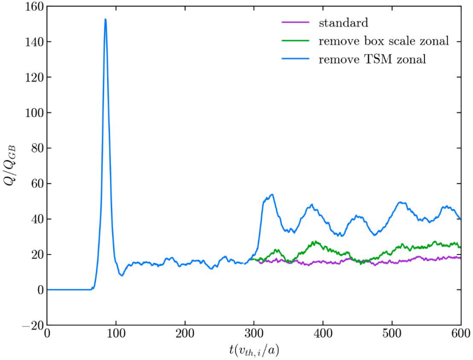
Ongoing Work – 2. TSM in W7-X

- TSM has been observed in W7-X when strongly driven (see work by Richard: Saturation of magnetized plasma turbulence by propagating zonal flows)
- Heat flux in W7-X is shown to have a linear scaling with temperature gradient, following "grand critical balance" predicted by Richard



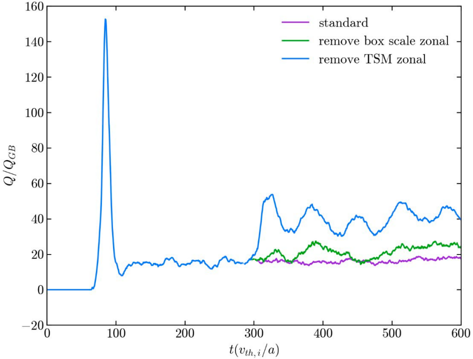
Ongoing Work – 3. How Far Can We Push It?

- In the strongly driven regime?



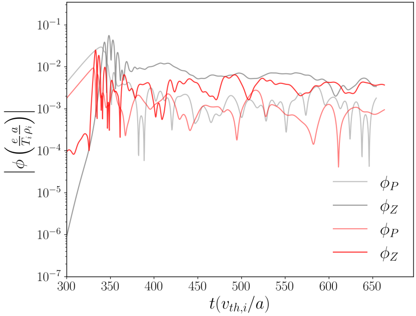
Ongoing Work – 3. How Far Can We Push It?

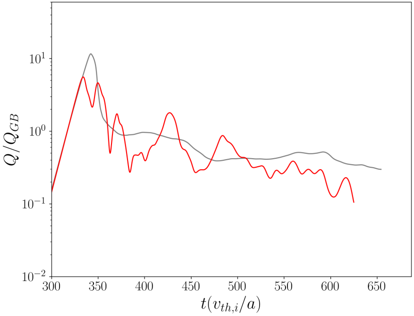
- In the strongly driven regime?
- Not surprising? Large-scale zonal becomes decoupled. Need to resolve the TSM



Ongoing Work – 3. How Far Can We Push It?

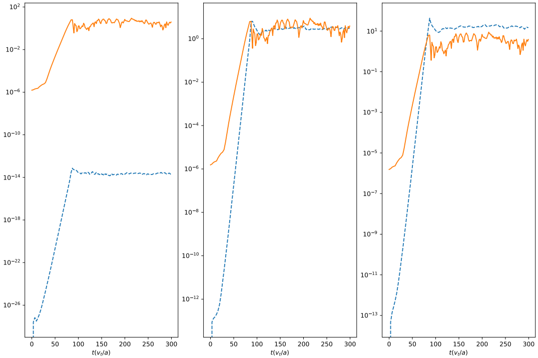
- In the strongly driven regime?
- Not surprising? Large-scale zonal becomes decoupled. Need to resolve the TSM
- Near marginality, potentially?
- Minimal model "blows up" around `tprim` where TSM would kick in



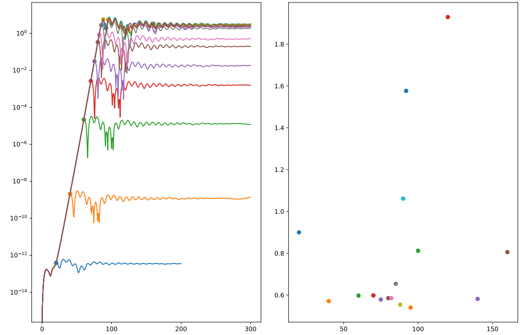


Thanks for listening

δn, $\delta u_\parallel$, δT



Residual fraction vs. time



Sensitivity of Residual