Optimizing stellarators for improved energetic particle stability and confinement

Elizabeth Paul

July 30, 2025

… & my thoughts on AI/ML for device design

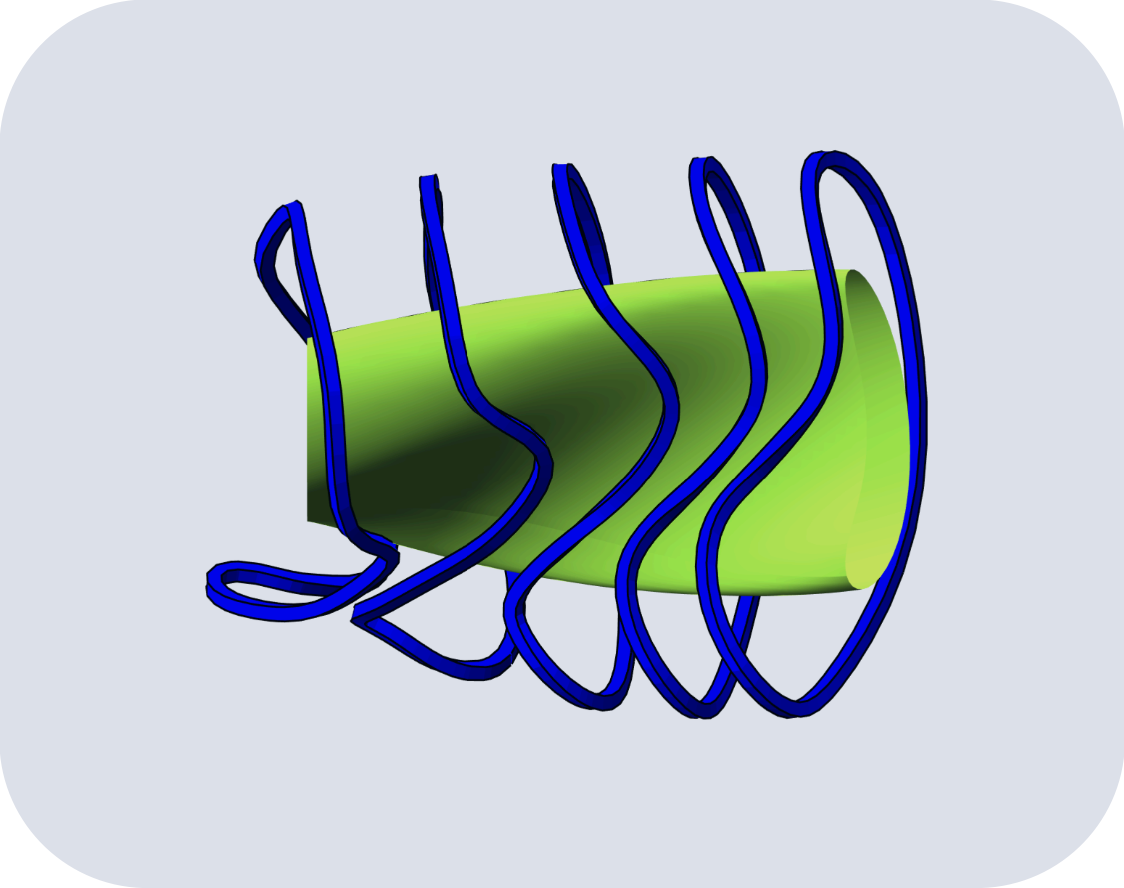
*Acknowledgements: A. Hyder, A. Knyazev, R. Jorge,
E. Rodriguez, D. Spong, H. Yoon*

E. J. Paul et al, *J. Plasma Phys.*, 91 (2025). https://doi.org/10.1017/S0022377825100524

16th Plasma Kinetics Working Group Meeting

Outline

- ***Stellarator optimization and AI/ML***
- *Perturbation theory for shear Alfvén continuum*
- *Can we manipulate the shear Alfvén continuum to avoid resonance?*



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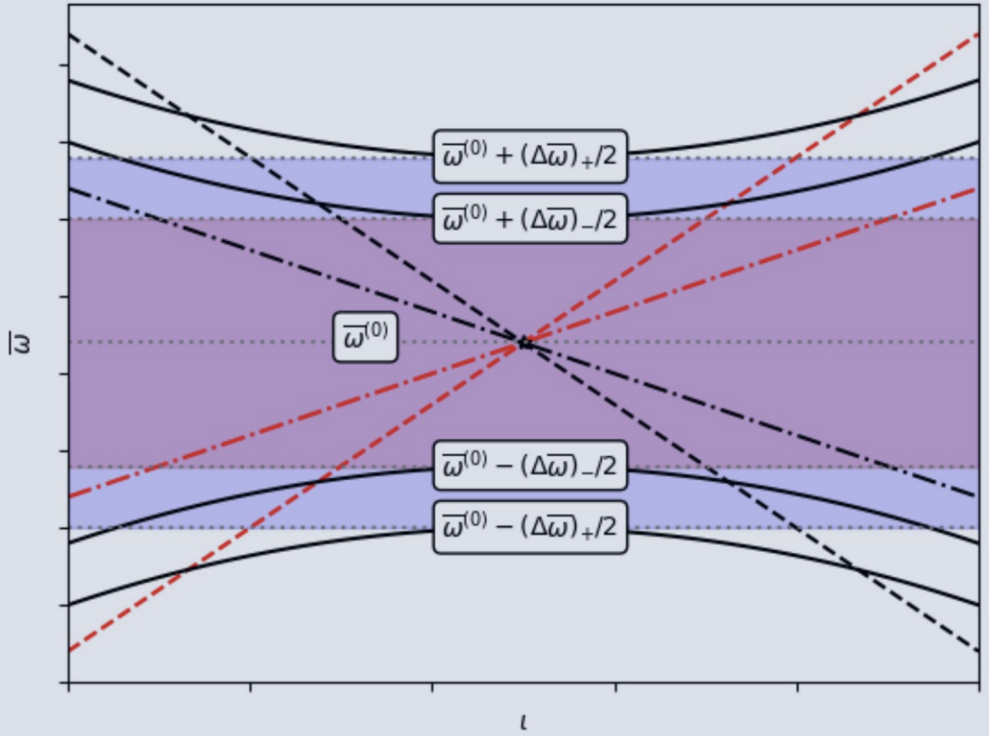

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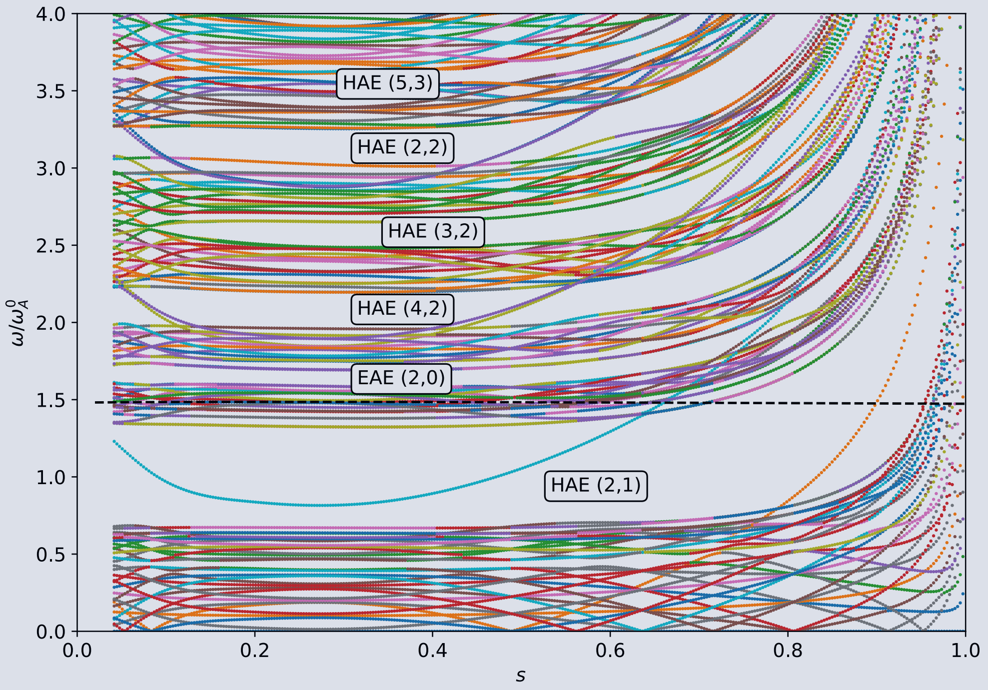
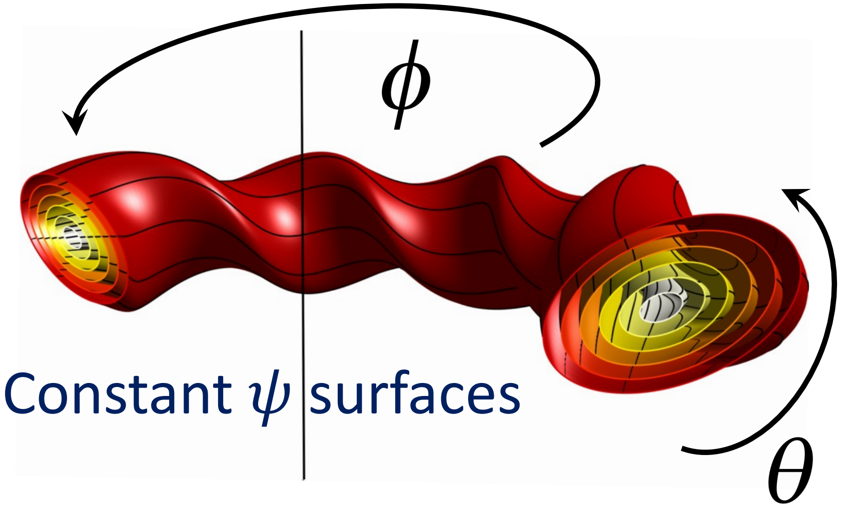


Symmetry of field strength yields particle confinement in 3D

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = m\frac{|\dot{\boldsymbol{x}}|^2}{2} + q\boldsymbol{A}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}$$

$$\Big\downarrow \text{ Guiding center}$$

$$\mathcal{L}(\psi, \theta, \phi, \dot{\psi}, \dot{\theta}, \dot{\phi}) = \mathcal{L}(\psi, B(\psi, \theta, \phi), \dot{\psi}, \dot{\theta}, \dot{\phi})$$



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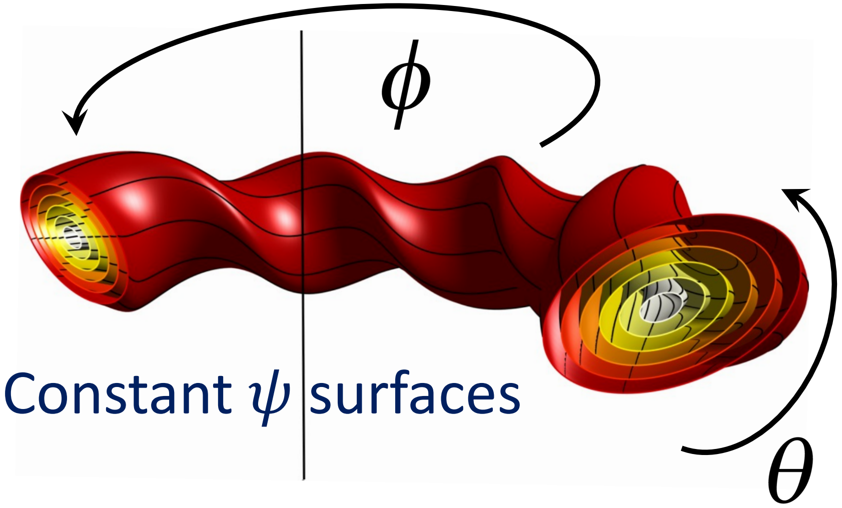

$$\frac{\partial B(\psi, \theta, \phi)}{\partial \phi} = 0 \longrightarrow \frac{dp_\phi}{dt} = 0$$

Ignorable coordinate

Conserved momentum

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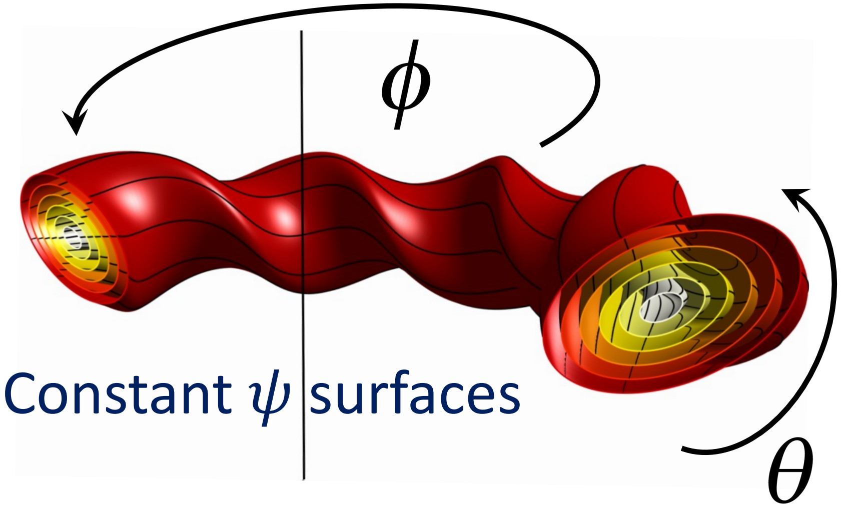
$$\Big\downarrow$$

Guiding center

$$\mathcal{L}(\psi,\theta,\phi,\dot{\psi},\dot{\theta},\dot{\phi}) = \mathcal{L}(\psi,{\color{red}B(\psi,\theta,\phi)},\dot{\psi},\dot{\theta},\dot{\phi})$$



Constant ψ surfaces

$$\frac{\partial B(\psi,\theta,\phi)}{\partial\phi} = 0 \longrightarrow \frac{dp_\phi}{dt} = 0$$

Ignorable coordinate

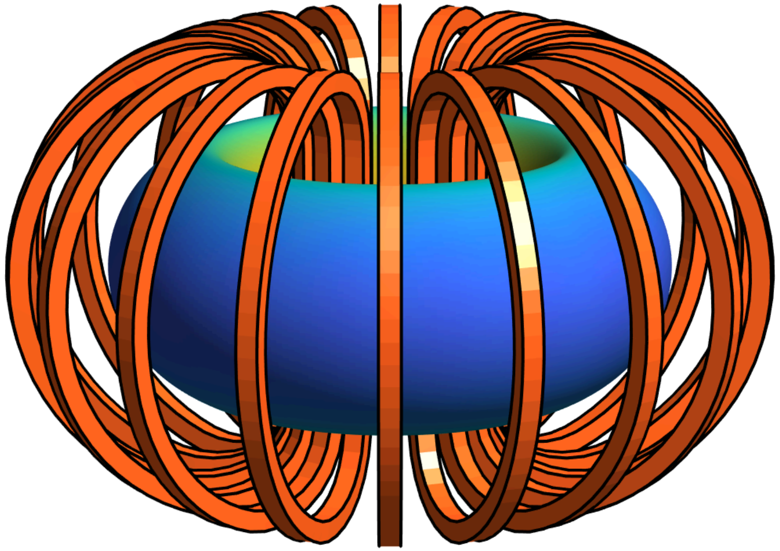
Conserved momentum

$$p_\phi = \cancel{\frac{mv_\phi}{qB}} + \underbrace{F(\psi)}$$

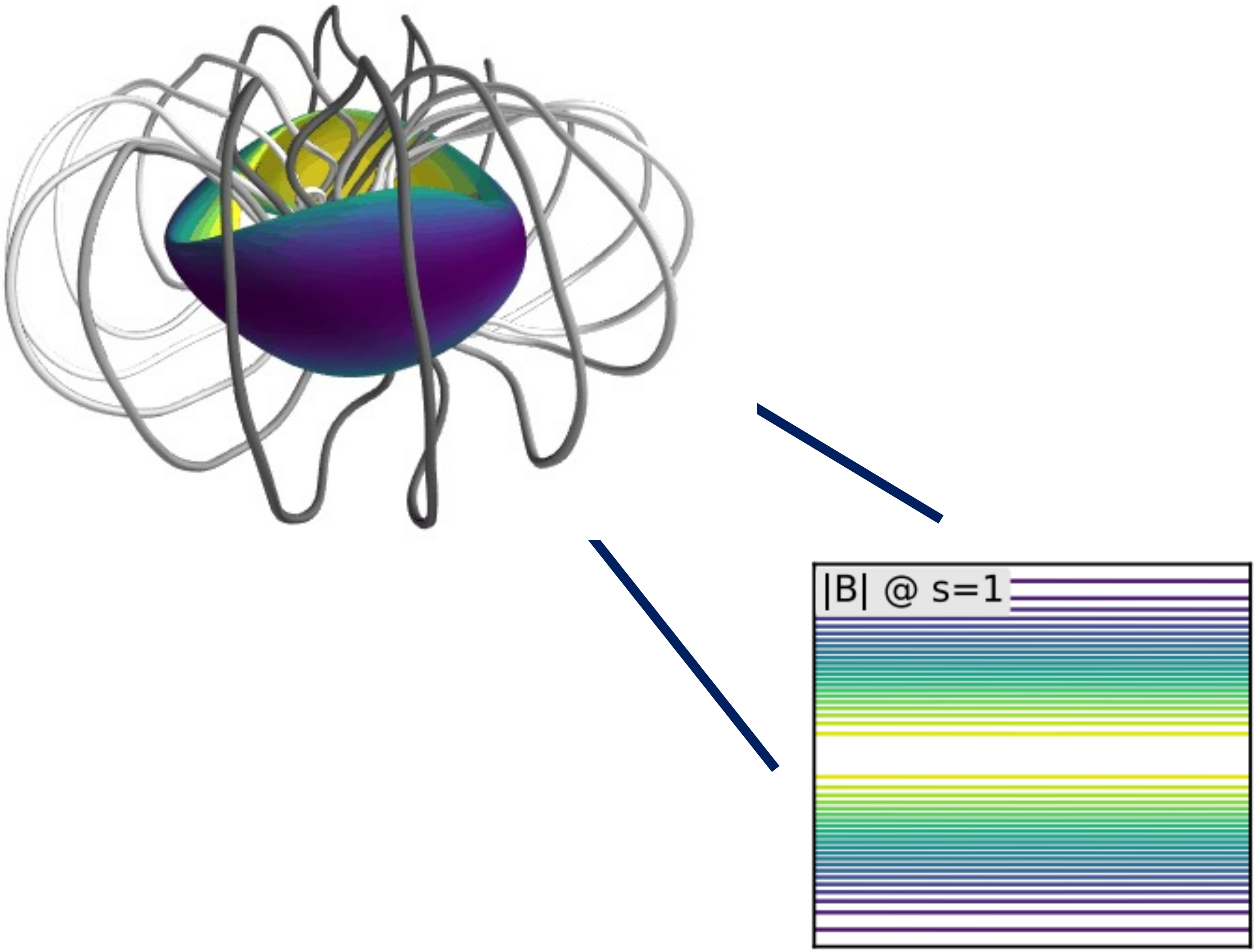
Particles stay confined to ψ surfaces

Quasisymmetry - a hidden symmetry of magnetic fields

Symmetric



Quasisymmetric



Ingredients of stellarator confinement

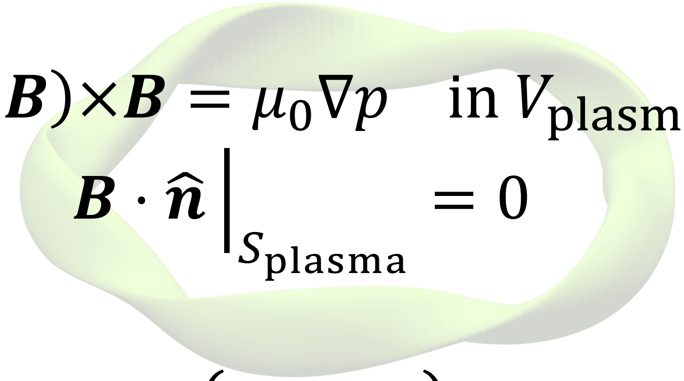
✓Integrable magnetic field
✓Energetic particle confinement
✓MHD stability
✓Collisional “bootstrap” current
✓Equilibrium β limit
✓Divertor configuration
✓Reduction of turbulent transport
✓Coil feasibility

ASG Superconductors

Traditional two-step optimization

1. MHD equilibrium optimization

Boundary of equilibrium optimized for confinement



$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \mu_0\nabla p \quad \text{in } V_{\text{plasma}}$$

$$\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\Big|_{S_{\text{plasma}}} = 0$$

$$\min_{S_{\text{plasma}}} f\left(\boldsymbol{B}\left(S_{\text{plasma}}\right), S_{\text{plasma}}\right)$$

Traditional two-step optimization

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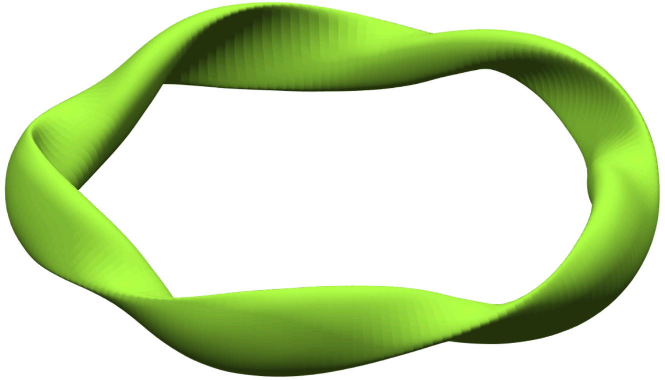
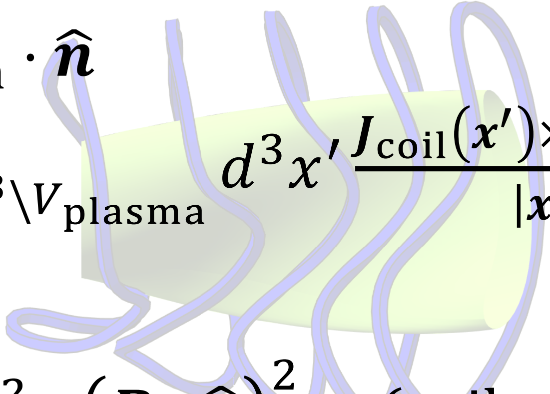


2. Coil design

Inverse problem solved to find coils to support equilibrium



$$\boldsymbol{B}\cdot\hat{\boldsymbol{n}} = \boldsymbol{B}_{\text{plasma}}\cdot\hat{\boldsymbol{n}} + \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3\setminus V_{\text{plasma}}} d^3x' \frac{\boldsymbol{J}_{\text{coil}}(\boldsymbol{x}')\times(\boldsymbol{x}-\boldsymbol{x}')\cdot\hat{\boldsymbol{n}}(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

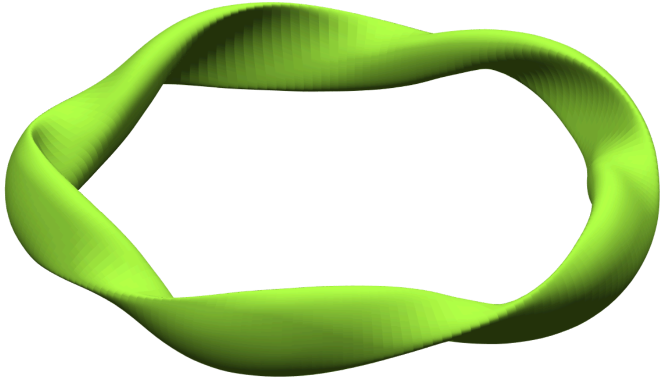
$$\min_{\boldsymbol{J}_{\text{coil}}}\left(\int_{S_{\text{plasma}}} d^2x\,\left(\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\right)^2 + (\text{coil complexity})\right)$$

Traditional two-step optimization

1. MHD equilibrium optimization

Boundary of equilibrium optimized for confinement

$(\nabla\times$



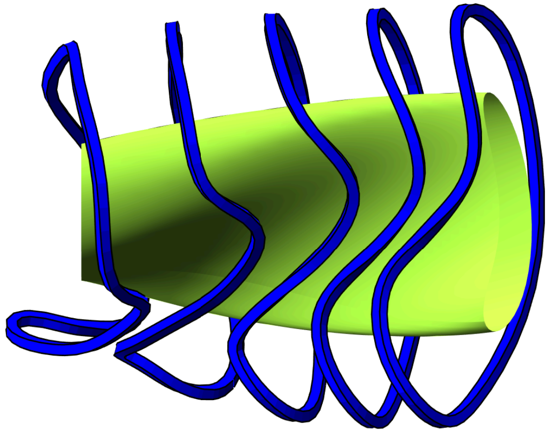
$\min_{S_{\text{pla}}}$

2. Coil design

Inverse problem solved to find coils to support equilibrium

$\boldsymbol{B}\cdot\hat{\boldsymbol{n}} = \boldsymbol{B}_{\text{plasma}}$

$+\frac{\mu_0}{4\pi}\int_{\mathbb{R}}$



$\frac{(\boldsymbol{x}-\boldsymbol{x}')\cdot\hat{\boldsymbol{n}}(\boldsymbol{x})}{-\boldsymbol{x}'|^3}$

$\min_{J_{\text{coil}}}\left(\int_{S_{\text{plasma}}} d \quad \text{omplexity})\right)$

Stellarator optimization at scale – Wendelstein 7-X

nature

Article

Demonstration of reduced neoclassical energy transport in Wendelstein 7-X

https://doi.org/10.1038/s41586-021-03687-w

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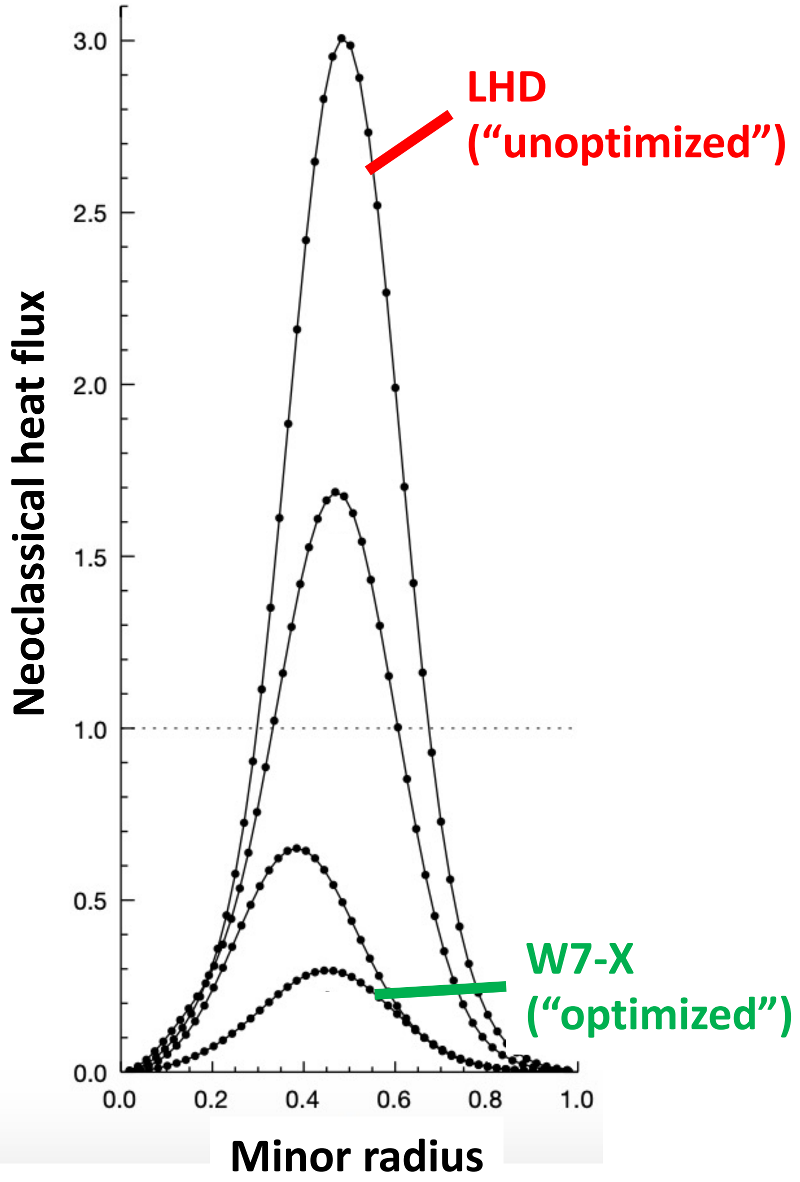
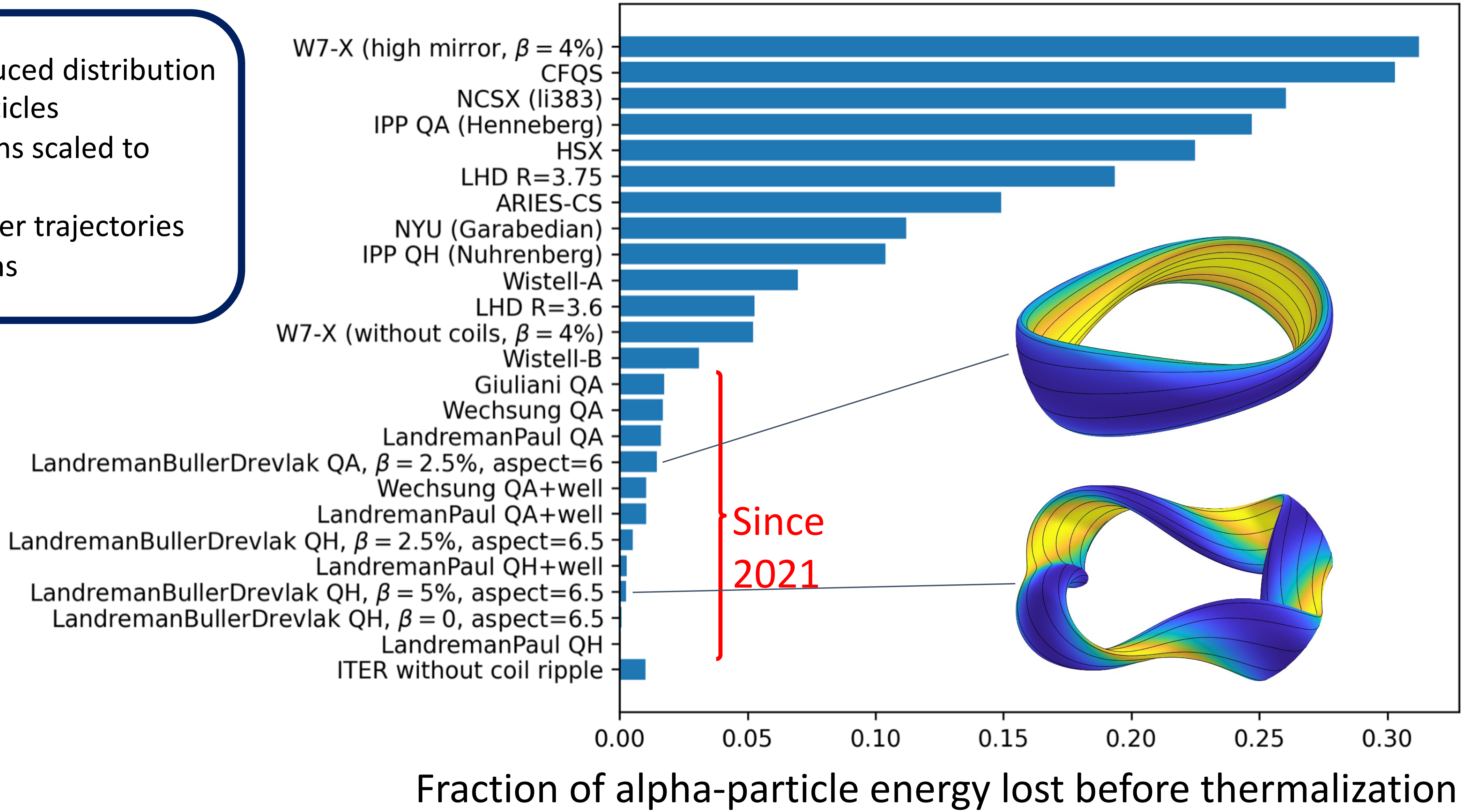
C. D. Beidler[1]✉, H. M. Smith[1], A. Alonso[2], T. Andreeva[1], J. Baldzuhn[1], M. N. A. Beurskens[1], M. Borchardt[1], S. A. Bozhenkov[1], K. J. Brunner[1], H. Damm[1], M. Drevlak[1], O. P. Ford[1], G. Fuchert[1], J. Geiger[1], P. Helander[1], U. Hergenhahn[1,5], M. Hirsch[1], U. Höfel[1], Ye. O. Kazakov[3], R. Kleiber[1], M. Krychowiak[1], S. Kwak[1], A. Langenberg[1], H. P. Laqua[1], U. Neuner[1], N. A. Pablant[4], E. Pasch[1], A. Pavone[1], T. S. Pedersen[1], K. Rahbarnia[1], J. Schilling[1], E. R. Scott[1], T. Stange[1], J. Svensson[1], H. Thomsen[1], Y. Turkin[1], F. Warmer[1], R. C. Wolf[1], D. Zhang[1] & the W7-X Team*



Through optimization, QS stellarators can confine fusion products

- Fusion-produced distribution of alpha particles
- Configurations scaled to reactor size
- Guiding center trajectories with collisions



Landreman & Paul PRL (2022), Wechsung et al PNAS (2022), Giuliani et al JPP (2022), Landreman, Buller & Drevlak PoP (2022)

Growing private investment in stellarators

THEA ENERGY

TYPE ONE ENERGY

STELLAREX

RENAISSANCE FUSION

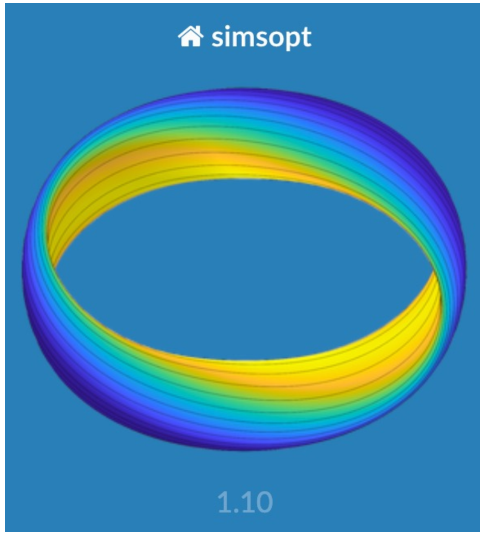
GAUSS FUSION

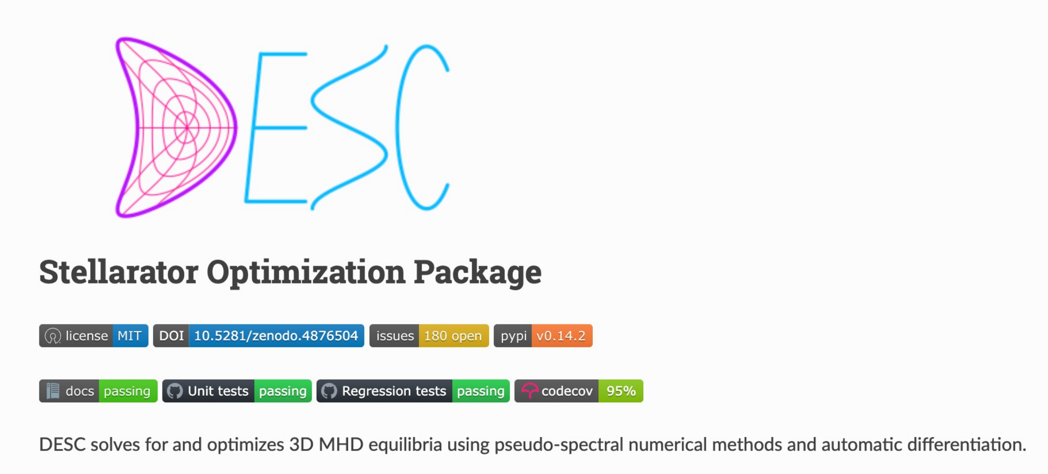
Proxima Fusion

Helical Fusion

Differentiable simulations for stellarator optimization

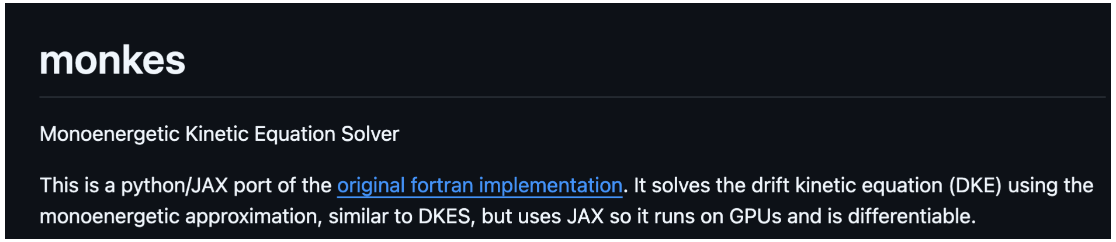
- Automatic differentiation (JAX)
  - ✗ Software must be rewritten with JAX-compatible functions
  - ✗ Potential poor memory scaling (intermediate values stored)
  - ✓ (relatively) straightforward to implement new objectives





Stellarator Optimization Package

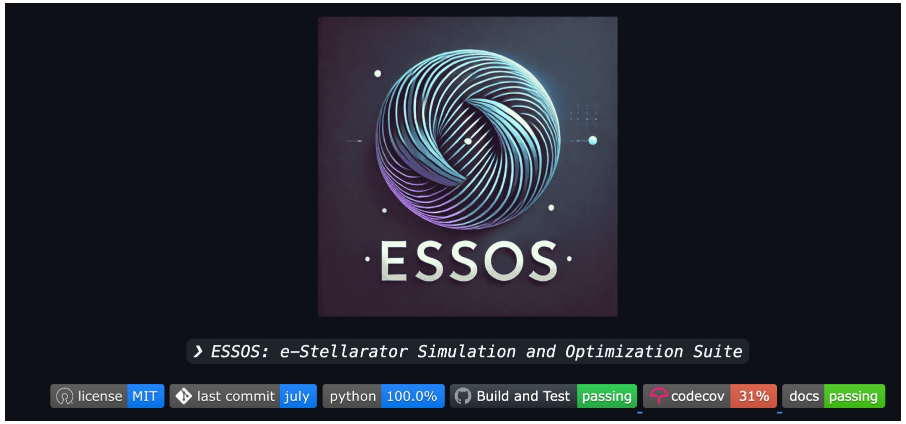
DESC solves for and optimizes 3D MHD equilibria using pseudo-spectral numerical methods and automatic differentiation.



monkes

Monoenergetic Kinetic Equation Solver

This is a python/JAX port of the original fortran implementation. It solves the drift kinetic equation (DKE) using the monoenergetic approximation, similar to DKES, but uses JAX so it runs on GPUs and is differentiable.



❯ *ESSOS: e-Stellarator Simulation and Optimization Suite*

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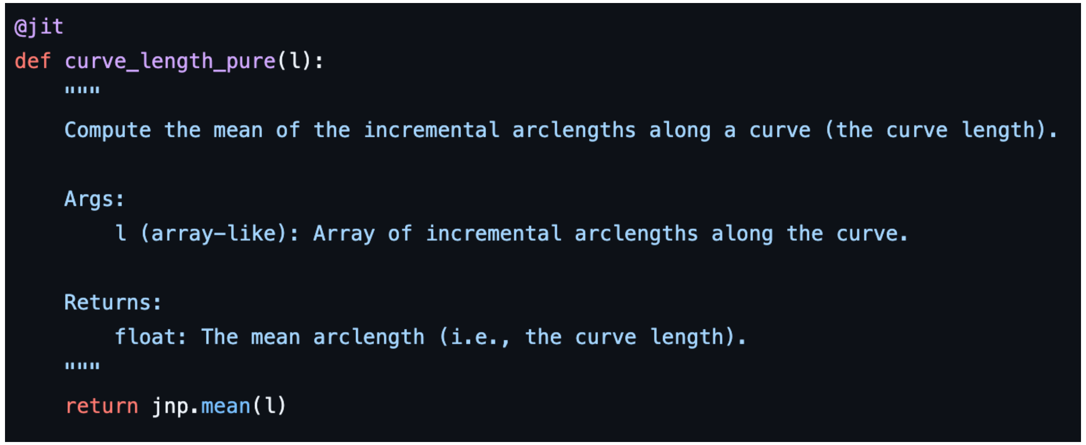
```
class CurveLength(Optimizable):
    r"""
    CurveLength is a class that computes the length of a curve, i.e.

    .. math::
        J = \int_{\text{curve}}~dl.

    """

    def __init__(self, curve):
        self.curve = curve
        self.dJ_dl = jit(lambda l: grad(curve_length_pure)(l))
        super().__init__(depends_on=[curve])

    def J(self):
        """
        This returns the value of the quantity.
        """
        return curve_length_pure(self.curve.incremental_arclength())

    @derivative_dec
    def dJ(self):
        """
        This returns the derivative of the quantity with respect to the curve dofs.
        """

        return self.curve.dincremental_arclength_by_dcoeff_vjp(
            self.dJ_dl(self.curve.incremental_arclength()))

    return_fn_map = {'J': J, 'dJ': dJ}
```



```
@jit
def curve_length_pure(l):
    """
    Compute the mean of the incremental arclengths along a curve (the curve length).

    Args:
        l (array-like): Array of incremental arclengths along the curve.

    Returns:
        float: The mean arclength (i.e., the curve length).
    """
    return jnp.mean(l)
```

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- Adjoint methods

- Enforce constraint with Lagrange multiplier:

$$\mathcal{L}(\Omega, u, q) = f(\Omega, u) + \langle q, L(\Omega, u) \rangle$$

- Solve ***adjoint*** equation:

$$\delta\mathcal{L}(\Omega, u, q; \delta u) = 0$$

- Compute derivative of cost function

$$\delta f(\Omega, u(\Omega), q; \delta\Omega) = \delta\mathcal{L}(\Omega, u(\Omega), q; \delta\Omega)$$

- u = state variables
- Ω = design parameters
- q = test function
- f = cost function
- $\langle \ldots \rangle$ = inner product

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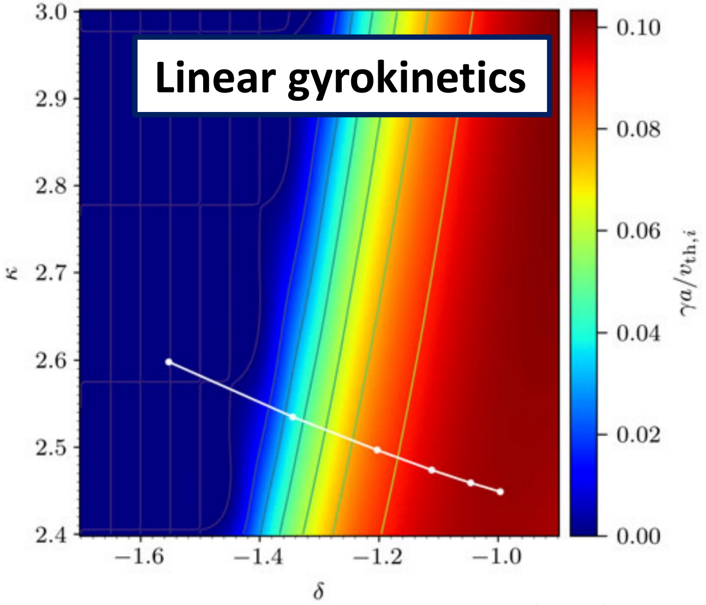
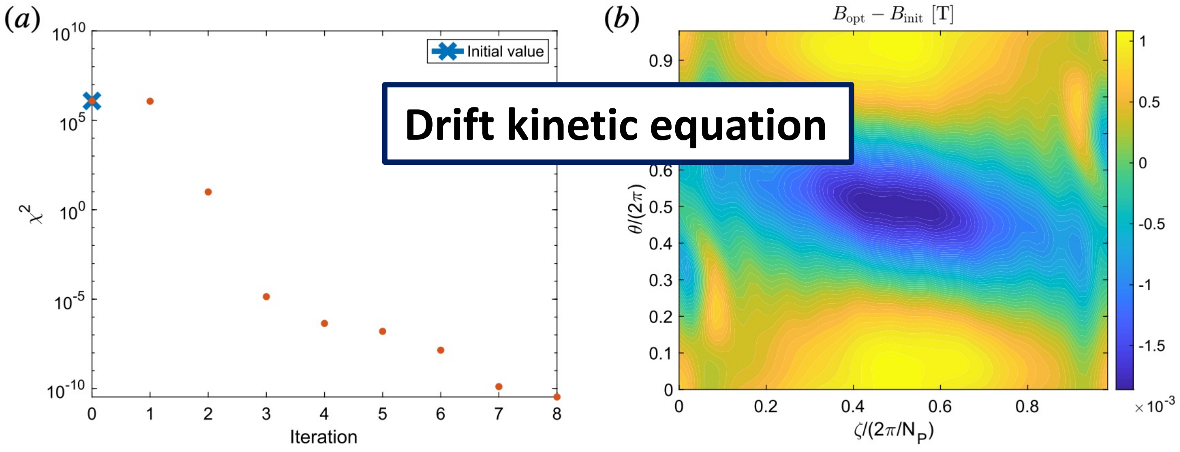
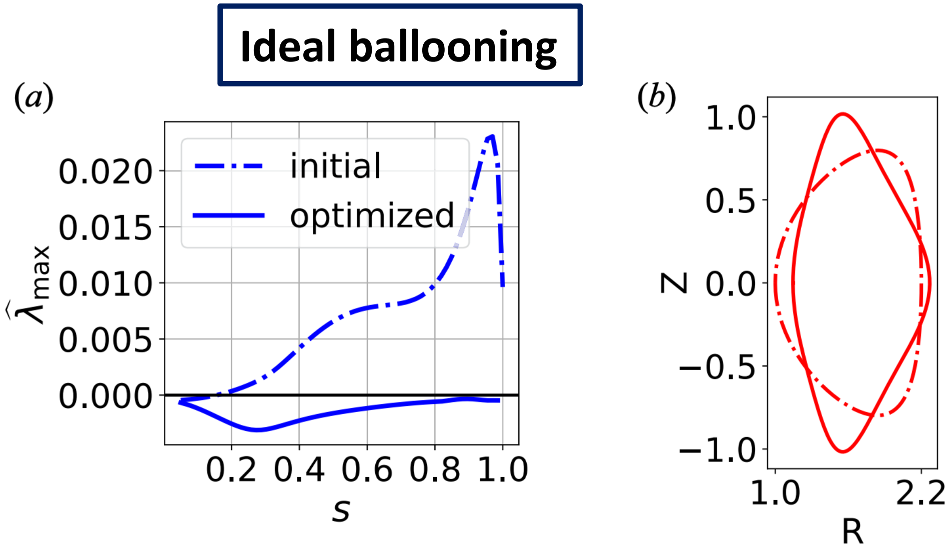
- Adjoint methods
  - ❌ Requires deriving new equations for new objectives
  - ✓ Reduction in memory overhead and computational cost (if # objectives << # parameters)
  - ✓ Legacy solvers can be reused (with modifications)



Acton et al, *J. Plasma Phys.* 90 (2024).



Paul et al, *J. Plasma Phys.* 85 (2019).



Gaur et al, *J. Plasma Phys.* 89 (2023).

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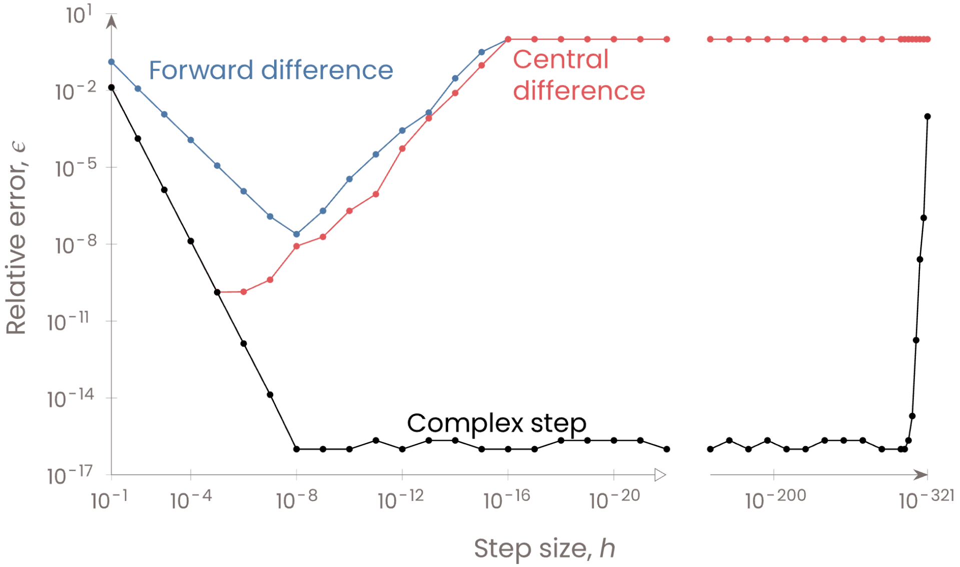
- Complex step differentiation
  - ✗ Requires complex analytic function
  - ✗ Expensive for high-dimensional gradient
  - ✓ Simple implementation

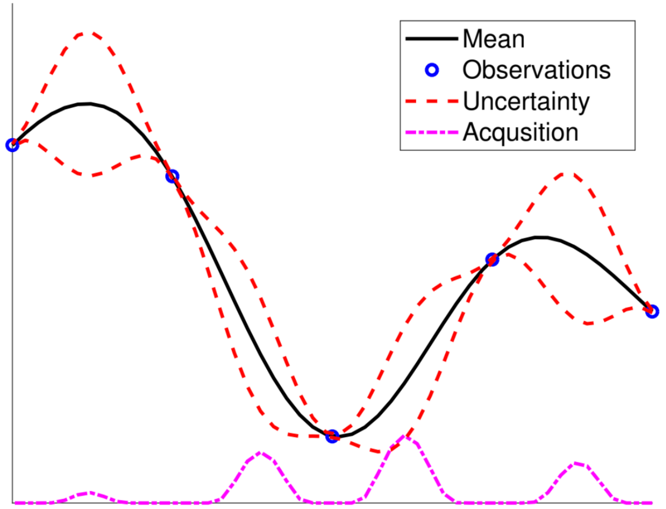
$$\frac{\partial f}{\partial x} \approx \frac{\mathrm{Im}[f(x+ih)]}{h}$$



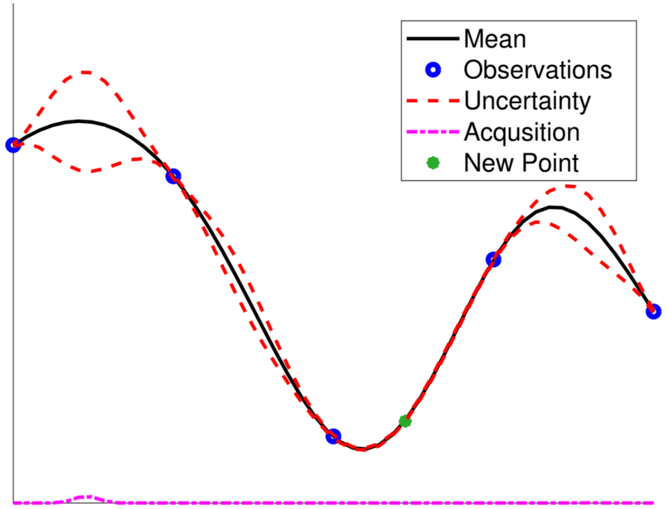
How can data-driven methods accelerate device optimization?

- Optimizing with data-driven surrogate models
  - **Bayesian optimization**:
    - Build Gaussian process surrogate based on limited function evaluations
    - Challenge: scaling to many ($\gtrsim 20$) dimensions often impractical
    - Incorporating gradients, several local BOs can help [Padidar, NeurIPS 2021]

BO Iteration 5



BO Iteration 6



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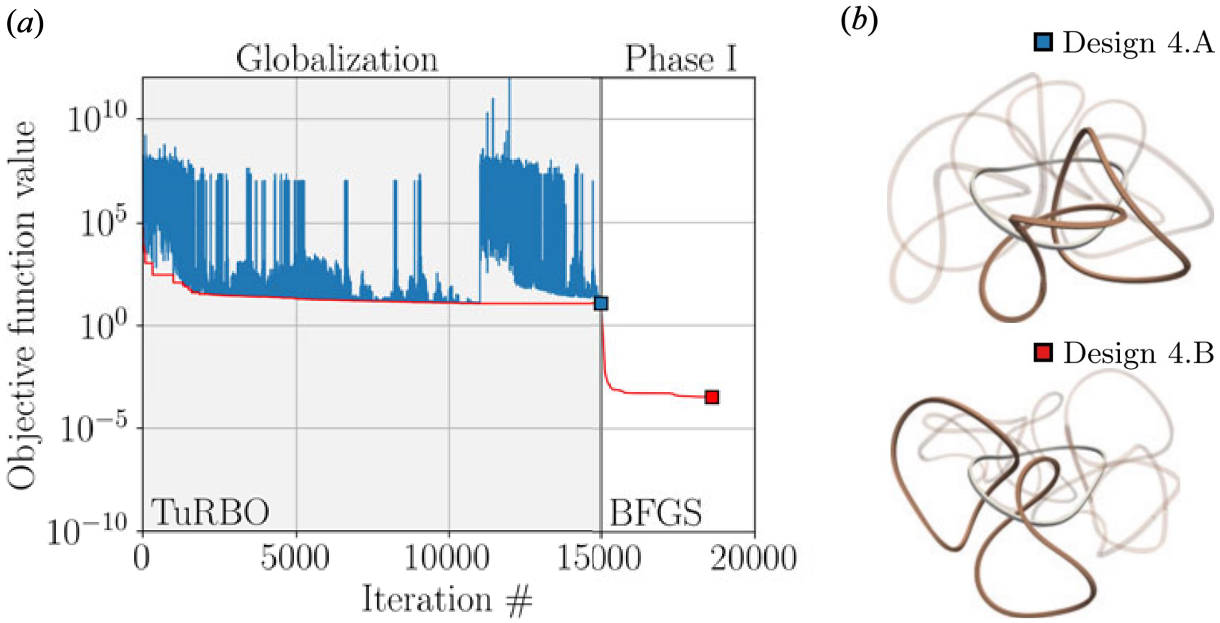
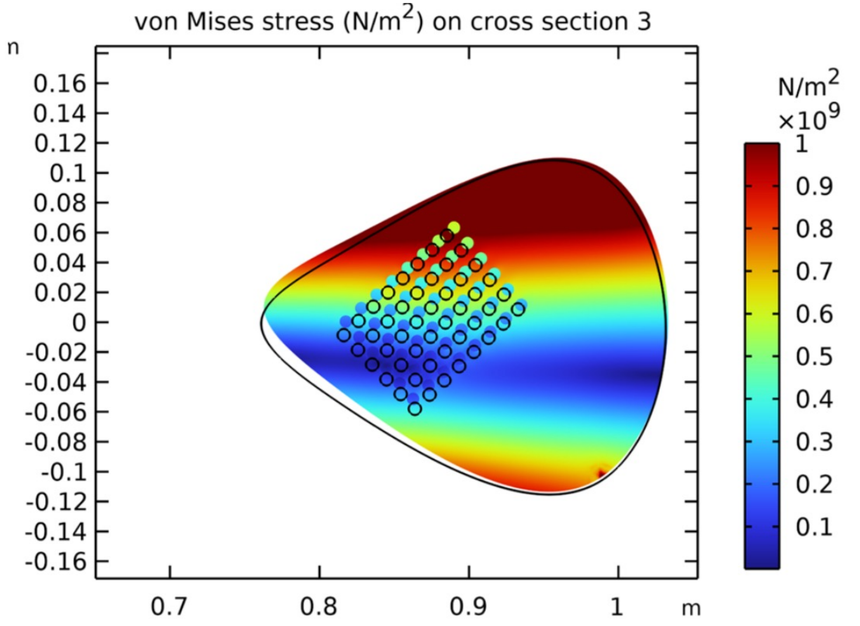
Initial stage coil exploration



Giuliani et al, *J. Plasma Phys.* 90 (2024).

Coil winding pack optimization



Packman et al, *J. Fusion Energy* 44 (2025).

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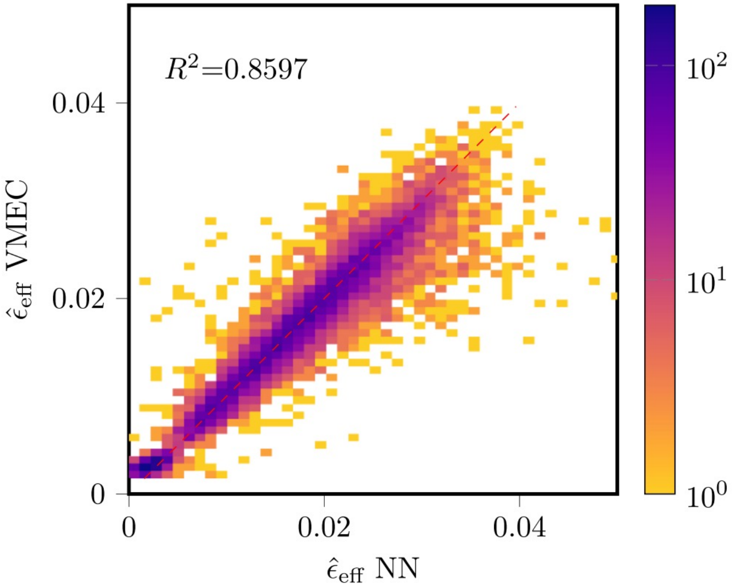
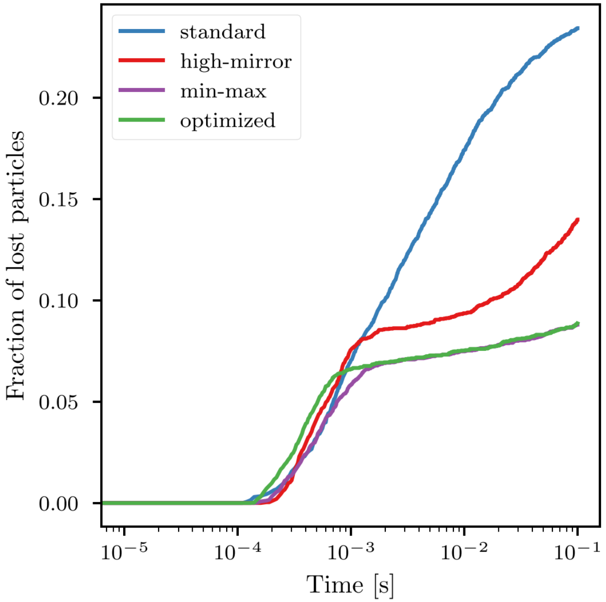
  - **Neural network surrogates:**
    - Provides differentiable model
    - Likely only feasible within a limited configuration space (e.g., W7-X operating space)





Merlo et al, *Nuclear Fusion* 63 (2023).

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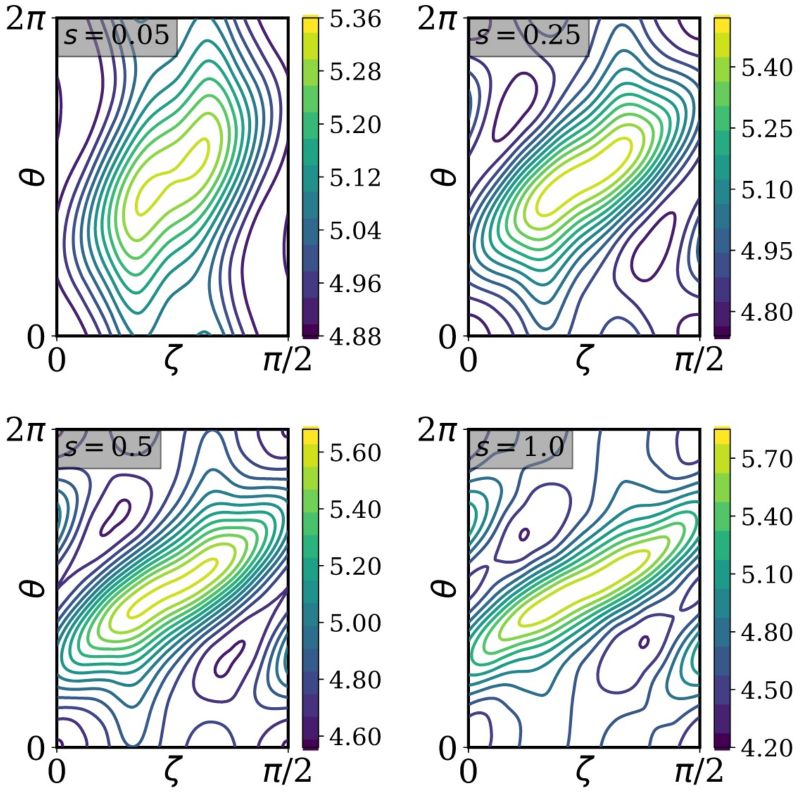
  - **Neural network surrogates:**
    - Provides differentiable model
    - Likely only feasible within a limited configuration space (e.g., W7-X operating space)
  - **Optimization with data-driven subgrid or fluid closure models:**
    - If geometry independent, may not suffer from curse of dimensionality
    - e.g., improved gyrofluid closure for EP instabilities

Recent progress largely made through "physics-reduced" models

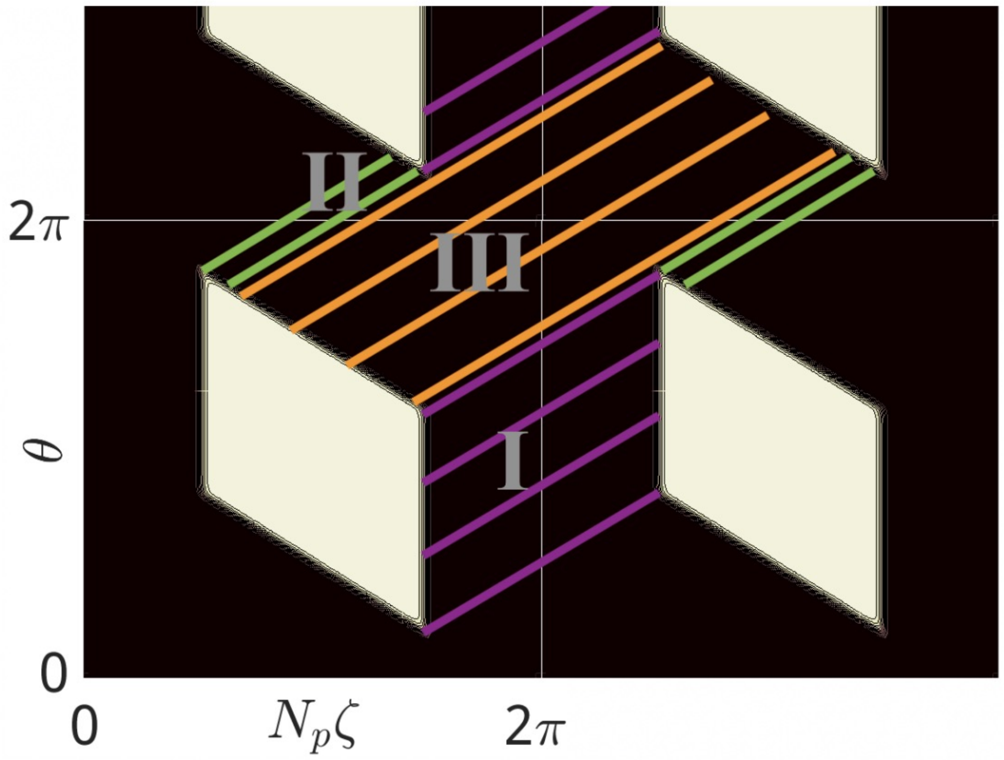
… but sometimes it pays off to attack the full problem

- Direct optimization of collisionless particle losses



Bindel et al, *PPCF* 65 (2024).

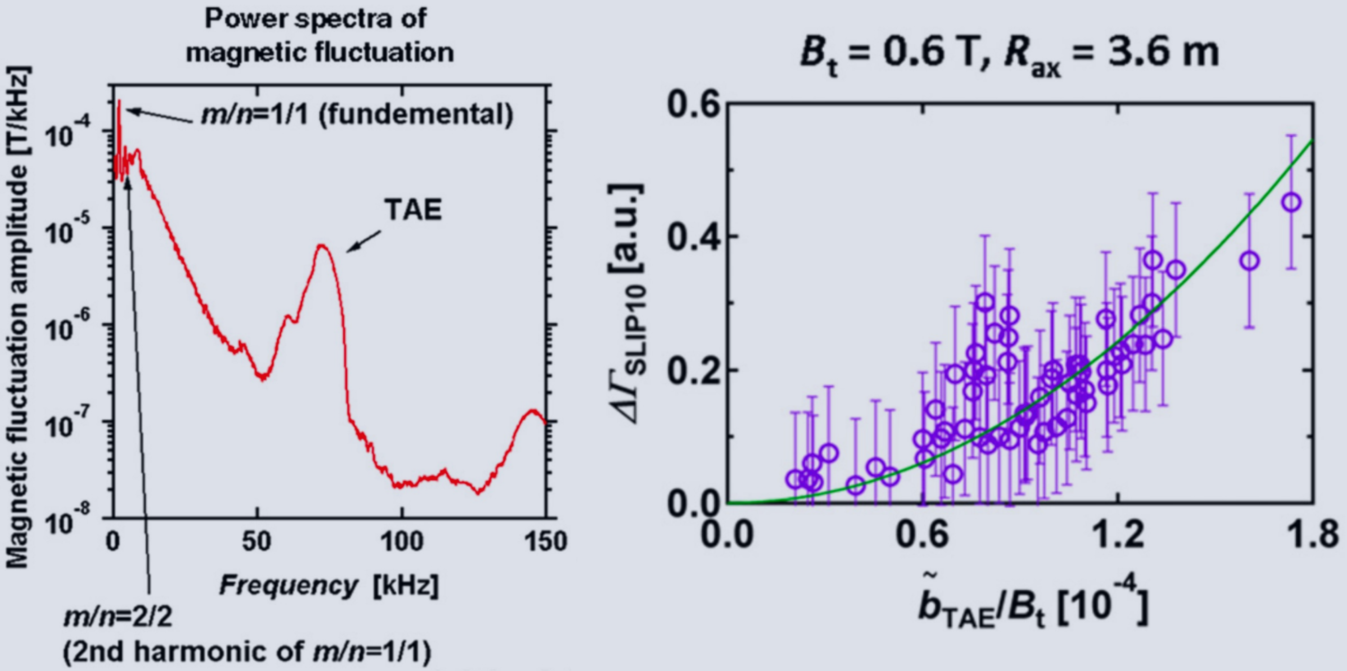
- Discovery of new classes of optimized stellarators → *piecewise omnigeneity*



Velasco et al, *PRL* 133 (2024).

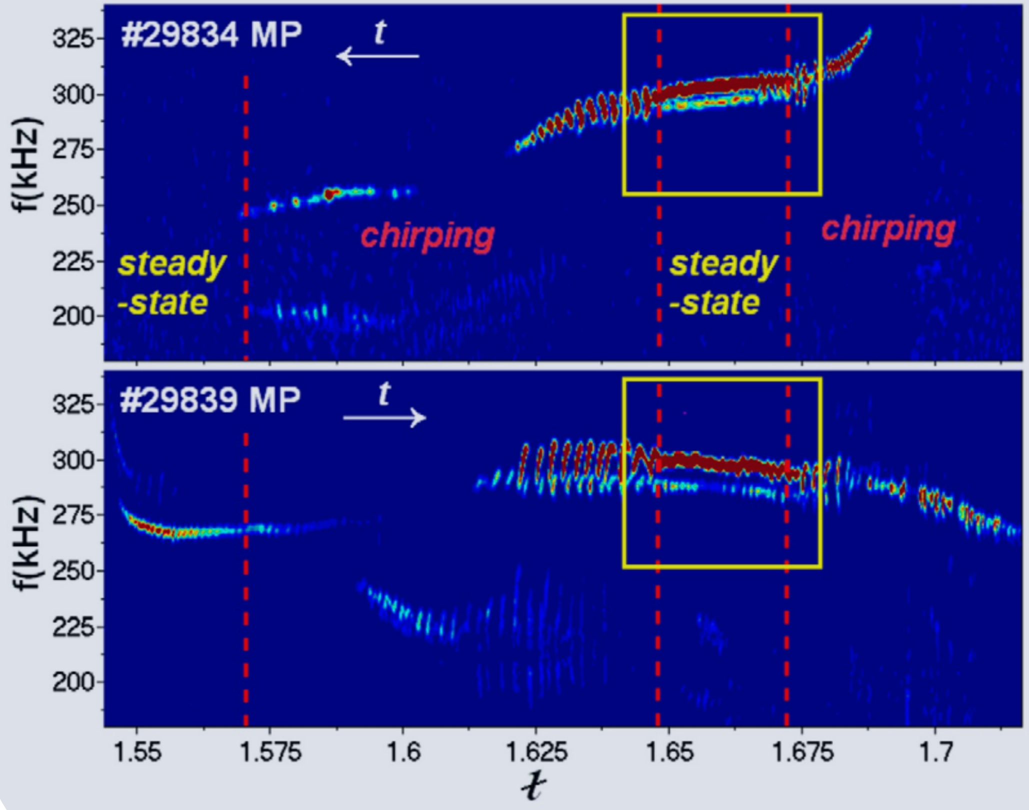
Resonance with AE will drive transport in optimized stellarators

TAE-induced diffusive transport on LHD



Ogawa et al, *Nucl. Fusion* 50 (2010).

Transition between chirping and steady frequency on TJ-II

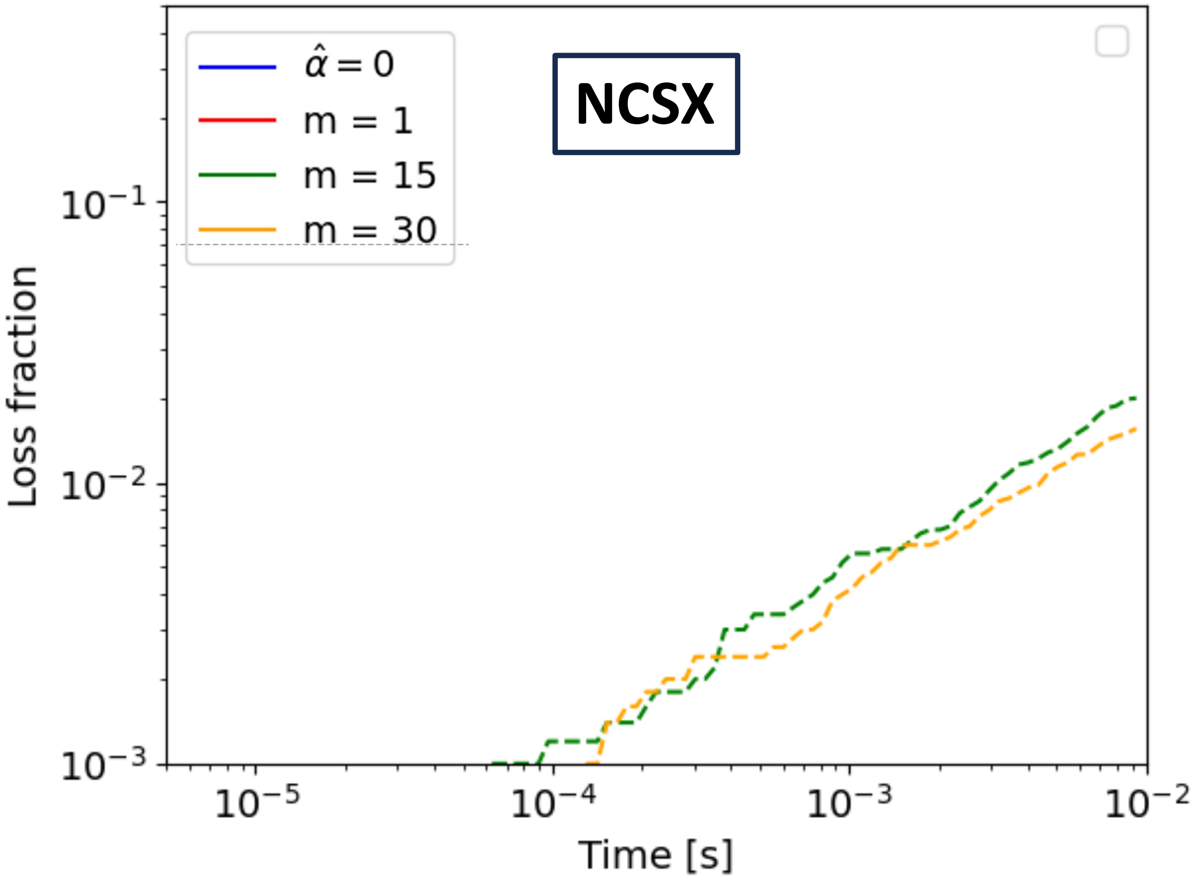


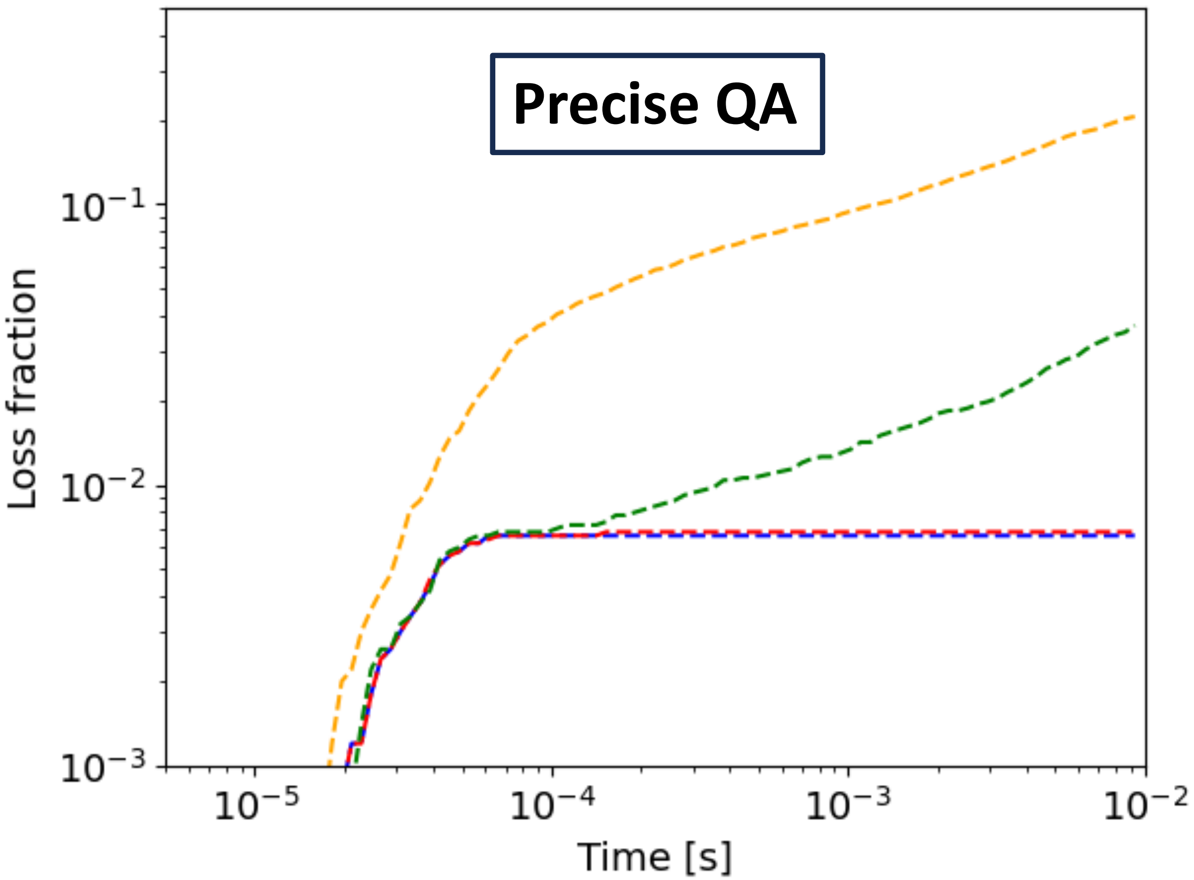
Melnikov et al, *Nucl. Fusion* 56 (2016).

Resonance with AE will drive transport in optimized stellarators

- Monte-Carlo analysis of collisionless 3.5 MeV alpha transport
- Resonant AE w/ $\delta \boldsymbol{B} \cdot \nabla\psi \sim 10^{-3}$ [Hsu & Sigmar, 1992]

Paul et al, *J. Plasma Phys.* 89 (2023).





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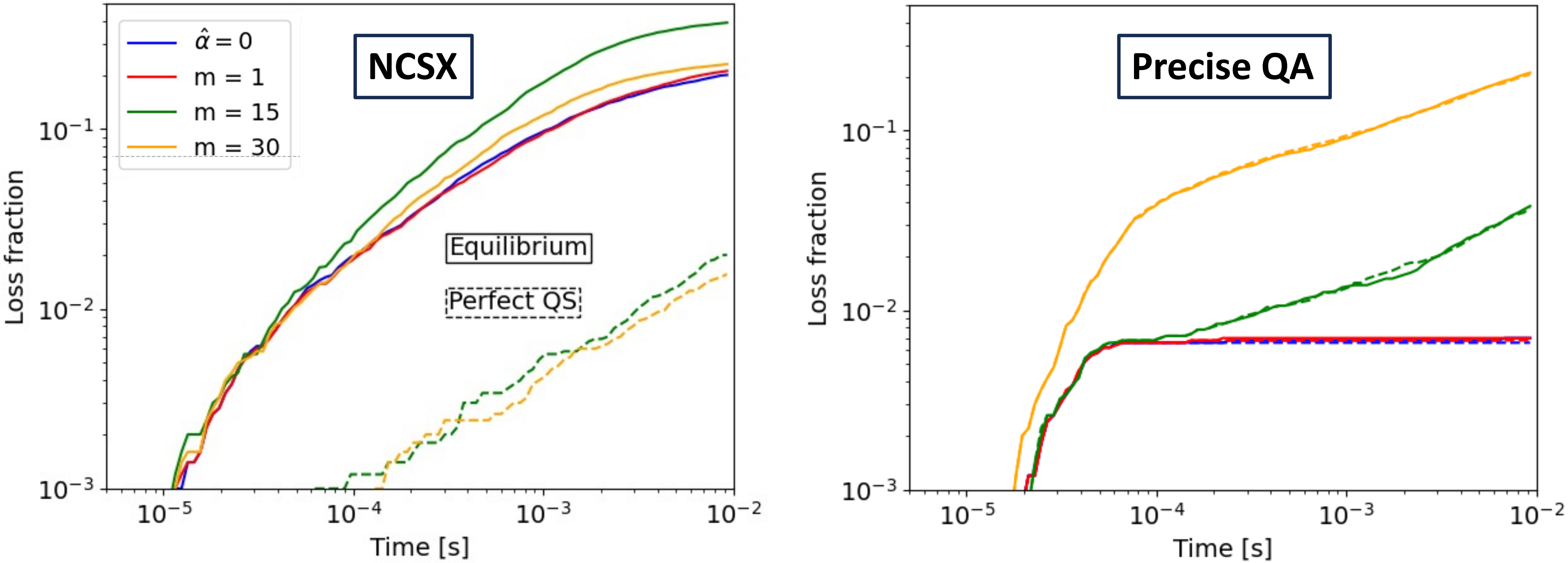
- Monte-Carlo analysis of collisionless 3.5 MeV alpha transport
- Resonant AE w/ $\delta \boldsymbol{B} \cdot \nabla\psi \sim 10^{-3}$ [Hsu & Sigmar, 1992]
- Equilibrium destruction of drift surfaces causes enhanced losses

Paul et al, *J. Plasma Phys.* 89 (2023).



Reduced MHD model for shear Alfvén waves

- The equations describing SAW can be derived from ideal MHD under the assumption of ”reduced MHD” ($^{k_{||}}/_{k_\perp} \ll 1$) and low beta [Salat & Tataronis, 2001]

$$B\,\nabla_{||}\left(\frac{\nabla_\perp^2\left(\nabla_{||}\Phi\right)}{B}\right)+\frac{\omega^2\nabla_\perp^2\Phi}{v_A^2}=0$$

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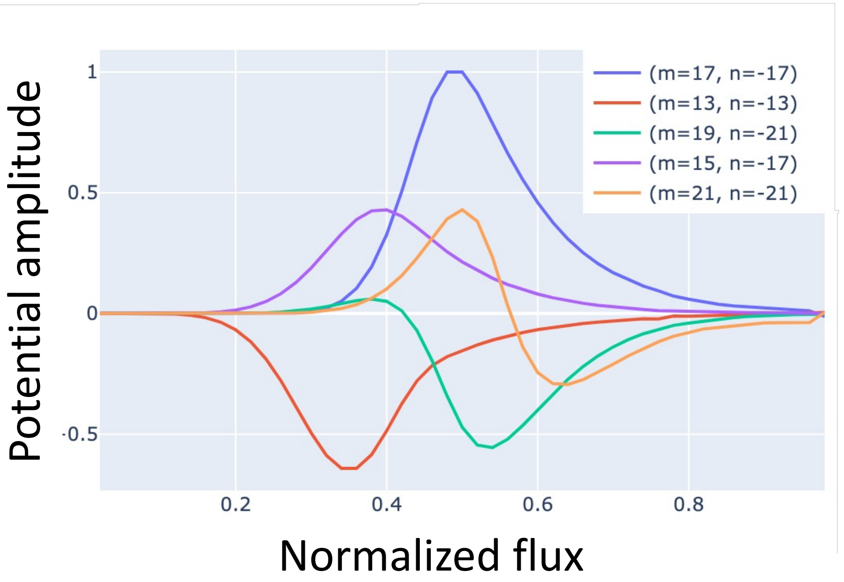
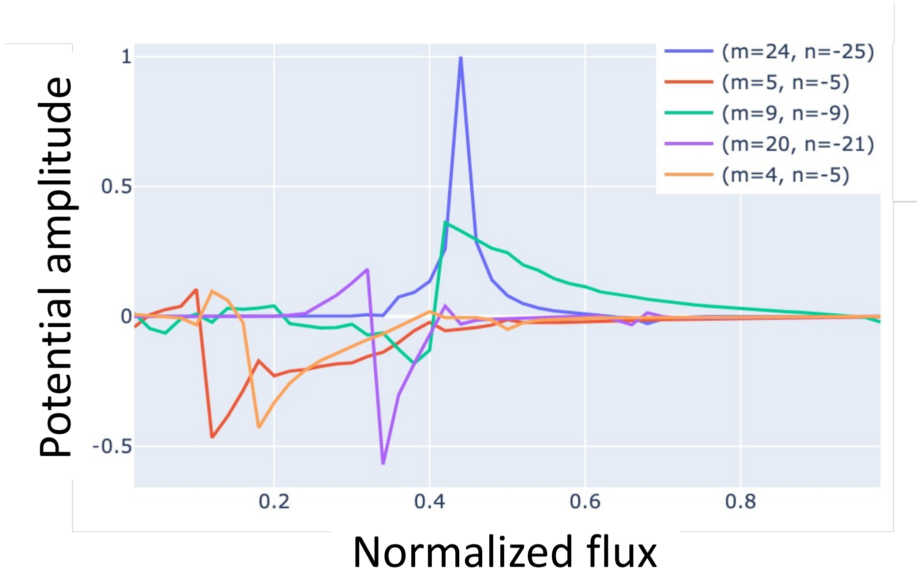
- Eigenvalue problem for potential, Φ, with solutions that are **global** (i.e., extending in radius) or **local** (like a delta-function)



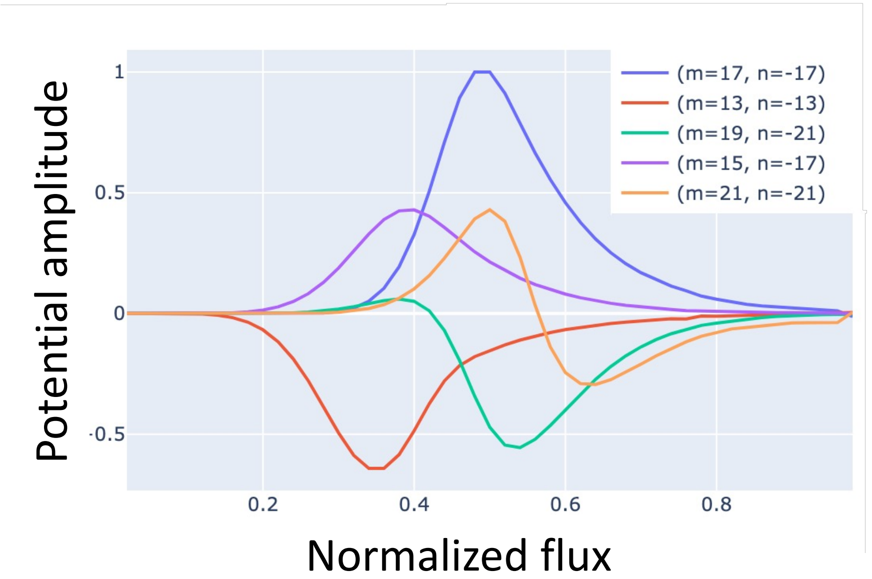


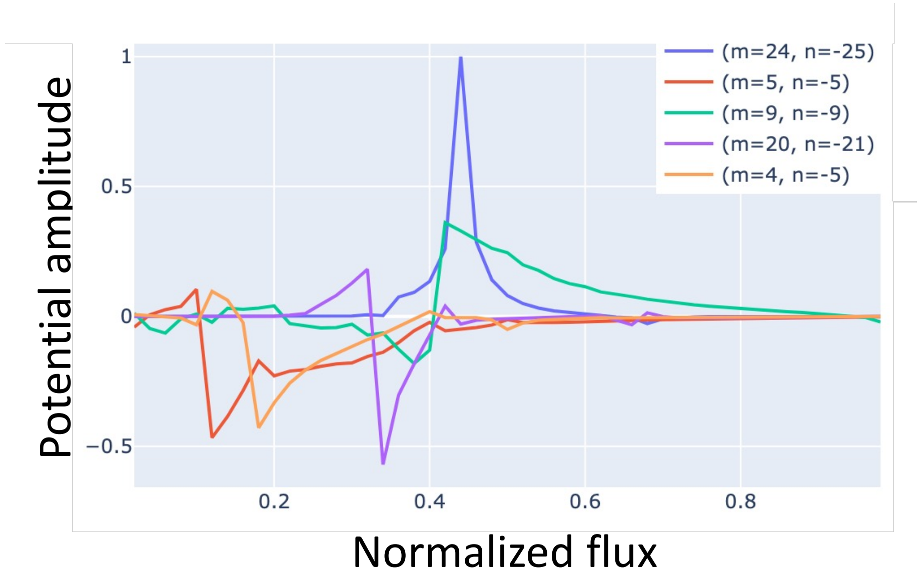
Shear Alfvén continuum structure informs global eigenmodes

$$\boxed{B\,\nabla_{||}\left(\frac{\nabla_\perp^2\left(\nabla_{||}\Phi\right)}{B}\right)+\frac{\omega^2\nabla_\perp^2\Phi}{v_A^2}=0} \longrightarrow \boxed{B\,\nabla_{||}\left(\frac{|\nabla\psi|^2}{B}\nabla_{||}\Phi\right)+\frac{\omega^2|\nabla\psi|^2}{v_A^2}\Phi=0}$$

- Reduced MHD ($k_\perp \gg k_{||}$) model for SAW
- Discrete modes **can be destabilized**

- Nullspace of second radial derivative
- Localized energy density → **heavily damped**

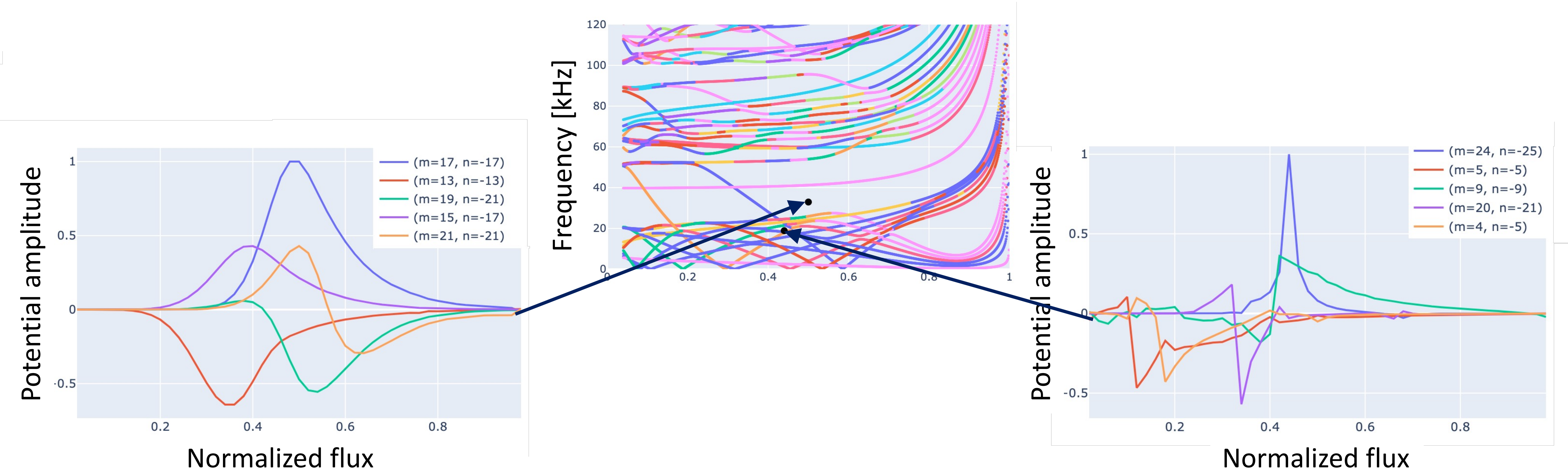




Shear Alfvén continuum structure informs global eigenmodes

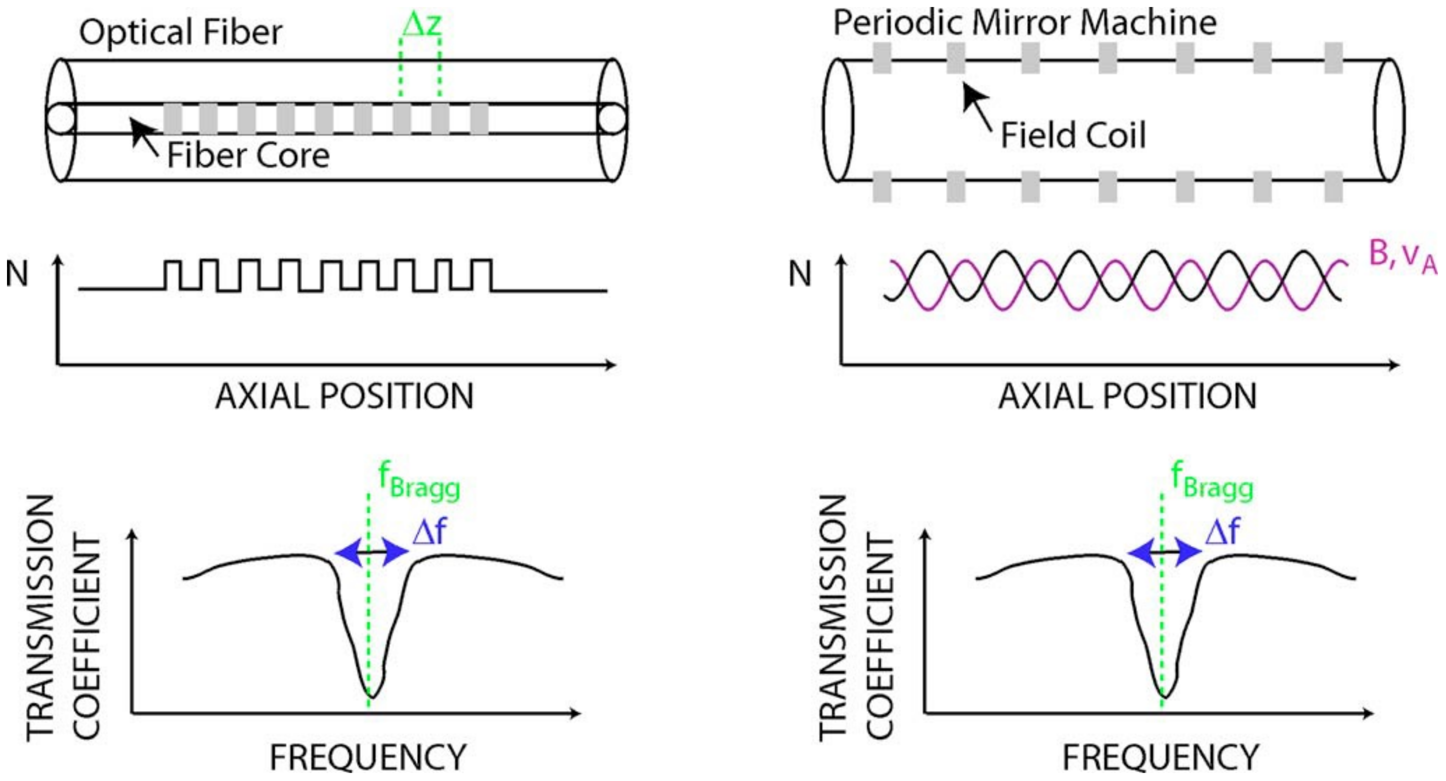
- We focus on localized continuum solutions, nullspace of highest-order radial derivative → **often the first step in assessing AE stability**

$$B\,\nabla_{||}\left(\frac{|\nabla\psi|^2}{B}\nabla_{||}\Phi\right)+\frac{\omega^2|\nabla\psi|^2}{v_A^2}\Phi=0$$



Analogous spectral optimization problems

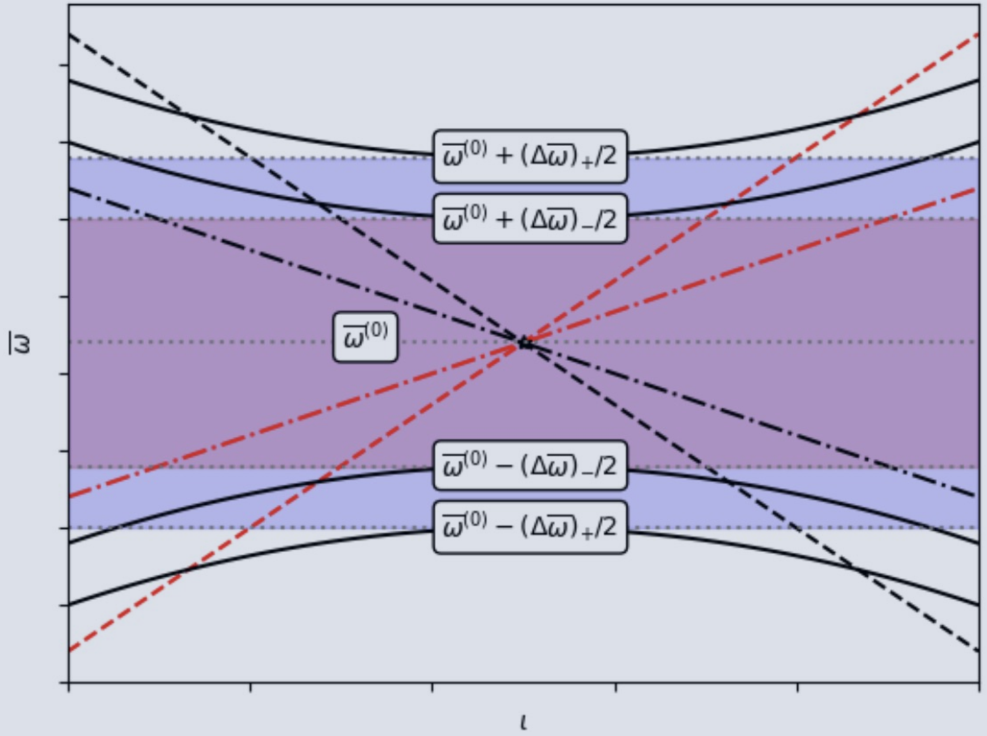
Heidbrink, *Phys. Plasmas* 15 (2008).



- Periodic modulation of index of refraction introduces **frequency gap**
  - Optical fiber → transmission gap
  - Semiconductors → band gap
- “Band gap engineering” by modifying material properties

Outline

- *Stellarator optimization and AI/ML*
- ***Perturbation theory for shear Alfvén continuum***
- *Can we manipulate the shear Alfvén continuum to avoid resonance?*



Perturbative approach to solving the SAW continuum

- Continuum equation in Boozer coordinates near magnetic axis:

$$\frac{1}{1+\epsilon}\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\left[(1+\epsilon)\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\Phi\right]+\overline{\omega}^2\Phi=0$$

- Expand in smallness of coupling parameter, $\epsilon = \frac{|\nabla\psi|^2}{\langle|\nabla\psi|^2\rangle} - 1$

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- Expand in smallness of coupling parameter, $\epsilon=\frac{|\nabla\psi|^2}{\langle|\nabla\psi|^2\rangle}-1$
- **Degenerate perturbation theory** is required (analogous to, e.g. Stark effect)
- Goal:
  - *Analytic expressions for the gap widths*
  - *Understand interaction between gaps in 3D geometry*

Lowest order: No coupling (i.e., cylinder)

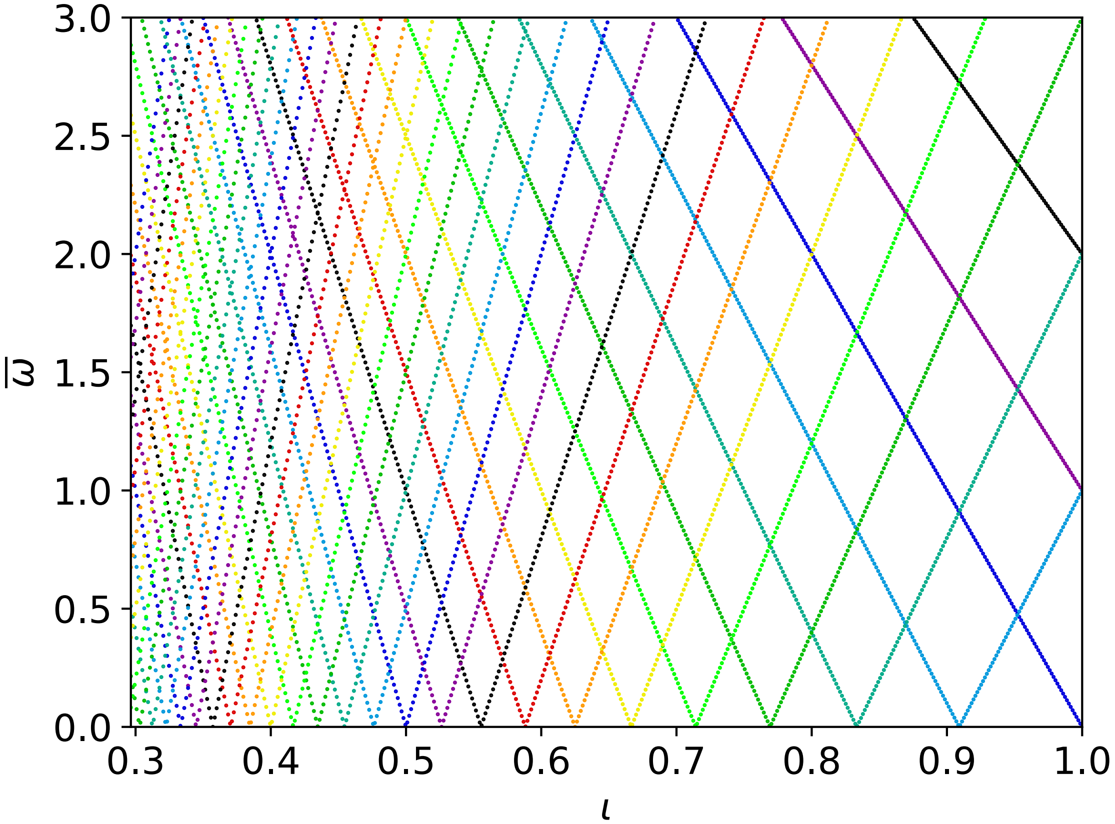
$$\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\left[\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\Phi_j^{(0)}\right]+\left(\overline{\omega}_j^{(0)}\right)^2\Phi_j^{(0)}=0$$

- Eigenfunctions:

$$\Phi_j^{(0)}=\Phi_{m_j,n_j}e^{i(m_j\theta-n_j\zeta+\omega t)}$$

- Normalized frequencies:

$$\left(\overline{\omega}_j^{(0)}\right)^2=\left(\iota_0 m_j-n_j\right)^2$$

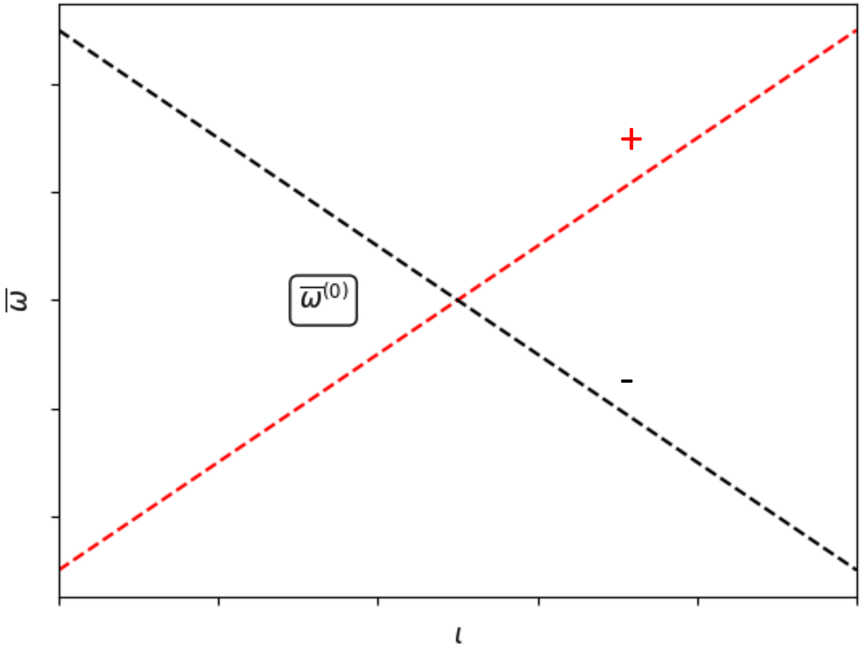


Linear order: Degenerate eigenvalues shifted

$$\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)^2\Phi^{(1)}+\left[\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\epsilon\right]\left[\left(\frac{\partial}{\partial\zeta}+\iota_0\frac{\partial}{\partial\theta}\right)\Phi^{(0)}\right]+\left(\overline{\omega}^{(0)}\right)^2\Phi^{(1)}+\left(\overline{\omega}^{(1)}\right)\Phi^{(0)}=0$$

- Degeneracy implies “crossing” of unperturbed frequencies: $\Phi^{(0)}=\alpha_j\Phi_j^{(0)}+\alpha_k\Phi_k^{(0)}$

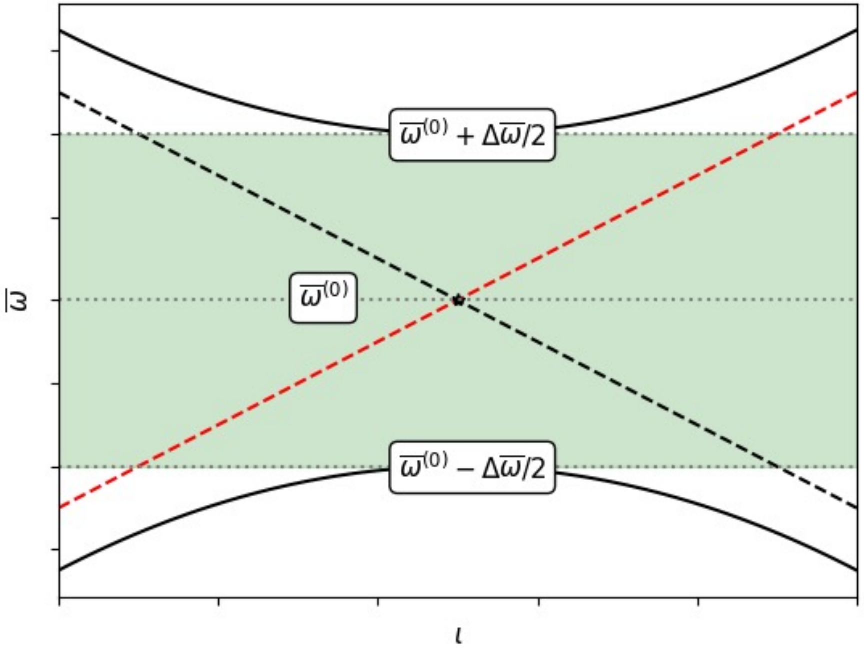
$$\left(\overline{\omega}_j^{(0)}\right)^2=\left(\overline{\omega}_k^{(0)}\right)^2\rightarrow\ \iota_0 m_j-n_j=\ \pm(\iota_0 m_k-n_k)$$



Linear order: Degenerate eigenvalues shifted

- For “co-propagating” degeneracy (+ root) → $\iota_0 = \Delta n / \Delta m$ **(only in 3D)**

$$\overline{\omega}^{(1)} = 0$$



Gap width:

$$\Delta\omega = 2\left|\epsilon_{\Delta m,\Delta n}\right| \overline{\omega}^{(0)}$$

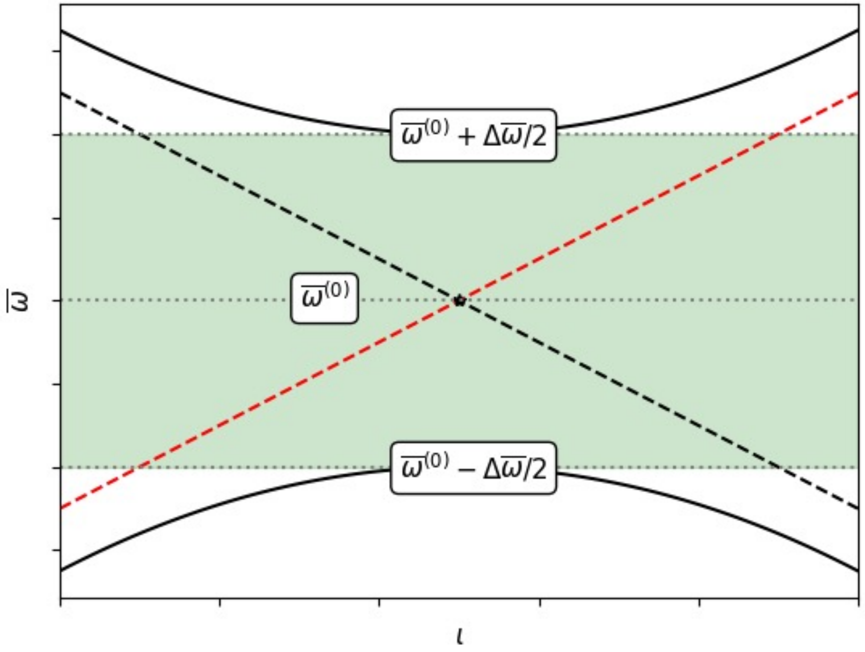
Linear order: Degenerate eigenvalues shifted

- For ”co-propagating” degeneracy (+ root) → $\iota_0 = \Delta n/\Delta m$ **(only in 3D)**

$$\overline{\omega}^{(1)} = 0$$

- For “counter-propagating” degeneracy (- root) → **frequency splitting**

$$\overline{\omega}^{(1)} = 2\left|\epsilon_{\Delta m,\Delta n}\right|\overline{\omega}^{(0)}$$

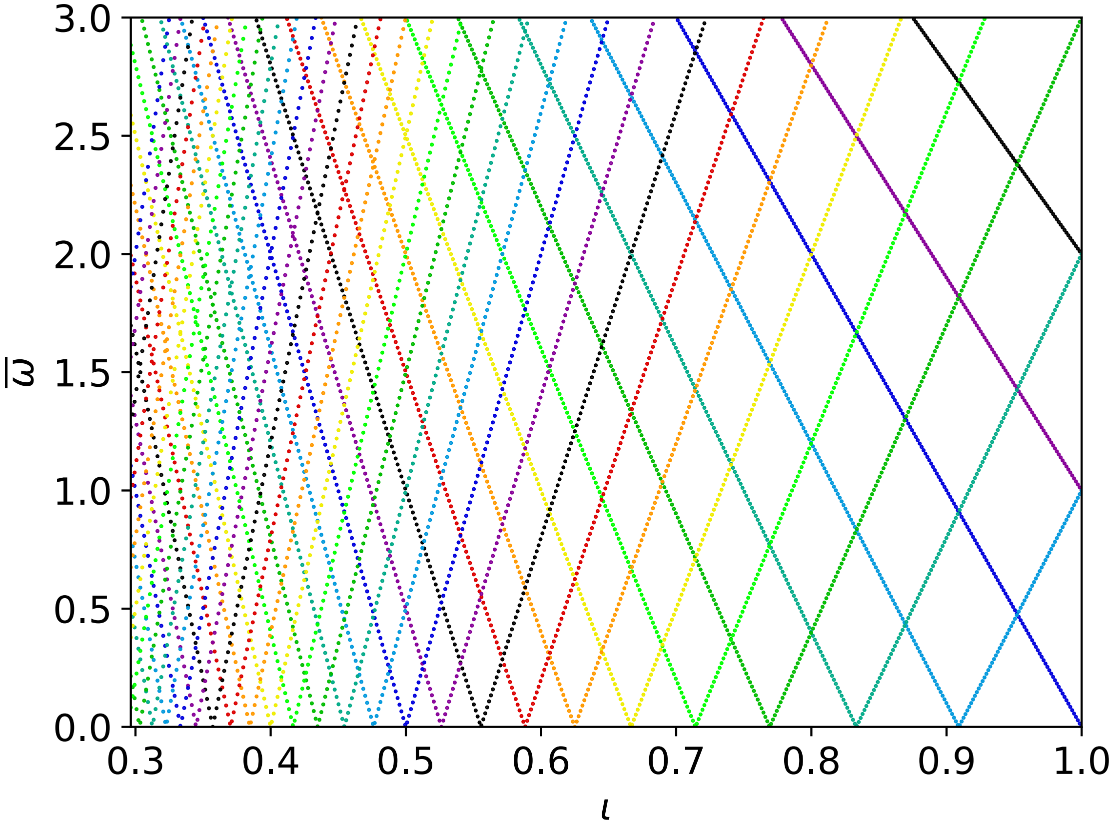


Gap width:

$$\Delta\omega = 2\left|\epsilon_{\Delta m,\Delta n}\right|\,\overline{\omega}^{(0)}$$

Classification of AE gap modes

- Classify gaps by coupling mode numbers $(\Delta m, \Delta n)$
- In axisymmetry:
  - TAE $(\Delta m = 1, \Delta n = 0)$
  - EAE $(\Delta m = 2, \Delta n = 0)$
  - (… higher-order poloidal shaping)
- In stellarators, also:
  - HAE $(\Delta m \neq 0, \Delta n \neq 0)$
  - MAE $(\Delta m = 0, \Delta n \neq 0)$



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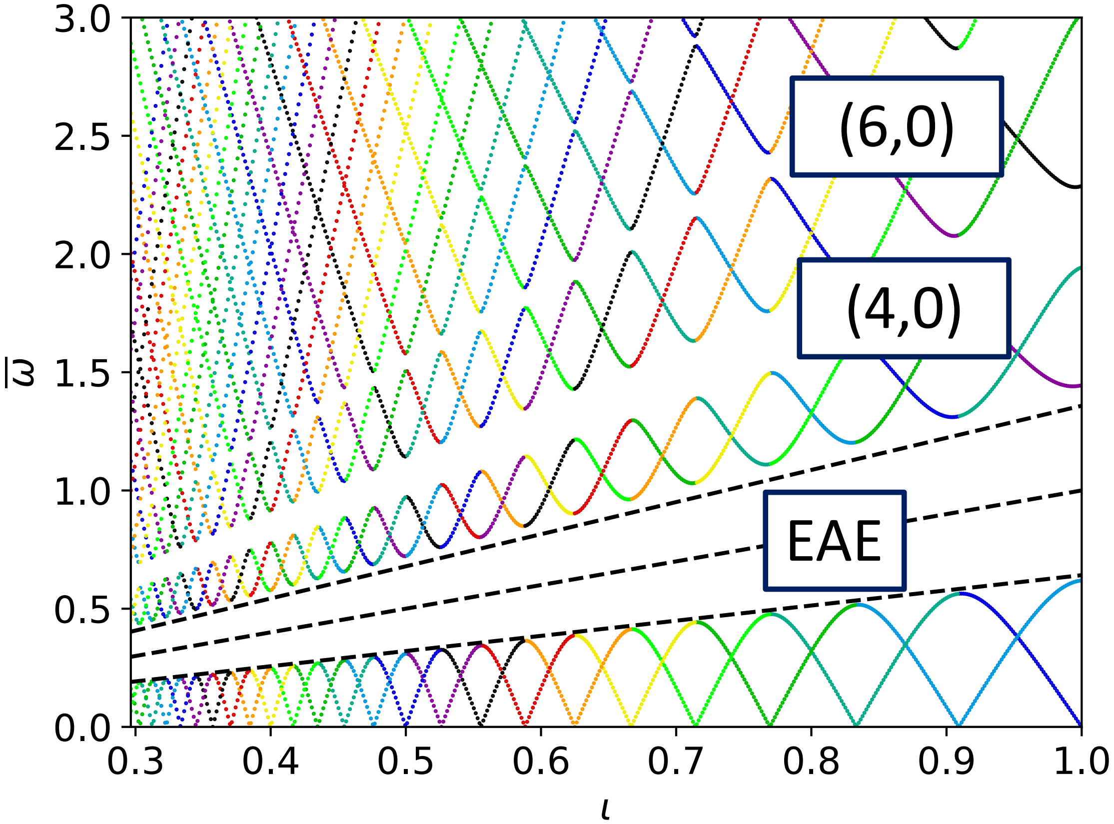
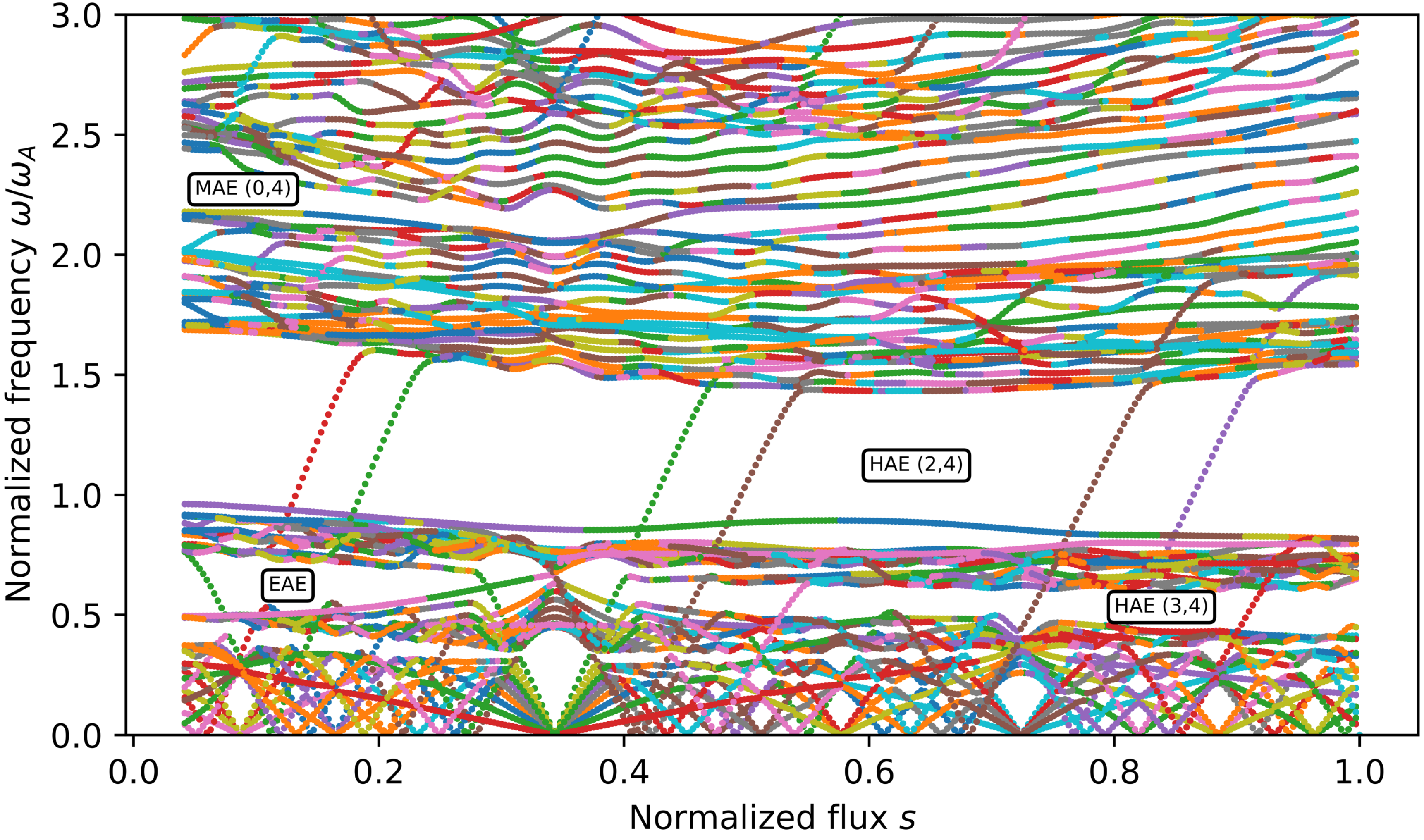
Gaps remain separated in 2D:

$$\overline{\omega}^{(0)} = |\iota_0 \Delta m - \Delta n|/2$$



Gap crossing in stellarators [Goodman, 2024]

Stellgap [Spong, 2003] calculations by A. Hyder



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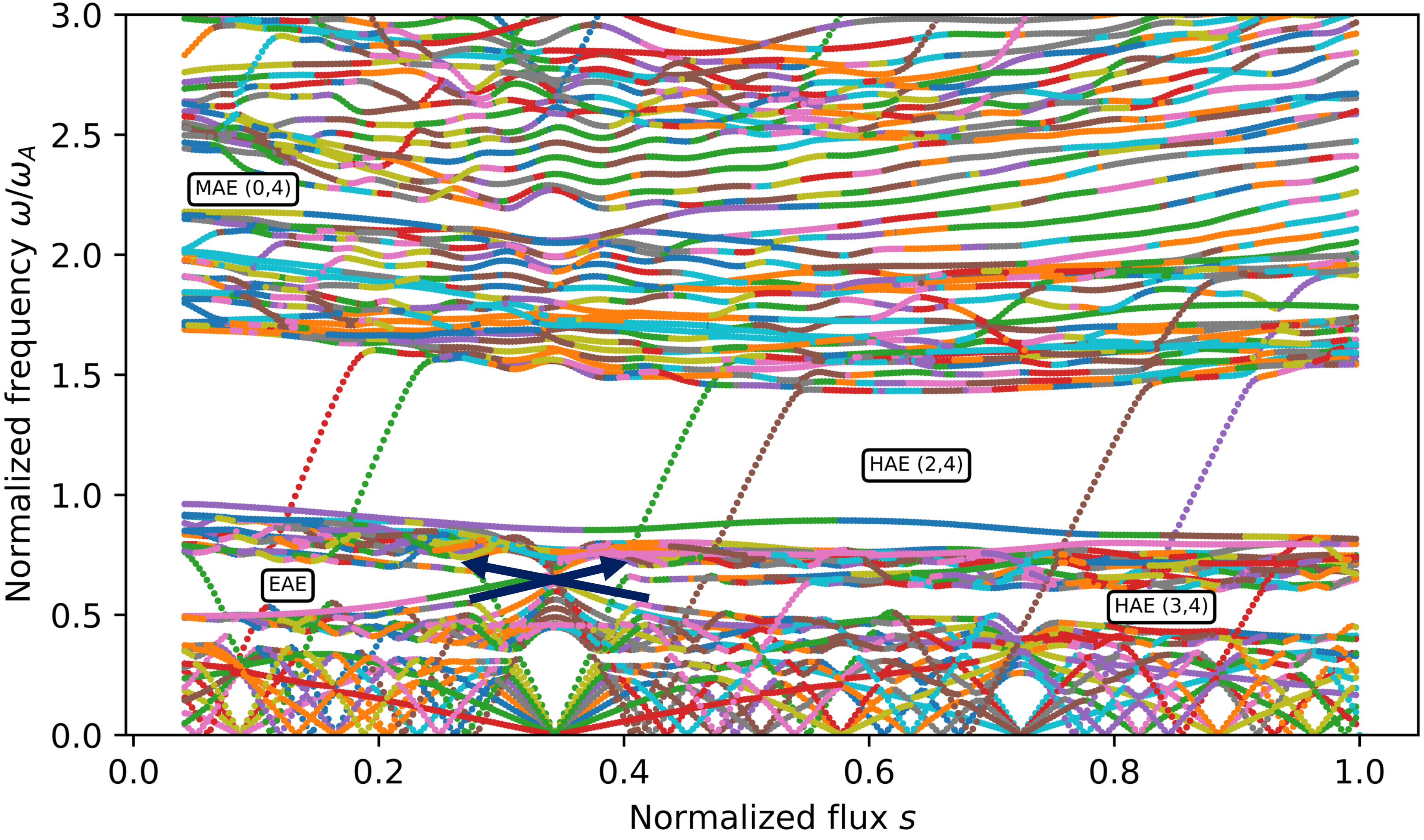
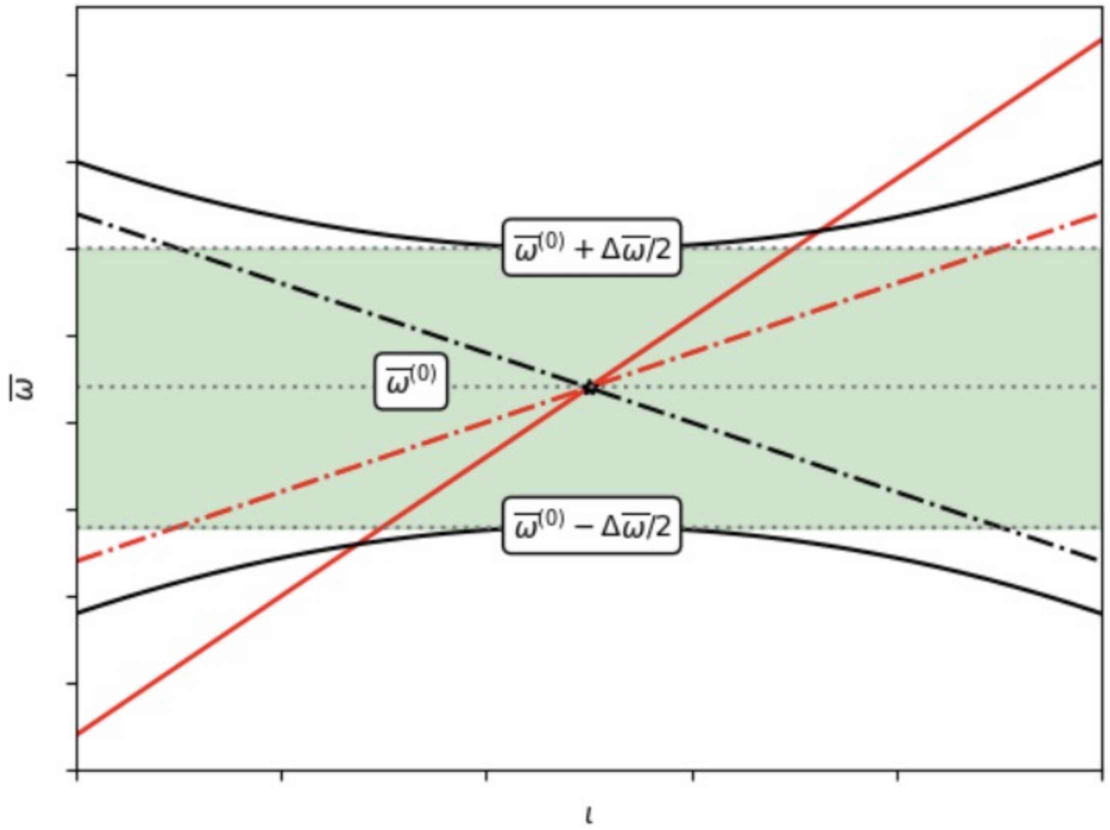
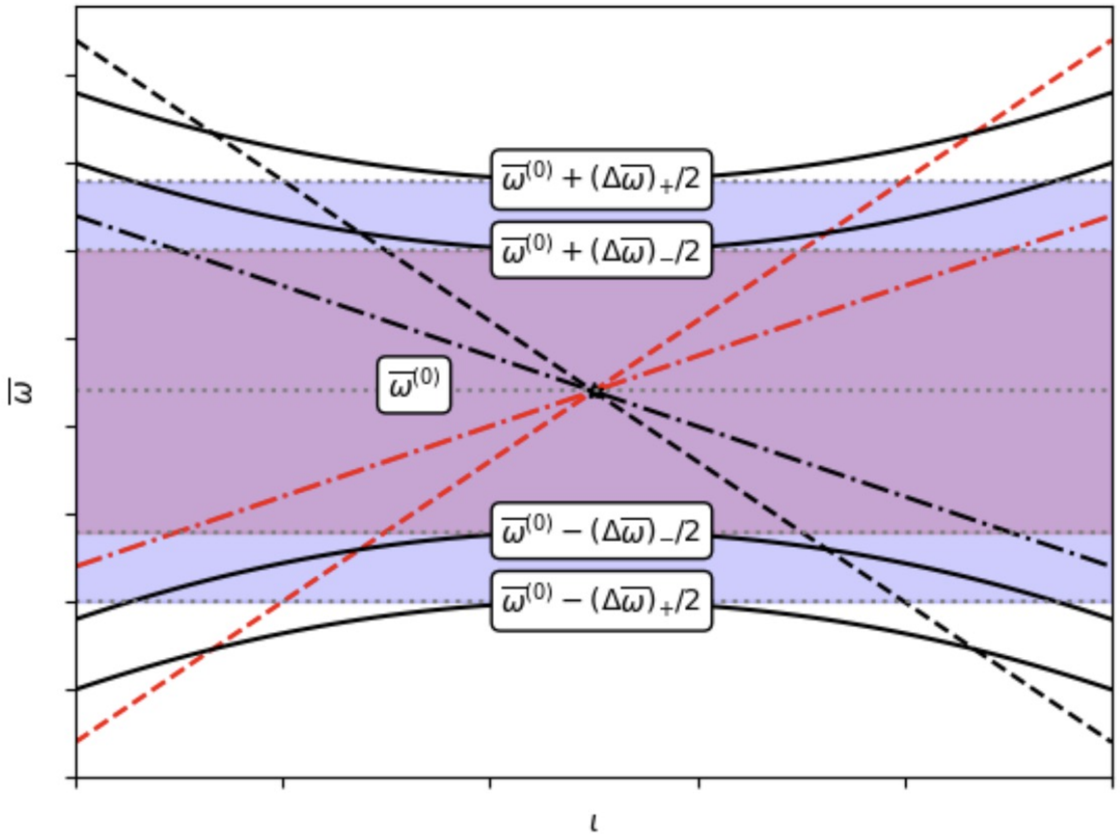


Higher-order crossings possible in 3D systems

Two co-propagating, third counter-propagating: one **“gap crossing”** mode

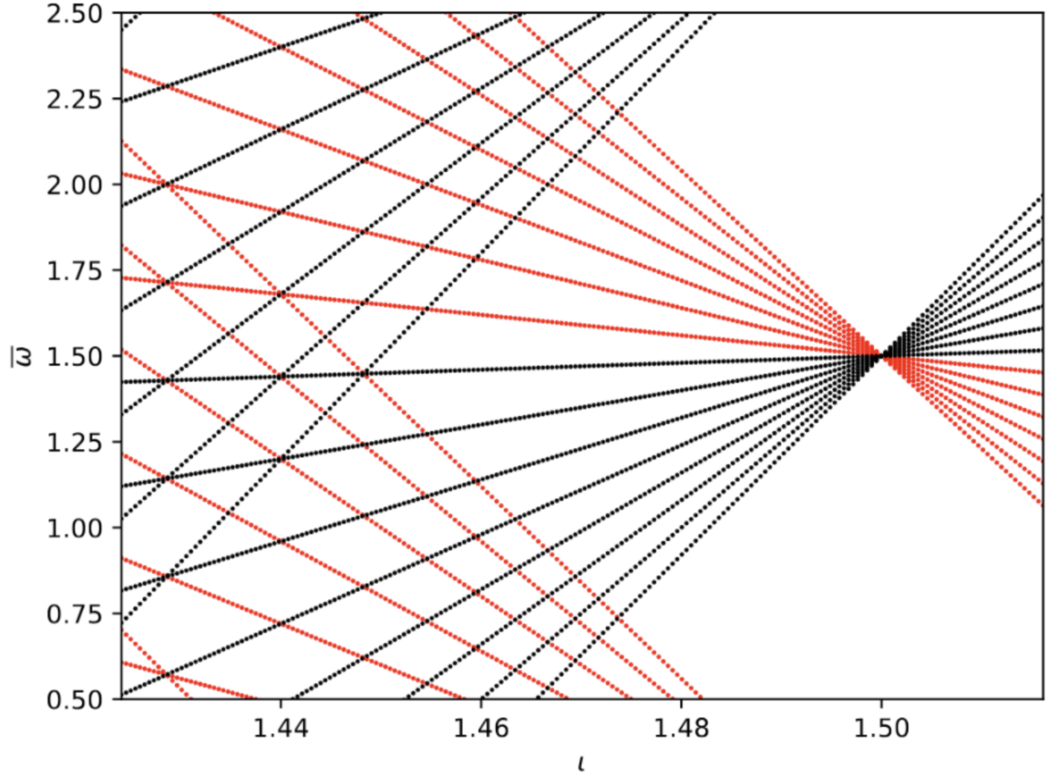


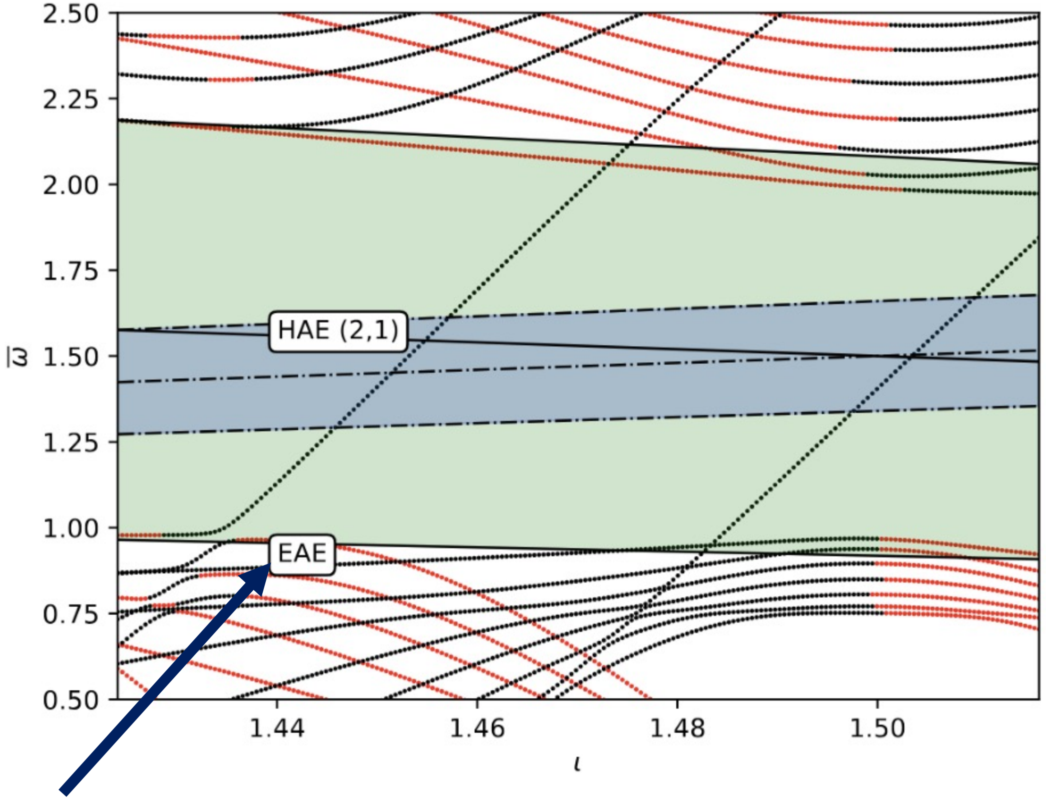
Two counter-propagating pairs: frequency of each shift in **“nested gaps”**



Nührenberg-Zille configuration: 15-way crossing

J. Nuhrenberg and R. Zille, *Phys. Lett. A* (1988)





”Gap repulsion” [Y. I. Kolesnichenko, 2001]

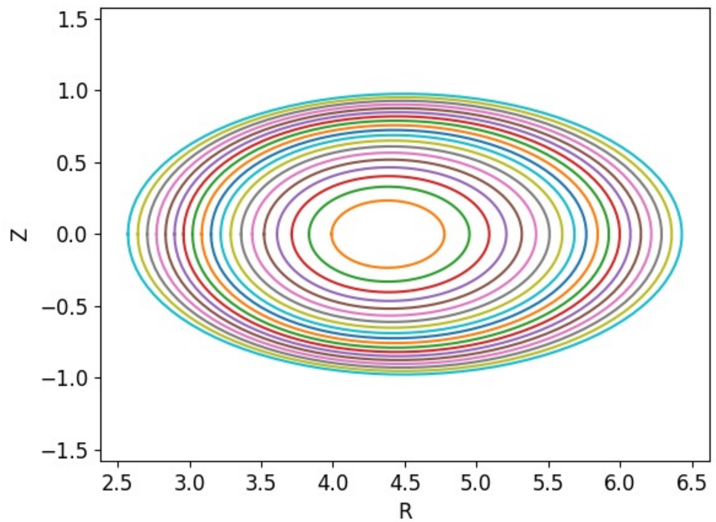
Rotating ellipticity dominates near-axis spectral content

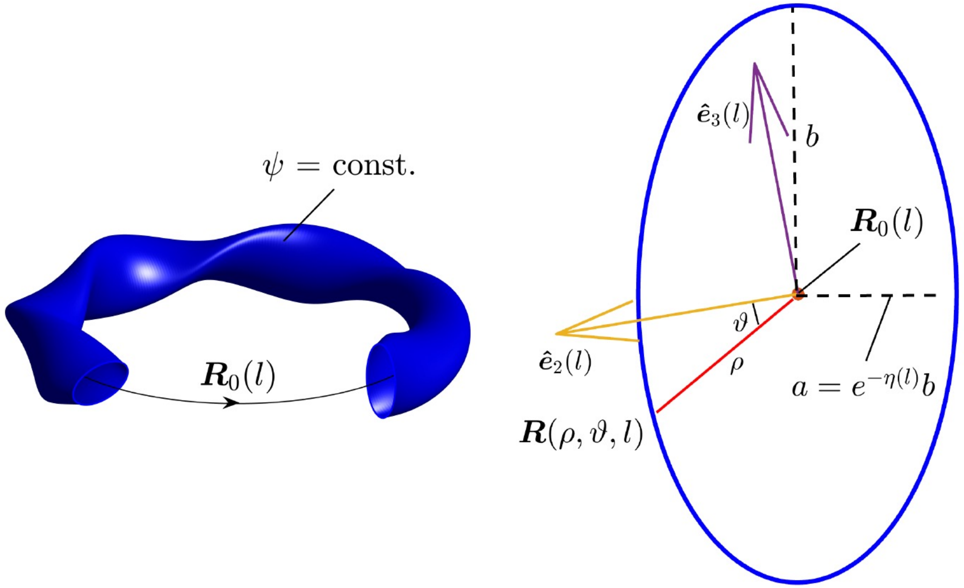
- Near-axis model for the flux-surface compression factor, $|\nabla\psi|^2$

$$|\nabla\psi|^2 = r^2\Psi_2 + r^3\Psi_3 + \mathcal{O}(r^4)$$

- Defining coordinate system oriented with ellipse axes, $x = a\cos\vartheta, y = b\sin\vartheta$

$$\Psi_2 = B_0^2\,\frac{p\ -\sqrt{p^2-4}\cos 2\vartheta}{2}$$





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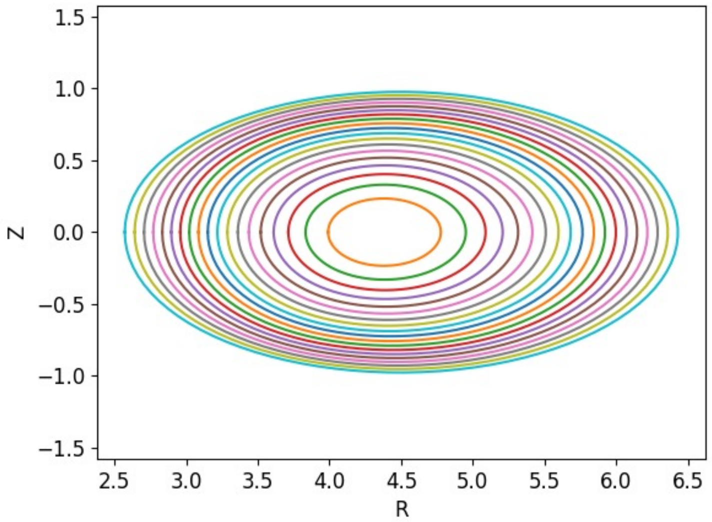
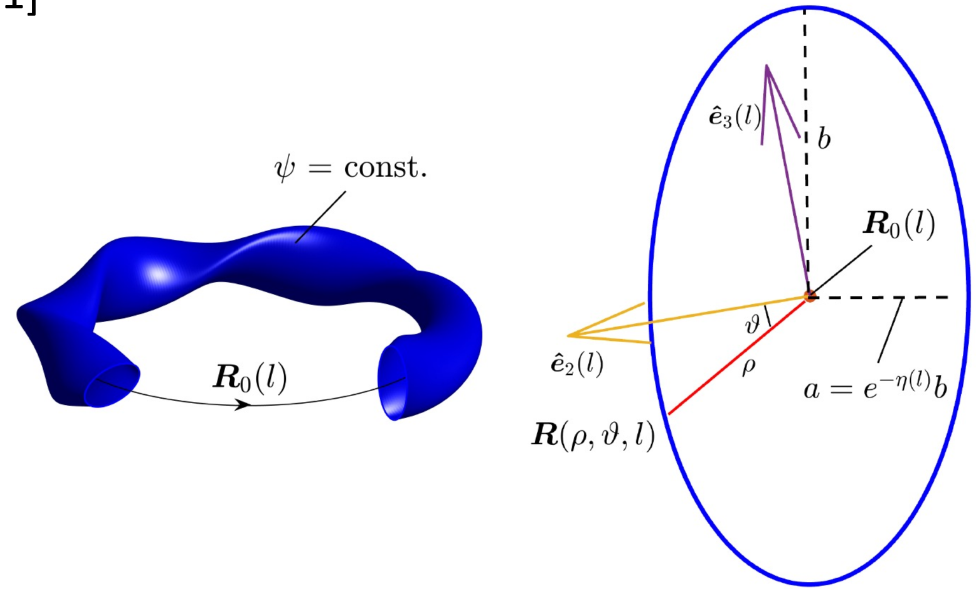
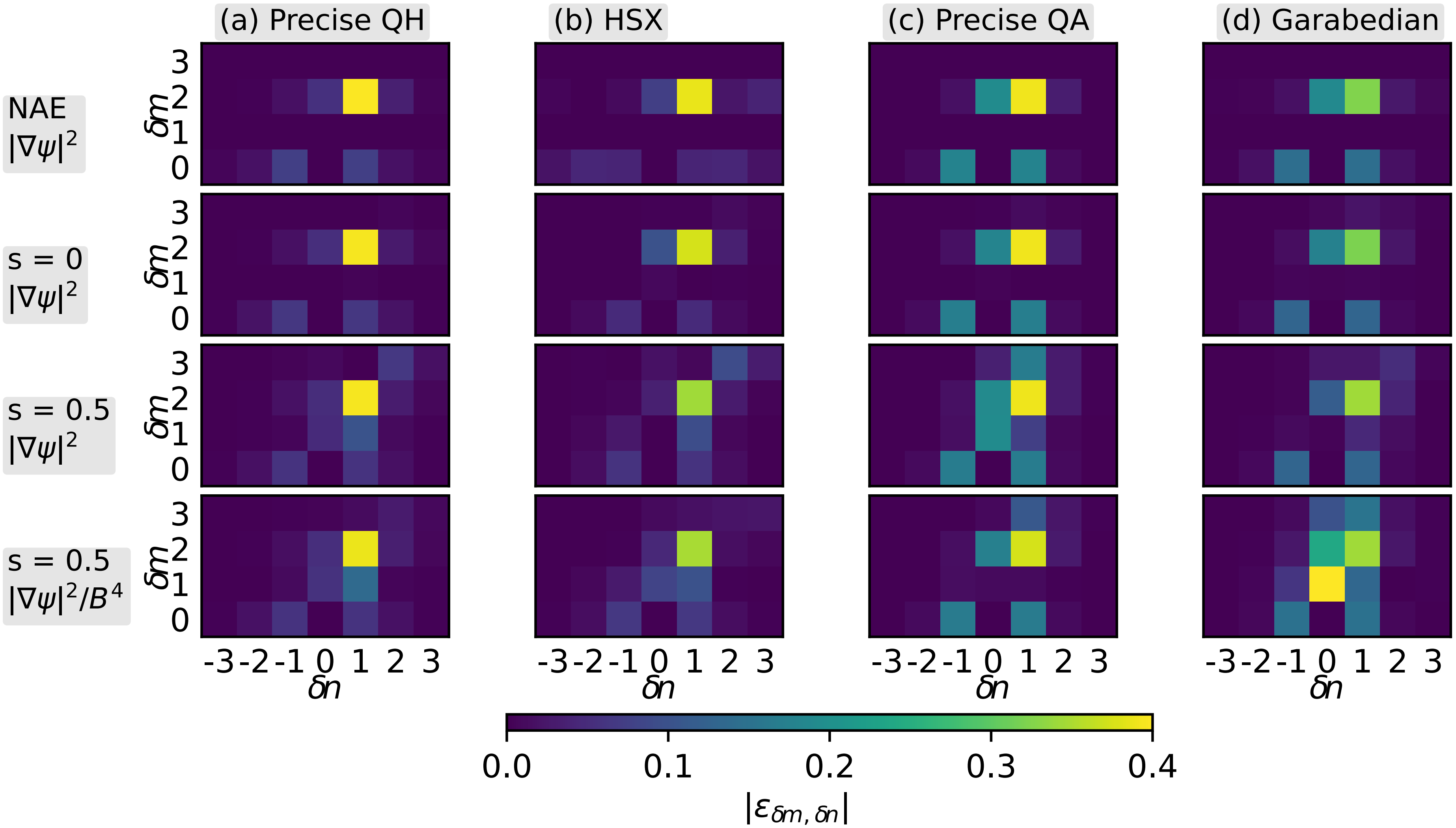
- Given that ellipse typically makes one half-rotation per field period, **produces helical ($\boldsymbol{m=2, n=N_P}$) coupling** [Kolesnichencko, 2001]





Rotating ellipticity dominates spectral content

- Toroidal variation of elongation and curvature drive MAE and EAE modes



Outline

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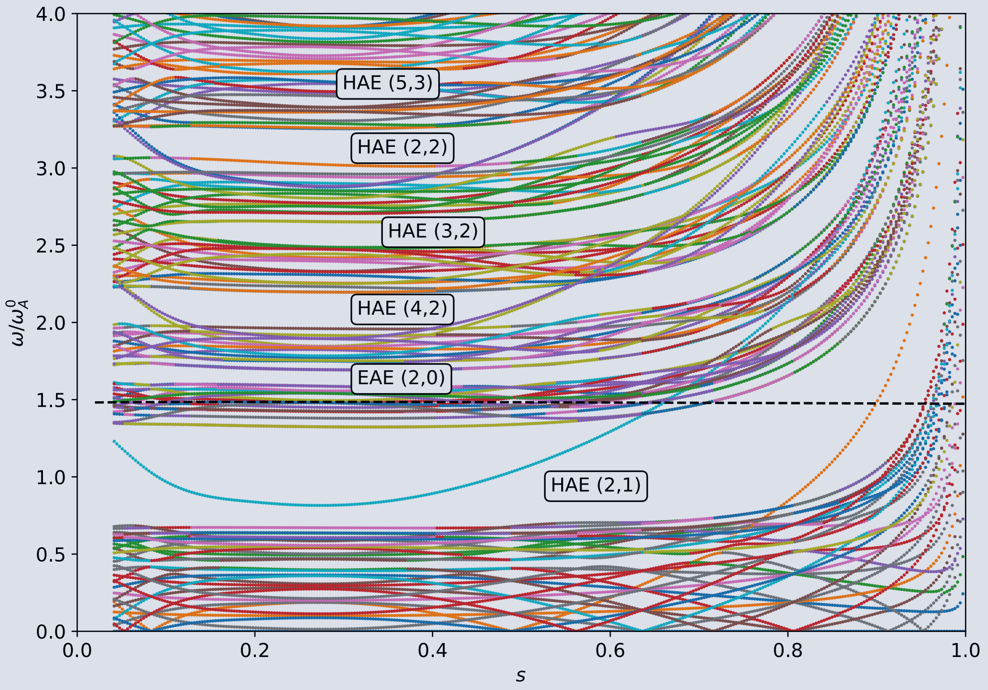

Resonance analysis for configurations close to QS [Paul, 2023]

- Assume configuration close to QS with $B(\psi, \chi = \theta - N\zeta)$
- Passing particle dynamics characterized by transit frequency profiles $\omega_\theta, \omega_\zeta, \omega_\chi$

e.g., $\omega_\theta = \langle \dot{\theta} \rangle$

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- Passing particle dynamics characterized by transit frequency profiles $\omega_\theta, \omega_\zeta, \omega_\chi$

$$\text{e.g., } \omega_\theta = \langle \dot{\theta} \rangle$$

- Resonance condition with SAW s.t. wave phase is constant in particle frame:

$$\Omega_l = (m + l)\omega_\chi - (n - Nm)\omega_\zeta + \omega = 0$$

  - Coupling through magnetic drifts introduces multiple resonant surfaces through l
  - Strongest resonance for $l = 0, 1$

A strategy for resonance avoidance [Paul, 2025]

- At typical reactor scale (e.g., HSR418), $n_i \approx 3 \times 10^{20}$ m/s, $B \approx 5$ T → ${}^{\omega_A}/_{\omega_\zeta} \approx 1/4$ at 3.5 MeV

  → Strong passing resonance requires $\omega/\omega_A > 1$

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  → Strong passing resonance requires $\omega/\omega_A > 1$

$$\boxed{\overline{\omega} > |\iota - N|/2}$$

- ***Strategy***:
  - Preferentially promote low-frequency gaps to
    - ✓ Avoid resonance at birth energy
    - ✓ Avoid wide gaps ($\Delta\omega \propto \omega$)
  - Reduce ω_A (e.g., high-density, low field)

Optimization of the shear Alfvén continuum

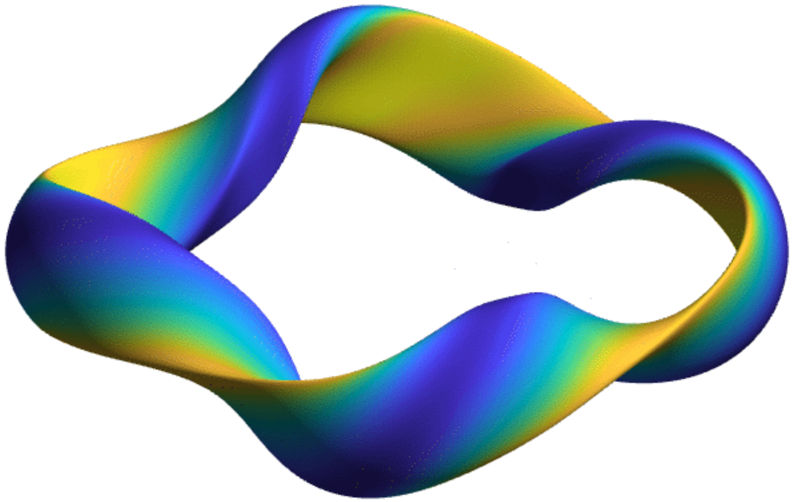
- Fixed-boundary optimization with SIMSOPT to reduce high-frequency gap spectral content (while maintaining QS, aspect ratio):

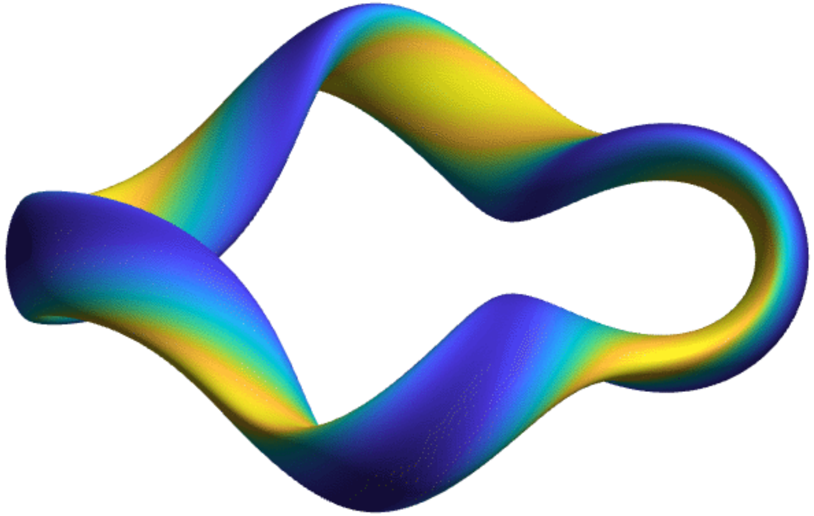
$$f(S_P) = (A(S_P) - A^*)^2 + f_{QS}(S_P) + f_\iota(S_P) + f_{\text{cont}}(S_P)$$

  - f_{QS}: two-term quasisymmetry error
  - f_ι: enforce $\iota \geq 1.03$
  - $f_{\text{cont}} = \sum_s \sum_{\underbrace{|\iota\delta m - N_P\delta n| > |\iota - N|}_{\text{Resonance condition}}} \underbrace{\left|\epsilon_{\delta m,\delta n}\right|^2 (\iota\delta m - N_P\delta n)^2}_{\propto(\text{Gap width})^2}$

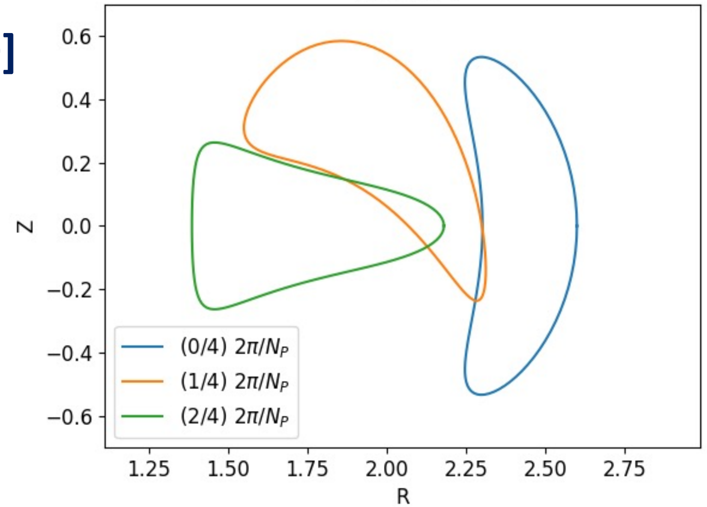
Optimization of the shear Alfvén continuum

- High-order shaping components reduced

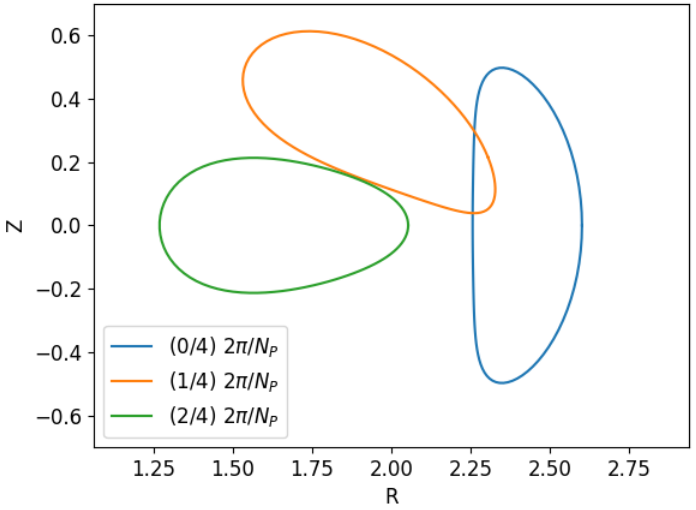




Wistell-A [Bader, 2020]

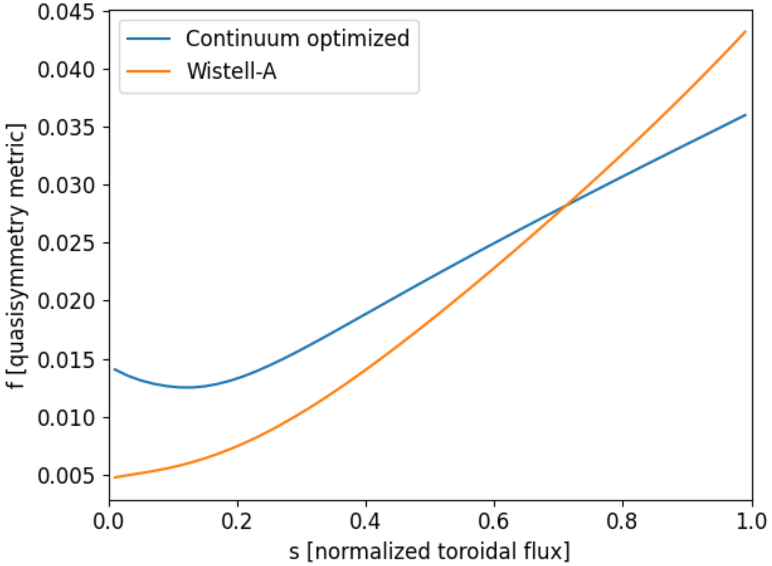


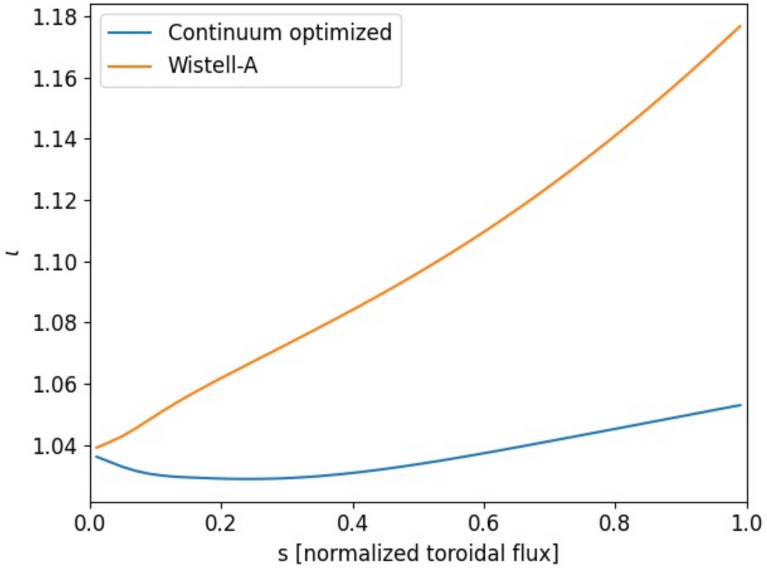
Optimized

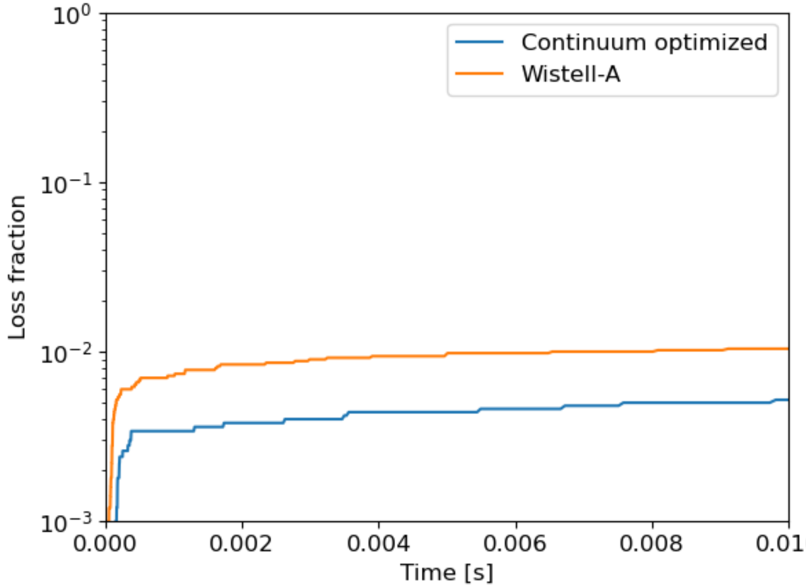


Optimization of the shear Alfvén continuum

- EP confinement (without MHD activity) maintained







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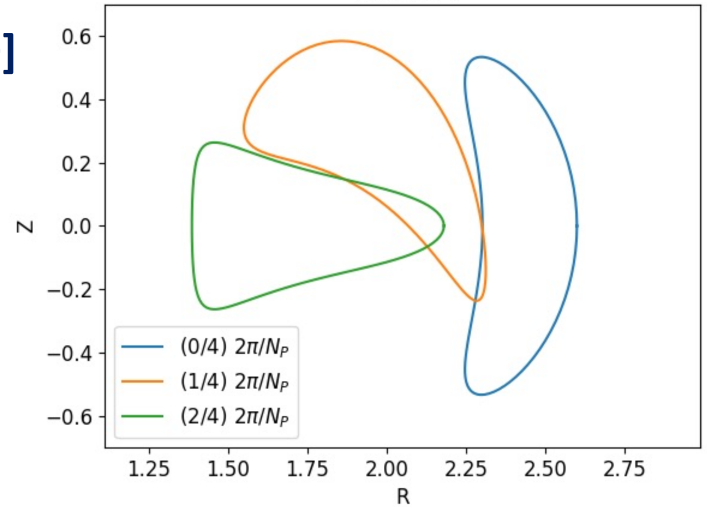

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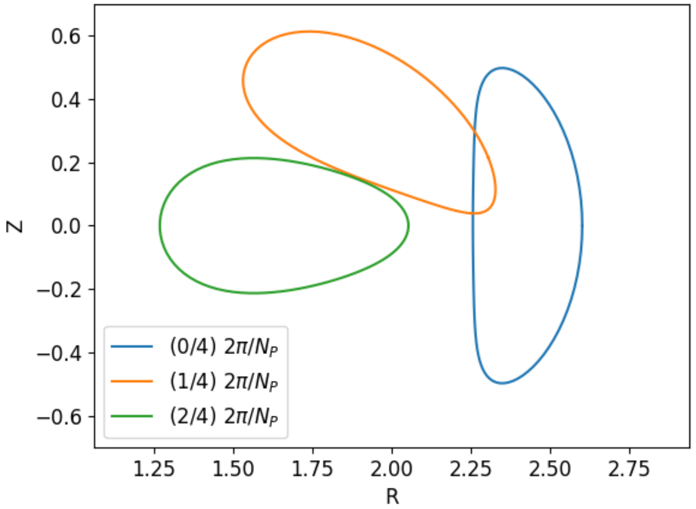
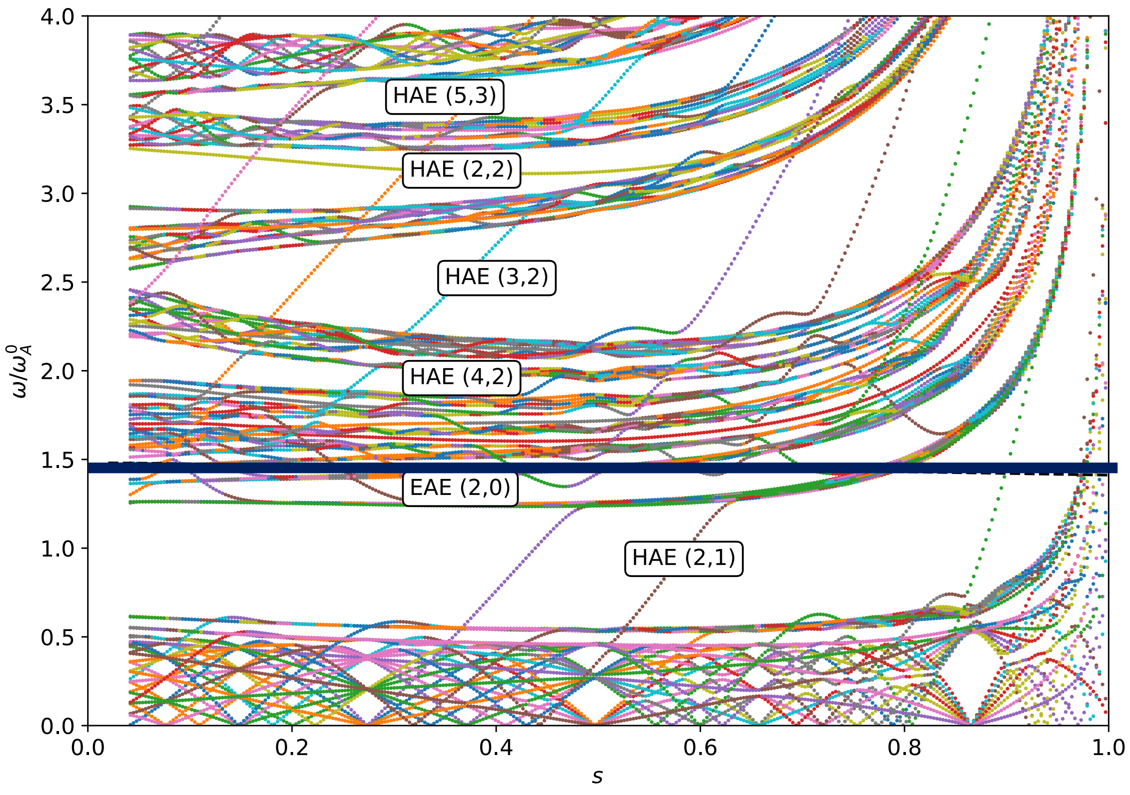
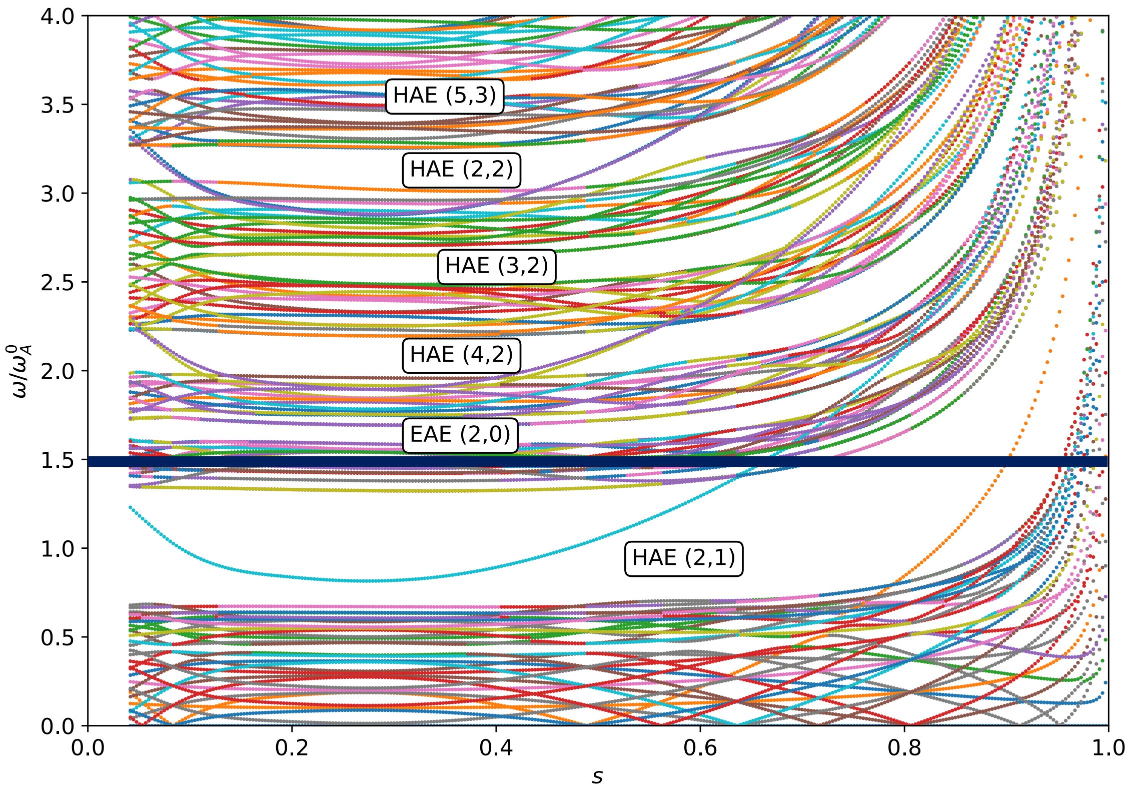


Optimization of the shear Alfvén continuum

- Stellgap calculations confirm width of high-frequency HAE gaps reduced



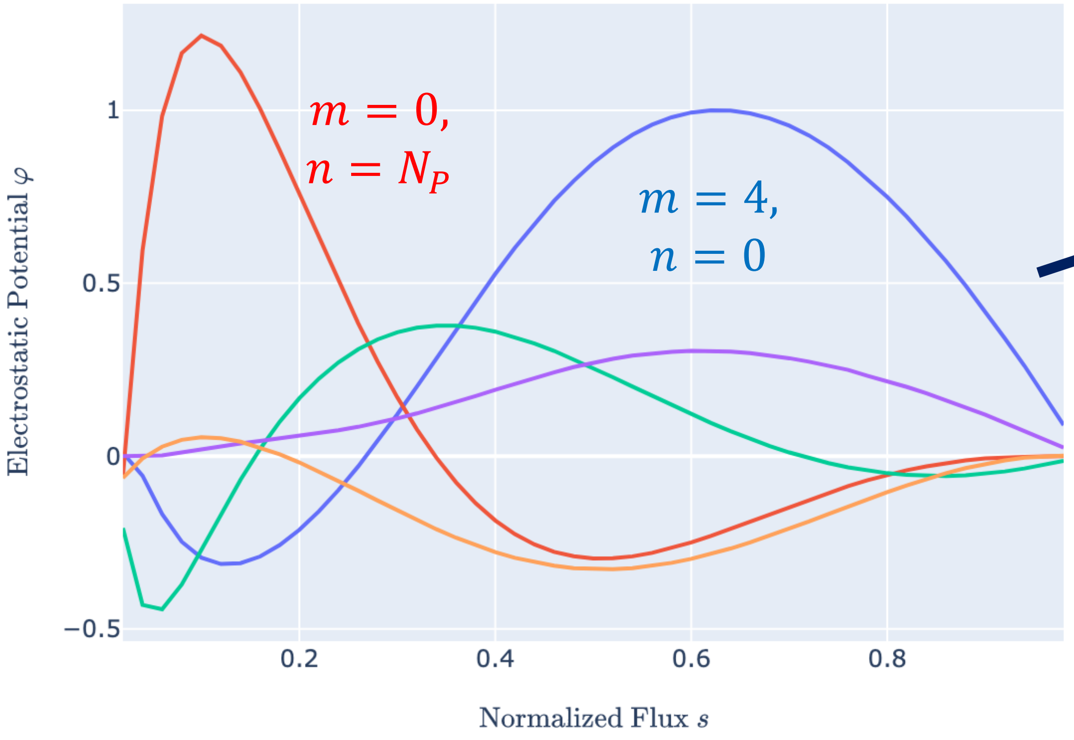
Wistell-A

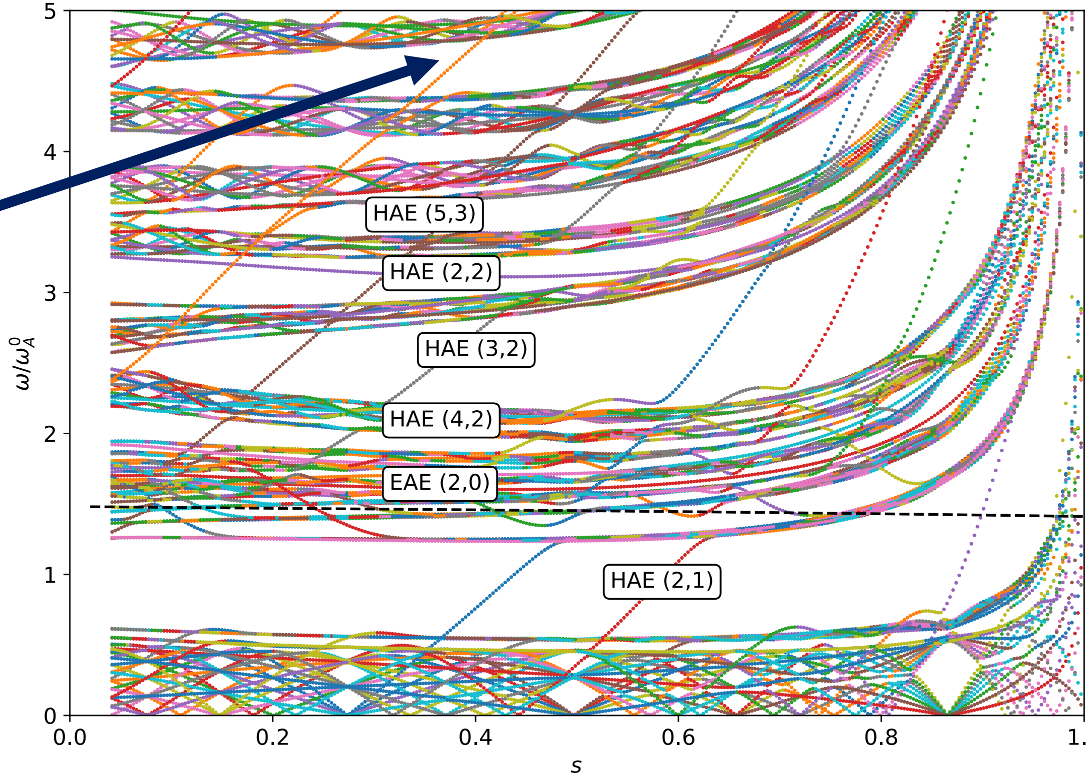


Optimized

Alfvén gap eigenmodes computed with AE3D [Spong, 2010]

$\overline{\omega}_A = 4.52$





Wistell-A

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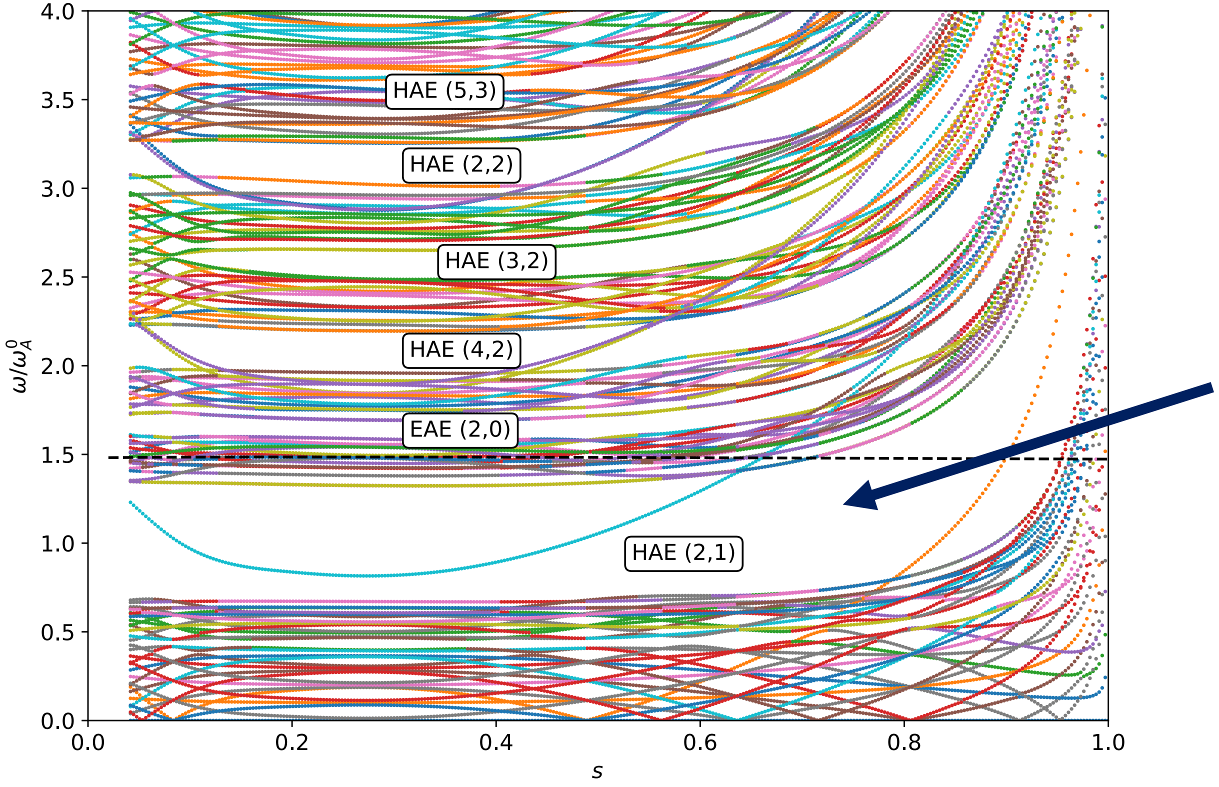
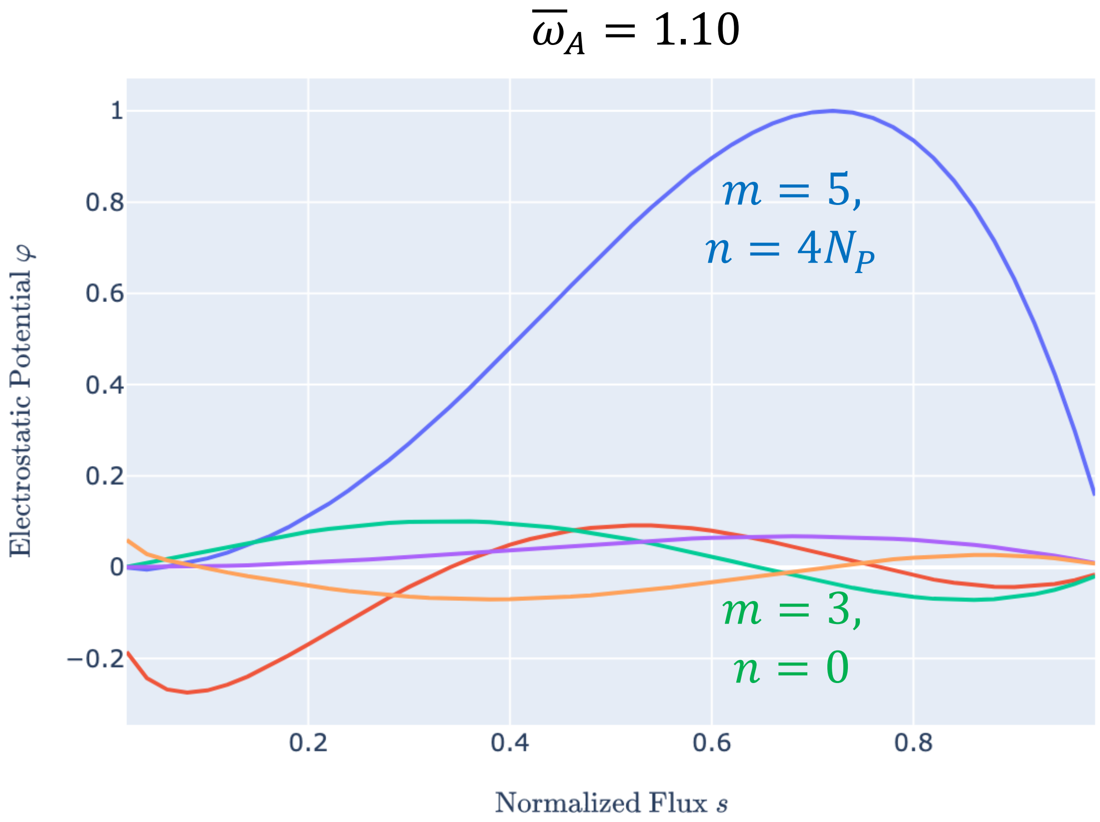
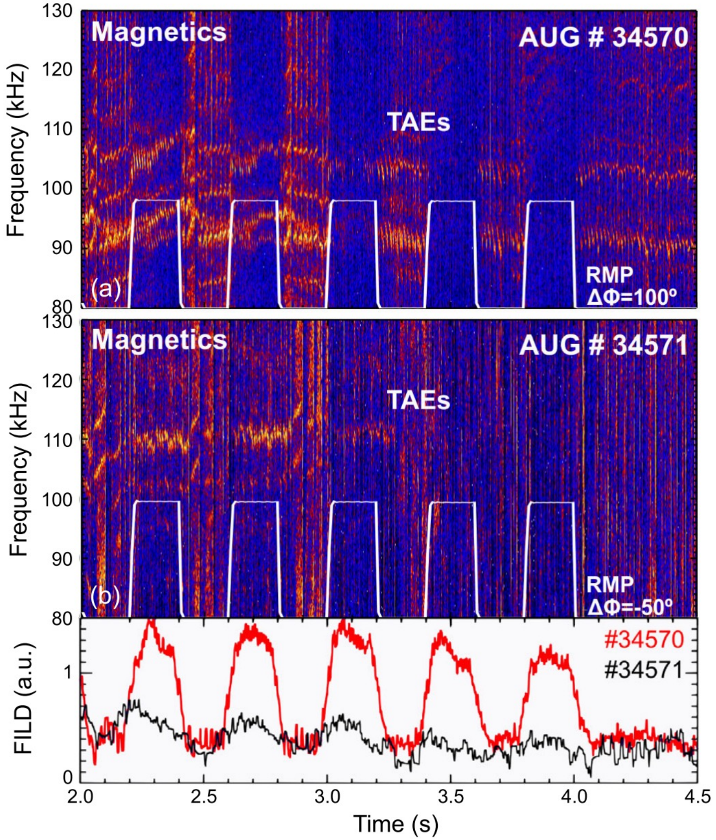




Optimized

Conclusions and future work

- Geometry can manipulate the continuum to promote stability, by reducing the width of high-frequency gaps which may strongly resonate with EPs
  - Application: control of AEs in tokamaks?



J. Gonzalez-Martin et al, *PRL* 130 (2023)

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  - Application: control of AEs in tokamaks?
  - Ongoing work: validation of optimization with FAR3D stability analysis
  - Future work: generalization of optimization approach to use spectral density [Weisse, 2006] with full continuum