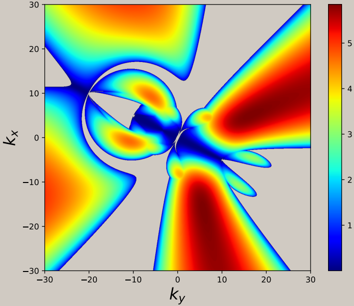
Informal update on the strange behaviour of TAI turbulence



Meyrand Romain

University of New Hampshire

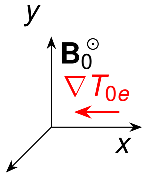
Motivation

- Temperature gradients in magnetized plasmas drive **microinstabilities**.
- These instabilities extract free energy from the background profile, generating **turbulence**.
- The resulting turbulence causes **anomalous heat transport**, often orders of magnitude above neoclassical predictions.
- **Why it matters:** This enhanced heat loss makes it significantly harder to maintain the plasma temperature required for fusion.
- Our focus is on turbulence driven by the **Thermo-Alfvénic Instability (TAI)**, studied within a local slab model of tokamak-like plasma.

Physical Setup

- **Magnetised, low-beta plasma** with $\beta_e \sim \frac{m_e}{m_i}$
- Governing equations derived via a **low-beta asymptotic limit** of gyrokinetics
- Electrons described by the **drift-kinetic approximation** (small ρ_e)
- Presence of an **equilibrium electron-temperature gradient**
- All other equilibrium quantities assumed **uniform**
- Focus on scales where **magnetic field lines are frozen into electron flow**, i.e. $k_\perp d_e < 1$
- Dynamics dominated by the **thermo-Alfvénic instability (TAI)**

Model Equations



(Zocco & Schekochihin 2011 + temperature gradient)

The dynamics are encoded in the drift-kinetic equation:

$$\frac{d\,\delta f_e}{dt} + v_\parallel \nabla_\parallel \delta f_e = \frac{e}{T_{0e}} \left(\frac{1}{c} \frac{\partial A_\parallel}{\partial t} + \nabla_\parallel \varphi \right) v_\parallel f_{0e} + C[\delta f_e]$$

+ background electron temperature gradient

Perturbations are advected by the $\boldsymbol{E} \times \boldsymbol{B}$ drift:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_E \cdot \nabla_\perp, \quad \boldsymbol{v}_E = \frac{\rho_e v_{\mathrm{the}}}{2} \boldsymbol{b}_0 \times \nabla_\perp \varphi, \quad \varphi = \frac{e\varphi}{T_{0e}}.$$

Their parallel motion is along the exact magnetic field:

$$\nabla_\parallel = \boldsymbol{b} \cdot \nabla = \frac{\partial}{\partial z} + \frac{\delta \boldsymbol{B}_\perp}{B_0} \cdot \nabla_\perp, \quad \frac{\delta \boldsymbol{B}_\perp}{B_0} = -\rho_e \boldsymbol{b}_0 \times \nabla_\perp \mathcal{A}, \quad \mathcal{A} = \frac{A_\parallel}{\rho_e B_0}.$$

Hermite Expansion

No explicit dependence on $v_\perp \Rightarrow$ it can be integrated out $\Rightarrow$ **reduced 4D description**

The kinetic equation can be recast as an **infinite hierarchy of Hermite moments**:

$$\frac{dg_m}{dt} + \frac{v_{\mathrm{the}}}{\sqrt{2}} \nabla_\parallel \left(\sqrt{m+1}\, g_{m+1} + \sqrt{m}\, g_{m-1} \right) = \mathrm{RHS}.$$

This formulation replaces the kinetic equation with a coupled set of fluid-like moment equations.

In the linear regime, the coupling via parallel streaming leads to phase mixing. Energy cascades to high Hermite modes, corresponding to increasingly fine velocity-space structure.

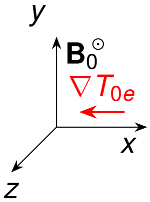
Electron Continuity Equation

$$\frac{d}{dt}\frac{\delta n_e}{n_{0e}} + \nabla_\parallel u_{\parallel e} = 0$$

The density perturbation evolves due to two main effects:

1. **Advection by the $\boldsymbol{E} \times \boldsymbol{B}$ drift**
2. **Compression or rarefaction from parallel electron flow along the perturbed magnetic field**

The Parallel Momentum Equation



The electron parallel momentum equation reads:

$$n_{0e} m_e \frac{du_{\|e}}{dt} = -\nabla_\| p_{\|e} - e n_e E_\| .$$

The two forces on the right-hand side are:

1. Parallel pressure gradient:

$$\nabla_\| p_{\|e} = \nabla_\| \delta p_{\|e} + n_{0e} \frac{\delta B_x}{B_0} \frac{dT_{0e}}{dx} = n_{0e} T_{0e} \left[\nabla_\| \left(\frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\|e}}{T_{0e}} \right) - \frac{\rho_e}{L_T} \frac{\partial \mathcal{A}}{\partial y} \right].$$

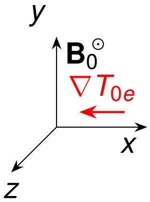
2. Parallel electric field:

$$E_\| = \mathbf{b} \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial A_\|}{\partial t} + \nabla_\| \varphi = -\frac{1}{c} \frac{dA_\|}{dt} + \frac{\partial \varphi}{\partial z}.$$

Relation between parallel current and electron velocity:

$$-e n_{0e} u_{\|e} = j_\| = \frac{c}{4\pi} \mathbf{b}_0 \cdot (\nabla_\perp \times \delta \mathbf{B}_\perp), \quad \Rightarrow \quad u_{\|e} = v_{\mathrm{the}} d_e^2 \nabla_\perp^2 \mathcal{A}.$$

Parallel Electron Temperature



The parallel electron temperature is

$$T_{\|e} = T_{0e} + \delta T_{\|e},$$

which is advected by the local $\mathbf{E} \times \mathbf{B}$ flow and modified by compressional heating or cooling from $u_{\|e}$, as well as by the perturbed parallel heat flux $\delta q_{\|e}$:

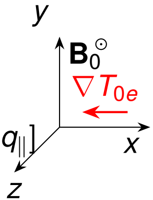
$$\frac{dT_{\|e}}{dt} = \frac{d\delta T_{\|e}}{dt} + \mathbf{v}_E \cdot \nabla_\perp T_{0e} = -\nabla_\| \frac{\delta q_{\|e}}{n_{0e}} - 2T_{0e} \nabla_\| u_{\|e}.$$

The equilibrium temperature gradient, advected by the $\mathbf{E} \times \mathbf{B}$ flow, drives temperature perturbations and extracts free energy,

$$\mathbf{v}_E \cdot \nabla_\perp T_{0e} = T_{0e} \frac{\rho_e v_{\mathrm{the}}}{2L_T} \frac{\partial \varphi}{\partial y},$$

leading to the development of the ETG instability.

Parallel Heat Flux



$$\frac{dq_\parallel}{dt} + \frac{v_{\rm the}}{\sqrt{2}}\left(2\,\hat{g}_4 + \sqrt{3}\,\delta T_{\parallel e}\right) = \sqrt{3}\,\frac{\rho_e v_{\rm the}}{2L_T}\frac{\partial A}{\partial y} + C[q_\parallel]$$

The TAI is kinetic and electromagnetic, unlike the ETG, which is fluid and electrostatic.

All higher-order moments satisfy:

$$\frac{dg_m}{dt} + \frac{v_{\rm the}}{\sqrt{2}}\nabla_\parallel\left(\sqrt{m+1}\,g_{m+1} + \sqrt{m}\,g_{m-1}\right) = C[g_m], \quad m \geq 4.$$

Quasineutrality

Particle density is related to φ via quasineutrality

$$\frac{\delta n_e}{n_{0e}} = \frac{\delta n_i}{n_{0i}} = -\frac{Z}{\tau}(1 - \hat{\Gamma}_0)\varphi \equiv -\bar{\tau}^{-1}\varphi \approx \begin{cases} \frac{Z}{2\tau}\rho_i^2 \nabla_\perp^2 \varphi, & k_\perp \rho_i \ll 1, \\ -\frac{Z}{\tau}\varphi, & k_\perp \rho_i \gg 1. \end{cases}$$

The operator $\hat{\Gamma}_0$ can be expressed in Fourier space as:

$$\hat{\Gamma}_0 = I_0(\alpha_i) e^{-\alpha_i},$$

where

$$\alpha_i = \frac{(k_\perp \rho_i)^2}{2},$$

and I_0 is the modified Bessel function of the first kind.

Summary of equations

Assembling together all of the above, we end up with the following systems of equations:

$$\frac{d}{dt}\left(\frac{\delta n_e}{n_{0e}}\right) + \nabla_\parallel u_{\parallel e} = 0, \qquad (1)$$

$$\frac{d}{dt}\left(A - \frac{u_{\parallel e}}{v_{\rm the}}\right) = -\frac{v_{\rm the}}{2}\left[\frac{\partial \varphi}{\partial z} - \nabla_\parallel \left(\frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}}\right) + \frac{\rho_e}{L_T}\frac{\partial A}{\partial y}\right], \qquad (2)$$

$$\frac{d}{dt}\left(\frac{\delta T_{\parallel e}}{T_{0e}}\right) + \nabla_\parallel \left(\frac{\delta q_{\parallel e}}{n_{0e} T_{0e}} + 2 u_{\parallel e}\right) + \frac{\rho_e v_{\rm the}}{2 L_T}\frac{\partial \varphi}{\partial y} = 0, \qquad (3)$$

$$\frac{dq_\parallel}{dt} + \frac{v_{\rm the}}{\sqrt{2}}\left(2\,\hat{g}_4 + \sqrt{3}\,\delta T_{\parallel e}\right) = \sqrt{3}\,\frac{\rho_e v_{\rm the}}{2 L_T}\frac{\partial A}{\partial y} + C[q_\parallel], \qquad (4)$$

$$\frac{dg_m}{dt} + \frac{v_{\rm the}}{\sqrt{2}}\nabla_\parallel\left(\sqrt{m+1}\,g_{m+1} + \sqrt{m}\,g_{m-1}\right) = C[g_m], \quad m \geq 4. \qquad (5)$$

$$\frac{\delta n_e}{n_{0e}} = -\bar{\tau}^{-1}\varphi, \qquad \frac{u_{\parallel e}}{v_{\rm the}} = d_e^2 \nabla_\perp^2 A$$

Instabilities

This sytem support a range of temperature-gradient-driven instabilities, both electrostatic and electromagnetic, distinguished by whether the magnetic field lines are frozen into the electron flow.

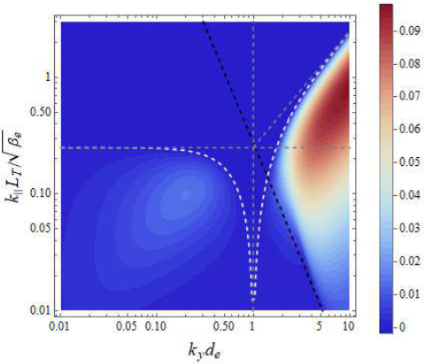


Figure: The growth rates of the collisionless instabilities (Adkins et al. 2022).

Free-energy cascades

The free energy is a nonlinear invariant of the system, i.e., it is conserved by nonlinear interactions , but can be injected into the system by equilibrium gradients, and is dissipated by collisions:

$$\frac{W}{n_{0e}T_{0e}} = \int d^3\frac{\mathbf{r}}{V}\left(\frac{\varphi\bar{\tau}^{-1}\varphi}{2} + |d_e\nabla_\perp A|^2 + \frac{1}{2}\frac{\delta n_e^2}{n_{0e}^2} + \frac{u_{\parallel e}^2}{v_{\mathrm{the}}^2} + \frac{1}{4}\frac{\delta T_{\parallel e}^2}{T_{0e}^2} + \dots\right)$$

$$\frac{1}{n_{0e}T_{0e}}\frac{dW}{dt} = \varepsilon_{TAI} + \varepsilon_{ETG} - D$$

$$\varepsilon_{ETG} = -\frac{1}{2L_T}\int\frac{d^3\mathbf{r}}{V}\frac{\delta T_{\parallel e}}{T_{0e}}\frac{\rho_e v_{\mathrm{the}}}{2}\frac{\partial\varphi}{\partial y}$$

$$\varepsilon_{TAI} = \frac{v_{\mathrm{the}}}{2L_T}\int\frac{d^3\mathbf{r}}{V}\frac{\delta q_{\parallel e}}{n_{0e}T_{0e}}\sqrt{3}\rho_e\frac{\partial A}{\partial y}$$

KAW turbulence

In the wave-number range where the ETG instability is suppressed by magnetic tension,

$$k_\perp d_e \ll 1,$$

the dominant plasma perturbations are KAW-like.

$$\omega_{\mathrm{KAW}} \sim k_{\|} v_{\mathrm{the}} k_\perp d_e \sim t_{\mathrm{nl}}^{-1} \sim\sim \rho_e v_{\mathrm{the}} k_\perp^2 \varphi_\perp .$$

A Kolmogorov-style constant-flux argument leads to the scaling of the amplitudes in the inertial range:

$$\bar{\tau}^{-1} \frac{\varphi_\perp^2}{t_{\mathrm{nl}}} \sim \varepsilon = \mathrm{const} \quad \Rightarrow \quad \varphi_\perp \sim \left(\frac{\varepsilon}{\Omega_e}\right)^{1/3} (k_\perp \rho_e)^{-2/3}.$$

This scaling translates into the following 1D spectrum:

$$E_{\varphi_\perp}(k_\perp) \sim \frac{\varphi_\perp^2}{k_\perp} \propto k_\perp^{-7/3}$$

TAI Turbulence

(Adkins et al. 2022)

In standard KAW turbulence, energy cascades from large to small scales.

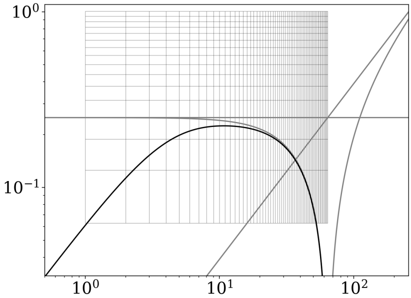
In contrast, TAI turbulence is driven directly at small (sub-ρ_i) scales by the TAI instability, which injects energy into KAWs at a rate:

$$\gamma \sim \omega_{\mathrm{KAW}} \sim \frac{v_{the}}{L_T\sqrt{\beta_e}}(k_\perp \rho_e)^2$$

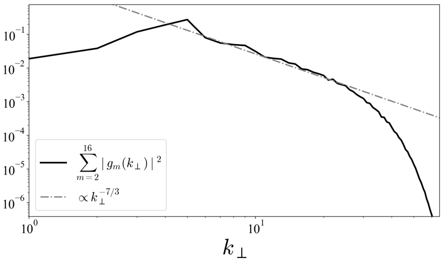
Because γ grows faster with $k_\perp$ than the nonlinear transfer rate $\Rightarrow$ **an inertial range cannot develop.**

Unless some mechanism exists to transfer energy back to larger scales?

Numerical Experiments



Grid layout



Energy spectrum

Wait — this looks just like standard KAW turbulence with energy injected at large scales!

Upscale Transfer in TAI Turbulence

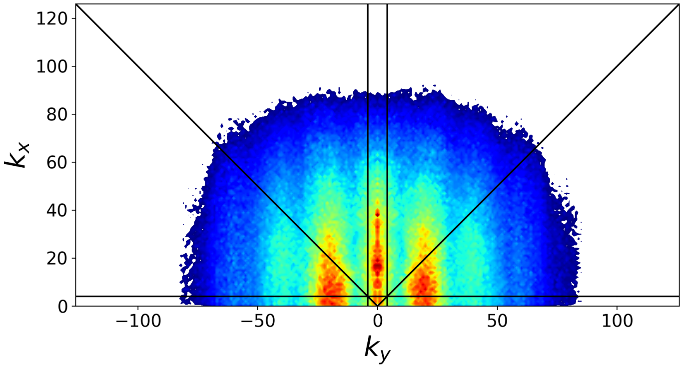
How does energy reach larger scales?

- Nonlinear modulation instability?
- Inverse cascade enabled by an additional inviscid invariant?

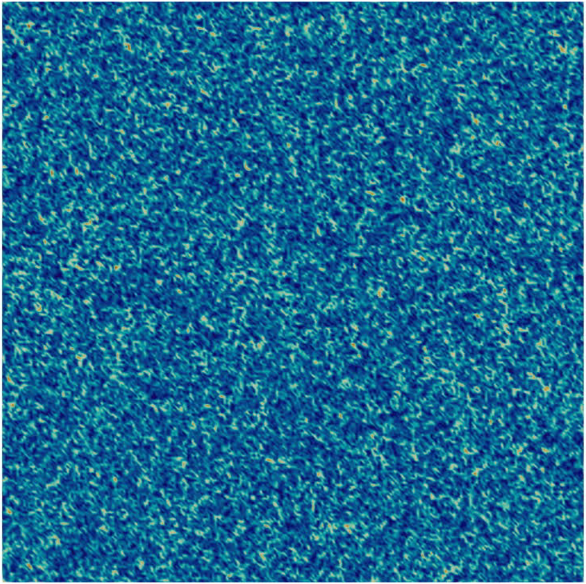
Zonation

Before saturation, a **modulational instability** drives **non-local energy transfer** in the k_y direction, funneling energy from large k_y modes directly to $k_y = 0$.

As a result, the system spontaneously self-organizes into coherent **zonal flows**. This process also occurs with Boltzmann ions.



Zonation Animation



Helicity and inverse cascade

In addition to the free energy, the electromagnetic system of equations has a second (nonlinear) “invariant” the **generalized helicity**:

$$H = \int \frac{\mathrm{d}^3\mathbf{r}}{V} \frac{\delta n_e}{n_{0e}} \left(A - \frac{u_{\parallel e}}{v_{\mathrm{the}}} \right)$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \varepsilon_H - D_H$$

where the generalized helicity injection rate is

$$\varepsilon_H = \frac{v_{\mathrm{the}}}{2} \int \frac{\mathrm{d}^3\mathbf{r}}{V} \frac{\delta n_e}{n_{0e}} \left(\nabla_\parallel \frac{\delta T_{\parallel e}}{T_{0e}} - \frac{\rho_e}{L_T} \frac{\partial A}{\partial y} \right)$$

The existence of this additional invariant could lead to an **inverse cascade** (see Adkins et al. 2025).

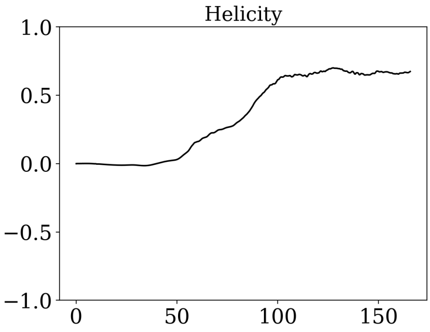
Parity Symmetry

The system under consideration possesses a parity symmetry:

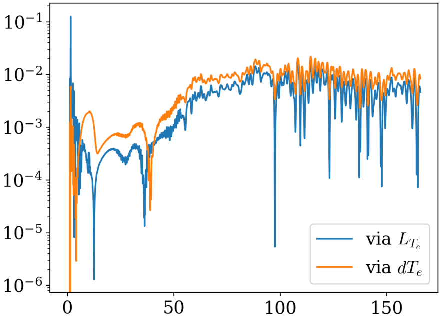
$$(x, y, z, v_{\|}, \delta f_e, \varphi, A, \delta B_{\|}) \ \rightarrow \ (-x, y, -z, -v_{\|}, -\delta f_e, -\varphi, A, -\delta B_{\|})$$

The generalized helicity is odd under the parity transformation $\varepsilon_H \ \rightarrow \ -\varepsilon_H$, and the same applies to the generalized helicity itself. Therefore, in a statistically homogeneous turbulent state, both quantities must vanish unless parity is broken, either by an external agent or spontaneously.

Spontaneous Symmetry Breaking

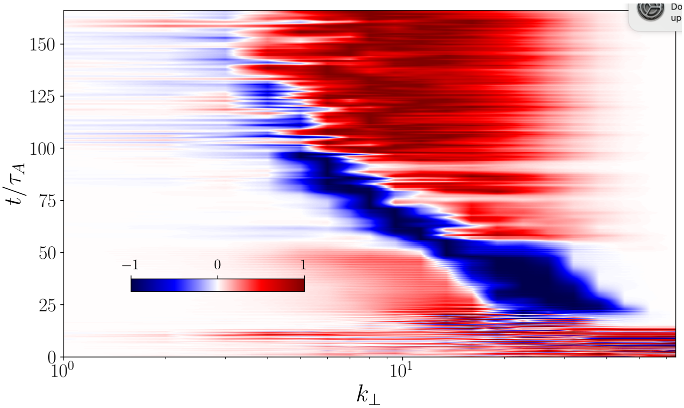


Helicity vs Time



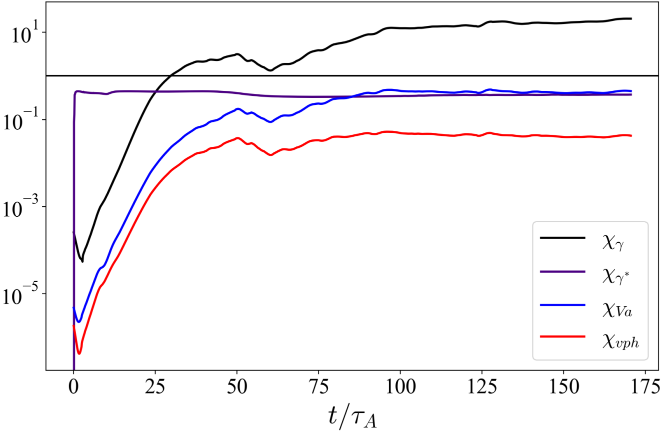
Helicity Injection

Energy Flux



Nomralized energy flux versus time. Blue corresponds to inverse cascade (energy transfer to larger scales), while red corresponds to direct cascade (energy transfer to smaller scales).

Saturation



The nonlinear rate, which here characterizes outer large scales, and the maximum linear growth rate, a small-scale quantity, are equal.

Looking Ahead

- What causes the spontaneous symmetry breaking?
- How does the heat flux scale with the temperature gradient?
- Could TAI turbulence interfere with or modify the dynamics of ITG turbulence?
- Any suggestions or insights?