ION-ACOUSTIC PHASE SPACE TURBULENCE

Rishin Madan, Matthew W. Kunz, Robert J. Ewart, Michael L. Nastac, Alexander A. Schekochihin, Vladimir Zhdankin

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Motivation: a simple example of a turbulent collisionless plasma interacting with microscales.

THE VLASOV-BOLTZMANN SYSTEM

Ions:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{1}$$

Gauss with isothermal Boltzmann Electrons:

$$\exp\left(\frac{e\phi}{T_e}\right) = \frac{n_i}{n_0} + \lambda_{De,0}^2 \frac{\partial^2}{\partial x^2}\frac{e\phi}{T_e} \tag{2}$$

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$\frac{e\phi}{T_e} \sim : \delta \ll 1.$

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$$\phi = \frac{n_i}{n_0} - 1 + \epsilon^2 \frac{\partial^2 \phi}{\partial x^2} + \mathcal{O}(\delta^2) \qquad (2)$$

$c_s := \sqrt{T_e/m} \to 1$

$\phi \sim: \delta \ll 1$

$\epsilon = \lambda_{De,0}/L \ll 1$

SIMULATION DETAILS

Ions through PIC (Pegasus++):

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- For $T_e/T_i \ll 1$, a phase space inertial range forms, possibly governed by a ponderomotive potential

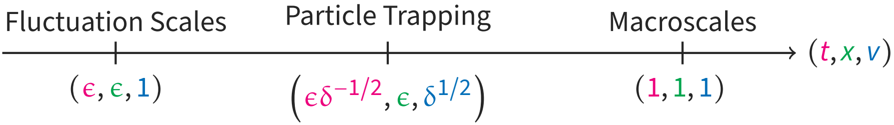
PHASE SPACE TURBULENCE: $T_e/T_i = 0.2$

MULTIPLE SCALES

Fluctuation Scales — Particle Trapping — Macroscales → (t, x, v)

- Fluctuation Scales: $(\omega_{pi}^{-1}, \lambda_{De}, c_s)$
- Particle Trapping: $\left(\lambda_{De}\sqrt{\frac{m}{e|\phi|}}, \lambda_{De}, \sqrt{\frac{e|\phi|}{m}}\right)$
- Macroscales: $(T_{\text{ext}}, L_{\text{ext}}, c_s)$

MULTIPLE SCALES



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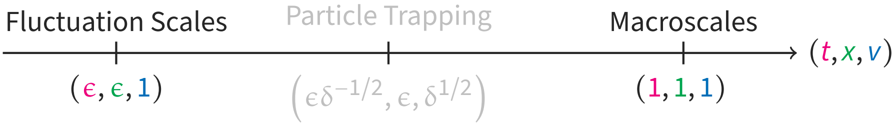

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$\tilde{f}, \tilde{\phi} = \mathcal{O}(\delta)$

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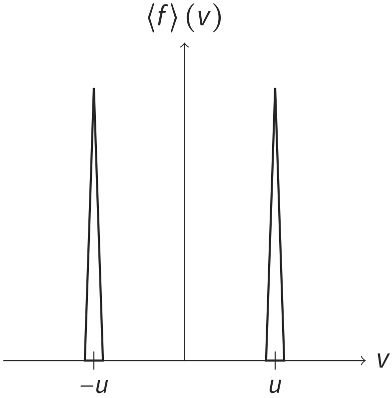

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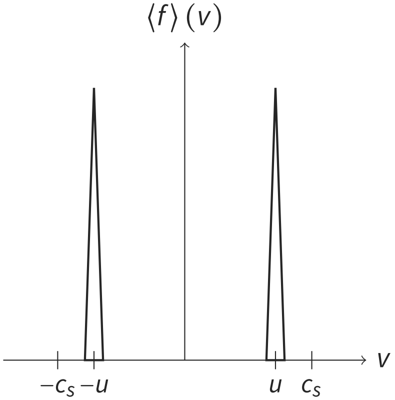

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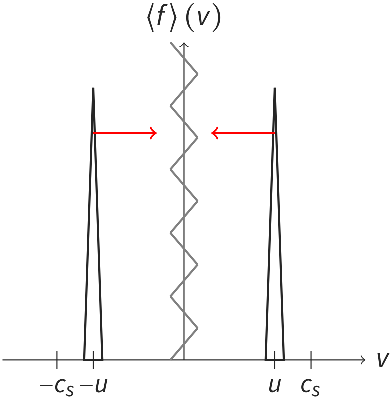

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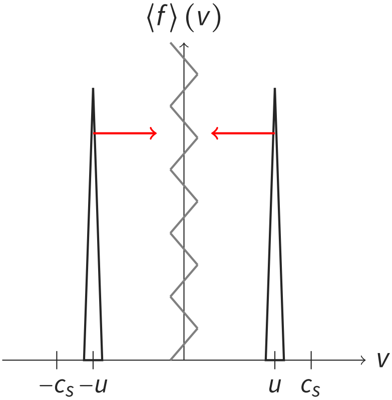

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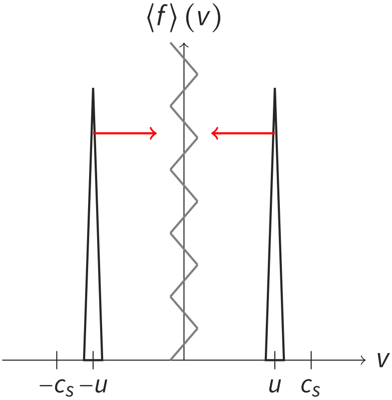

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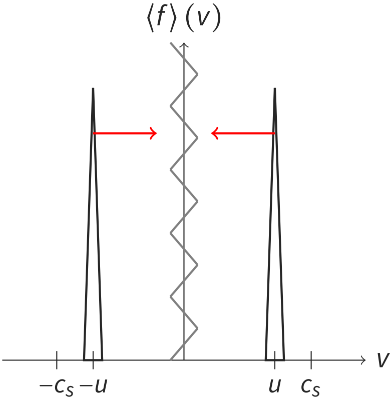

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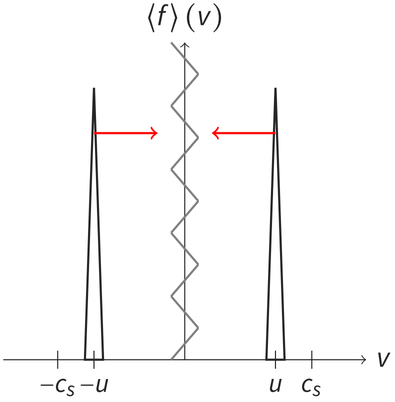

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Energy lost by beams goes into fastest growing mode:

$\Delta(u^2) \propto -\Delta((1 + \epsilon^2 k^2)|\tilde{\phi}_k|^2)$.

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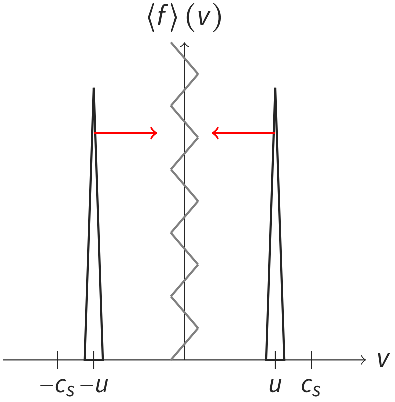

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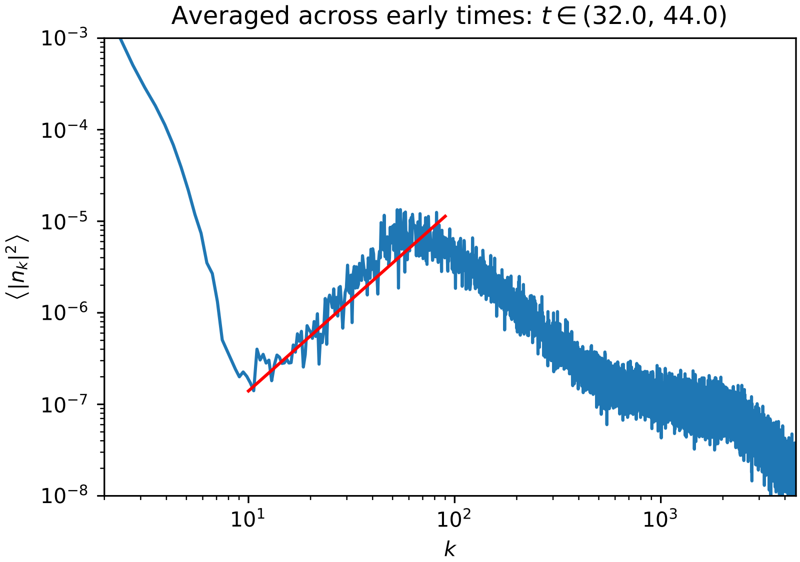
Prediction: $|\tilde{\boldsymbol{\phi}}_{\boldsymbol{k}}|^{\mathbf{2}} \propto \boldsymbol{k}^{\mathbf{2}}$ for for $L^{-1} \ll k \ll \lambda_{De}^{-1}$.

COMPARISON BETWEEN TWO-STREAM QL AND SIMULATION



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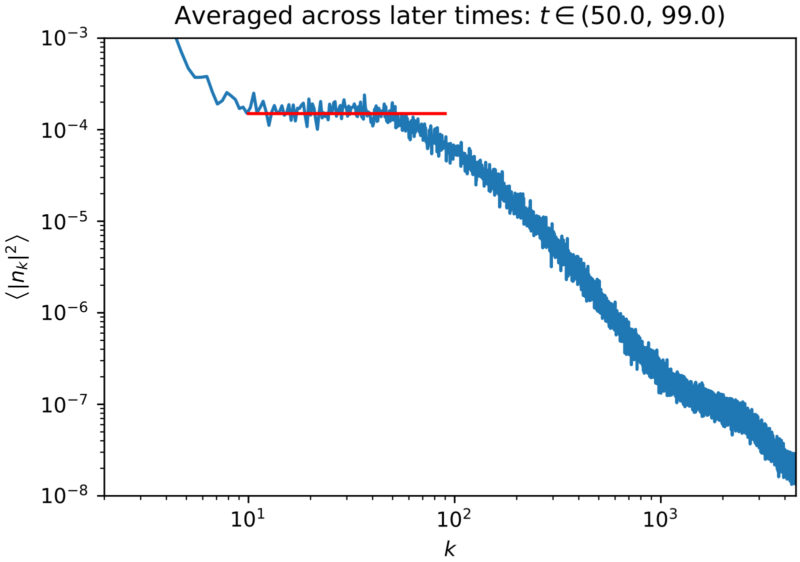
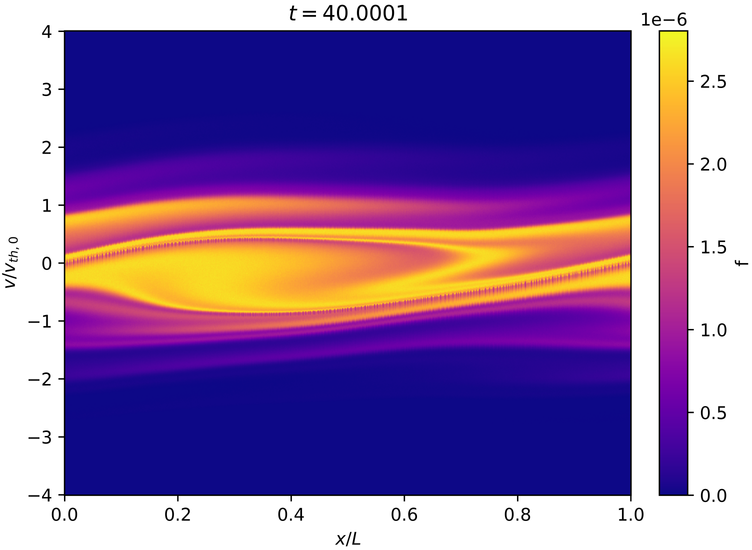


VISUAL EVIDENCE OF HOLES



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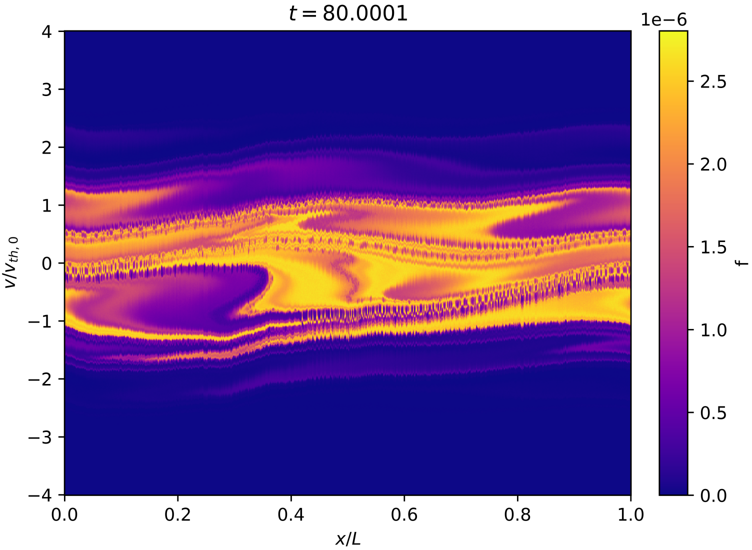

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- Analytical fix: for fluctuation Vlasov equation, include a boundary layer in velocity space at $v = v_p$

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- $\tilde{F} \sim \delta^{1/2}$, whilst $\tilde{f}_{\text{out}} \sim \delta$, so $\tilde{F}$ and $\tilde{f}_{\text{out}}$ contribute to $\tilde{\phi}$ equally

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- 'Diffusion' on scales $\Delta v \sim \delta^{1/2} c_s$ (averaging over holes)
- *An anomalous collisionality* that erases structures $\mathcal{O}(c_s)$ on trapping timescales.

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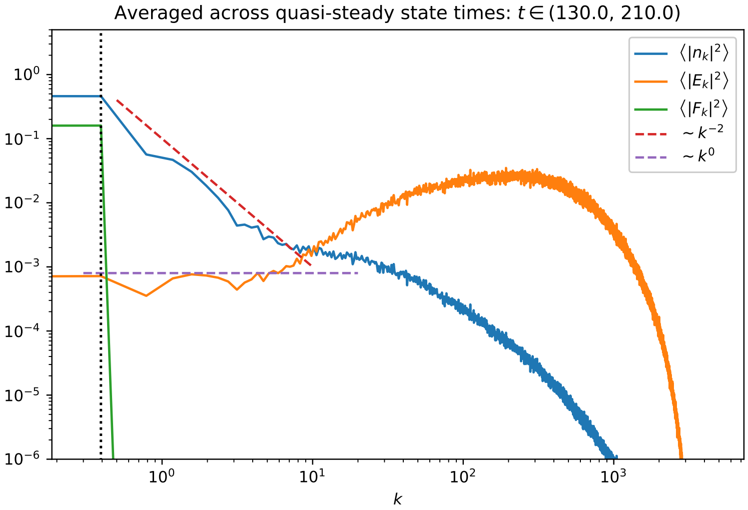
Can treat this as a cascade of entropy → theory for $|\hat{f}(k,s)|^2$ and thus $|\hat{n}(k)^2|$, à la Nastac *et al.* 2025?

- s is Fourier dual to v

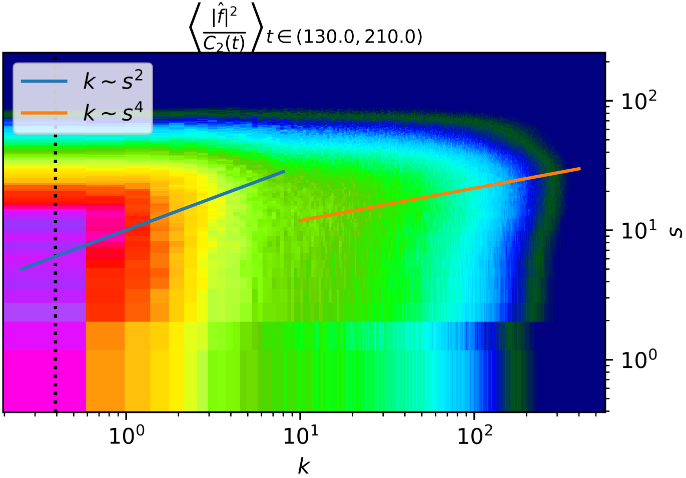
MYSTERY: $\langle \phi \rangle \ll \chi_{\mathrm{EXT}}$



Forcing is ‘critically balanced’: $k_{\mathrm{ext}} v_{\mathrm{th}} \sim k_{\mathrm{ext}} \sqrt{|\chi_{\mathrm{ext}}|} \sim \tau_{\mathrm{corr}}^{-1}$

SPECTRUM OF "ENTROPY"

Line of 'critical balance' is inconsistent with mixing dominated by outer scale forcing (Batchelor turbulence), as seen in Vlasov-Poisson (Nastac *et al.* 2025).



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- Puzzle: to construct a theory for $|\hat{f}(k,s)|^2$, need a theory for large-scale dependence of $|\tilde{\phi}|^2$
- Critical balance between linear and ponderomotive, nonlinear phase mixing $\implies \delta\left(|\tilde{\phi}|^2\right) \sim r^2$

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