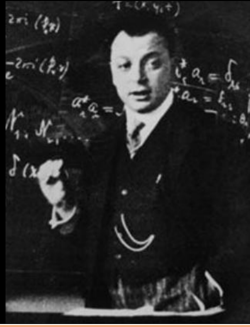


Cosmic-ray transport in inhomogeneous media

Robert J. Ewart

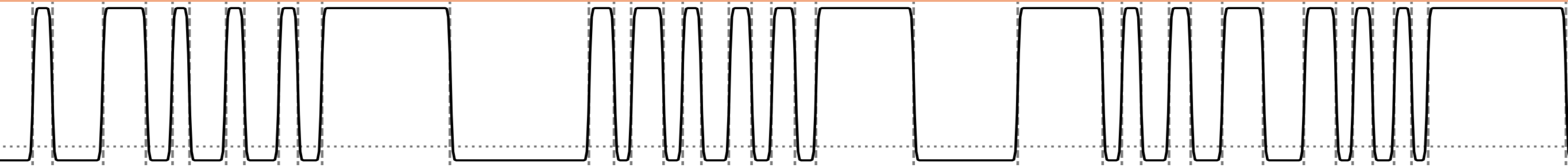
Collaborators: Patrick Reichherzer, Shuzhe Ren, Stephen Majeski, Francesco Mori, Michael L. Nastac, Archie F. A. Bott, Matthew W. Kunz, & Alexander A. Schekochihin

And the occasional question about how we could have done it better.

Reichherzer et al., 2025 Nature Astron. **9** 438
Ewart et al., 2024 MNRAS **532** 2098
Ewart et al., arXiv preprint 2507.19044
Kempski et al., arXiv preprint 2507.10651

Department of astrophysical sciences, Peyton Hall, Princeton University

16th Plasma kinetics working group meeting WPI, Vienna, July 26th 2025



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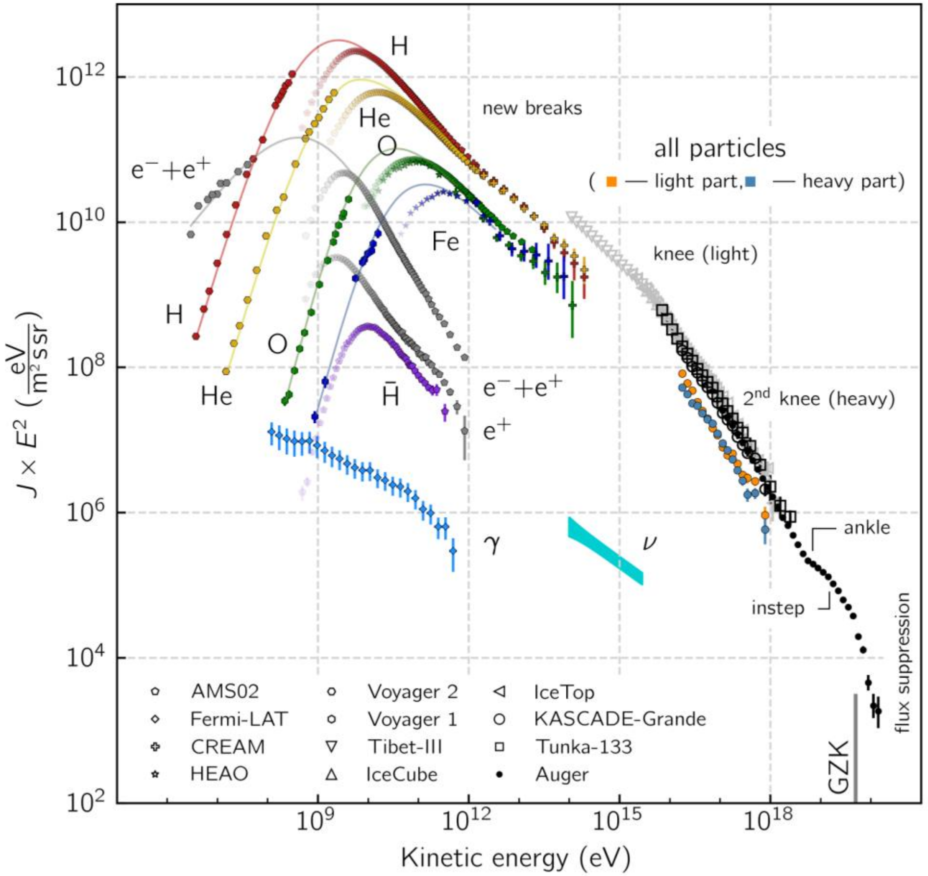
Facts of the case

Facts

- The cosmic-ray energy density is comparable to magnetic/thermal/turbulent energy densities in the galaxy. [1]
- The cosmic-ray spectrum reaching earth spans a large range of energies with a relatively constant spectral slope. [1]
- Simple models of cosmic-ray propagation in our Galaxy have, seemingly, no right to preform as well as they do. [2,3,4]

$$\frac{\partial n}{\partial t} = \kappa(E)\frac{\partial^2 n}{\partial x^2} + (\text{Sources/Sinks}) \qquad \kappa(E) \propto E^{0.3}$$

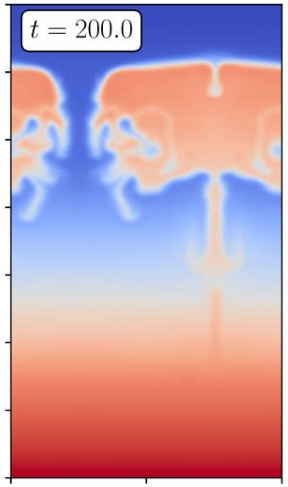


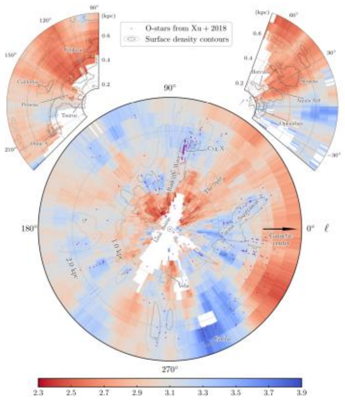
Question

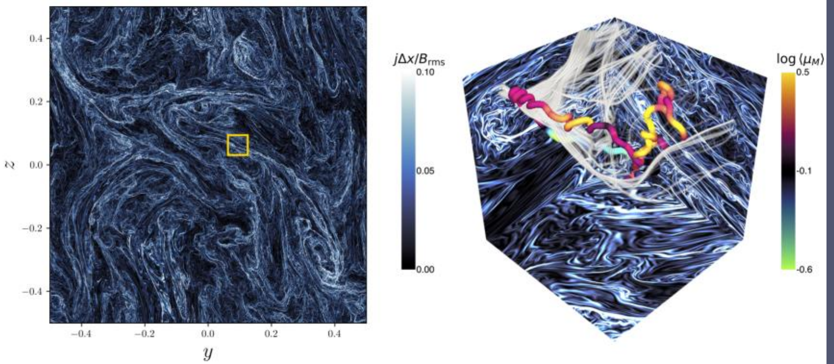
- How are cosmic-rays transported through The Galaxy/ICM?

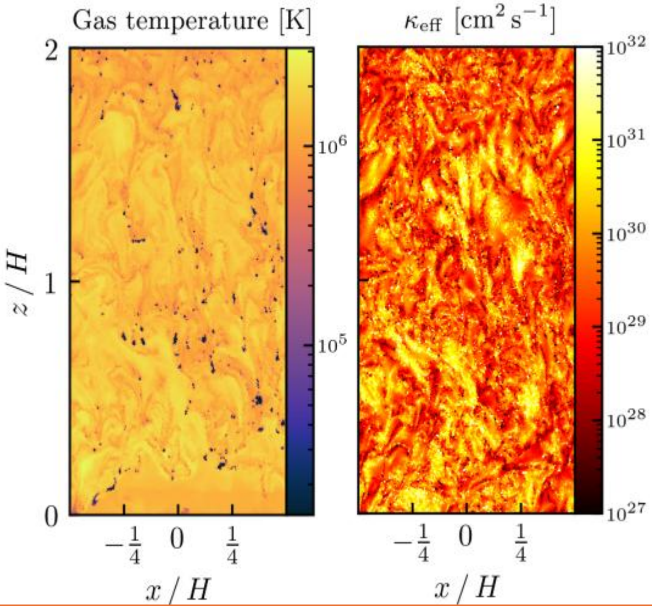
[1] Ruszkowski & Pfrommer, 2023, Astron. Astrophys. Rev. **31** 4
[2] Kempski & Quataert, 2022, Mon. Not. R. Astron. Soc. **514** 657
[3] Hopkins et al., 2022, Mon. Not. R. Astron. Soc. **517** 5413
[4] Evoli et al., 2008, JCAP **2008** 018
[5] Butsky et al., 2024, Mon. Not. R. Astron Soc., 528, 4245

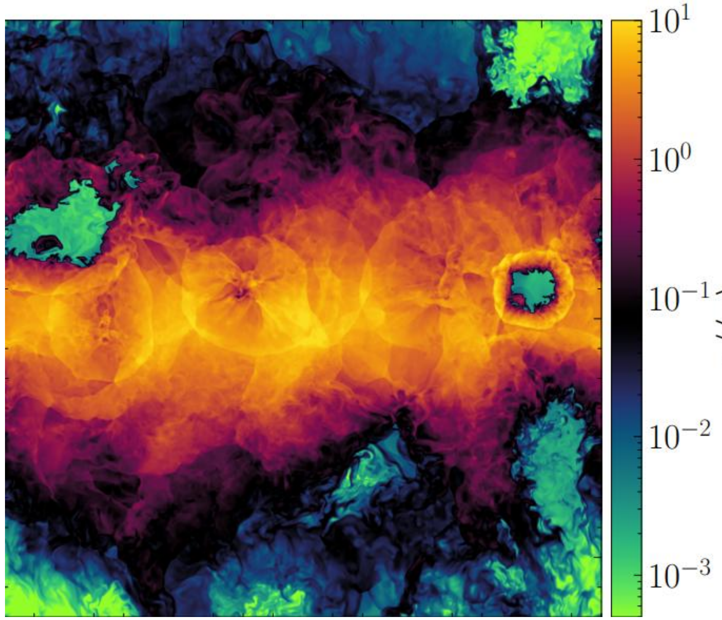
Simulations: motivation for multi-phase media

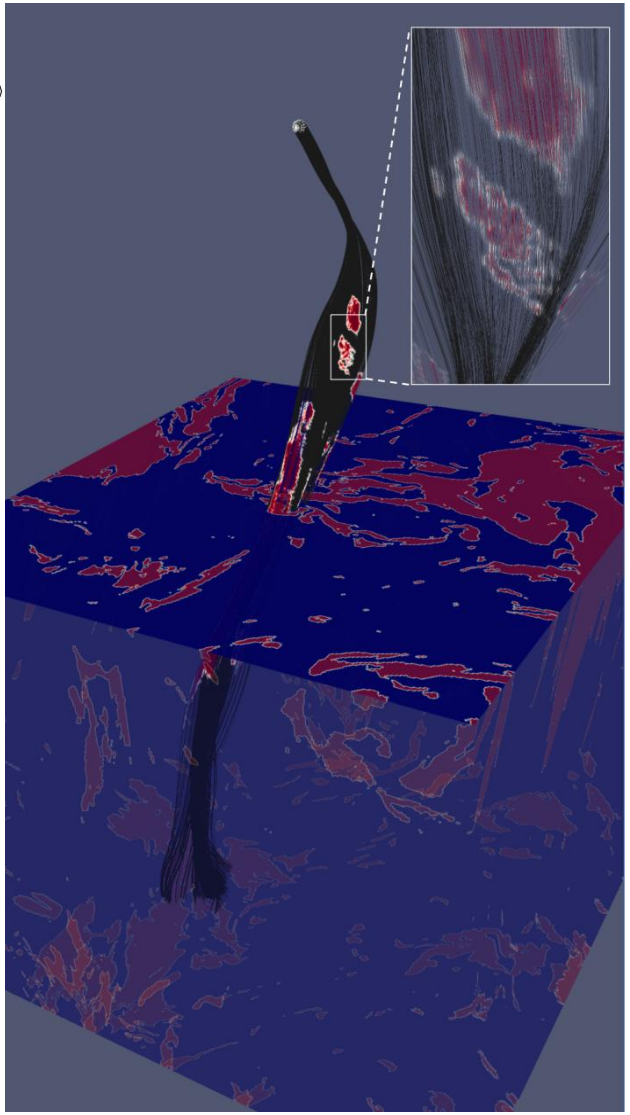


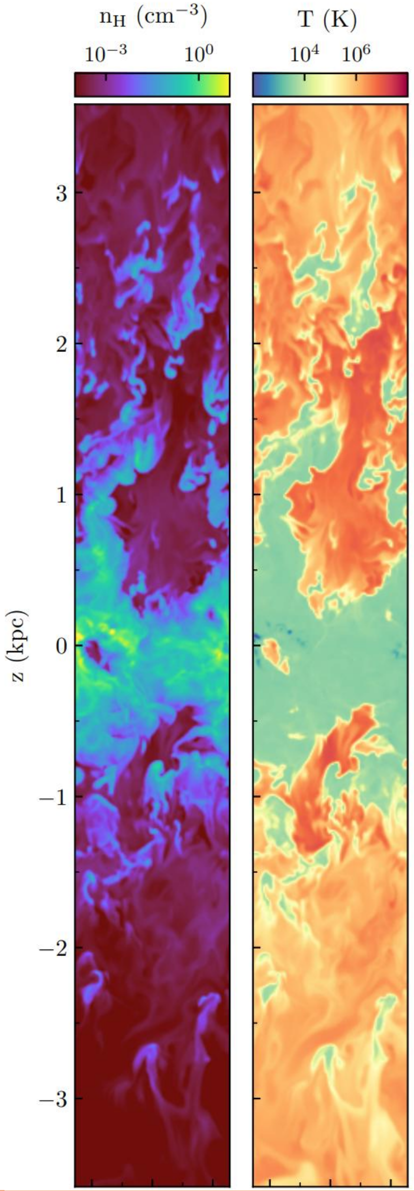












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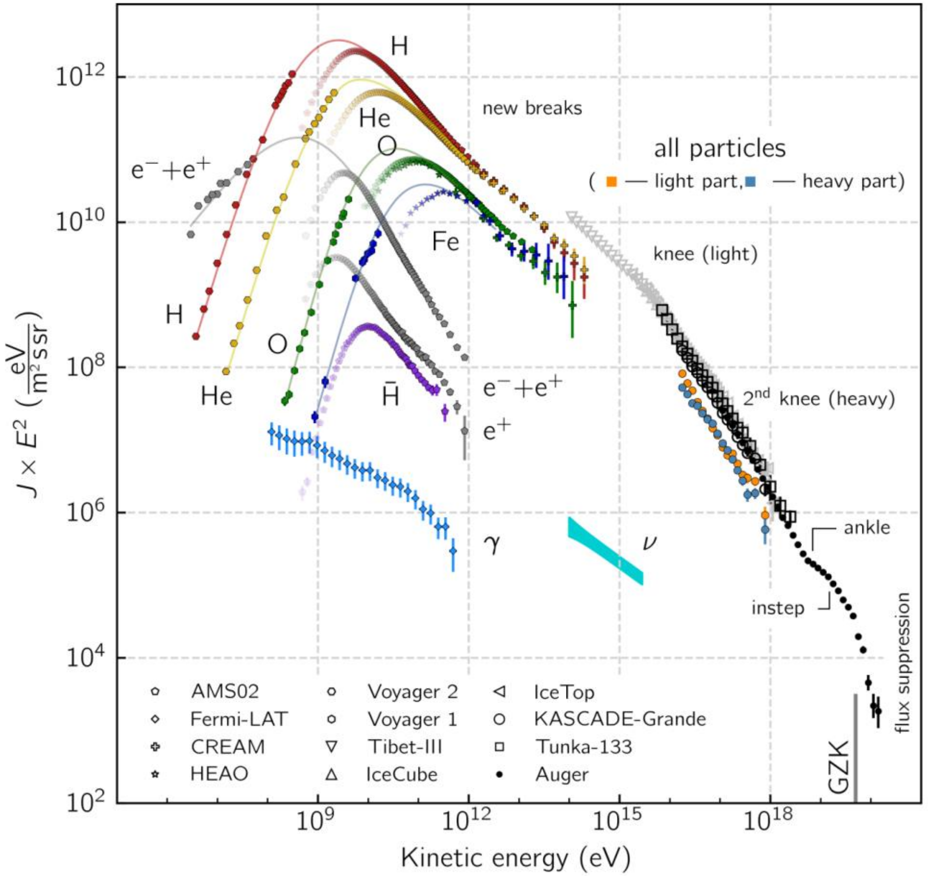

Questions

- How are cosmic-rays transported through The Galaxy/ICM?
- How do you describe "mean-field" transport in a system with more than one phase.
- What minimal set of quantities do we need access to be able to determine the nature of that mean-field (topology, filling fraction, intermittency, kitchen sink) [5]

[1] Ruszkowski & Pfrommer, 2023, Astron. Astrophys. Rev. **31** 4
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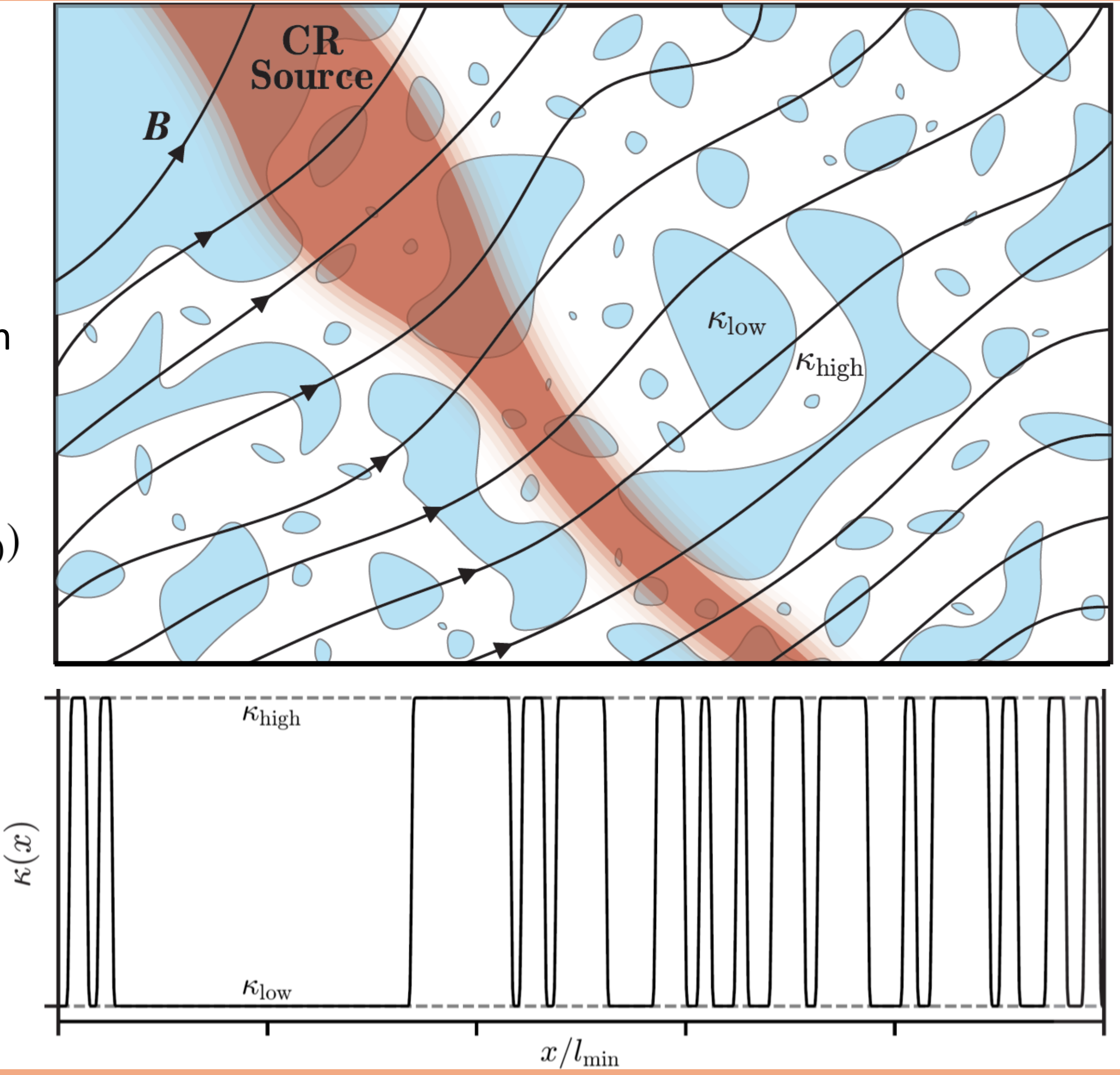
A toy diffusion model

- A two-phase medium with a spatially varying diffusion coefficient $\kappa(x)$.

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\kappa(x)\frac{\partial n}{\partial x}$$

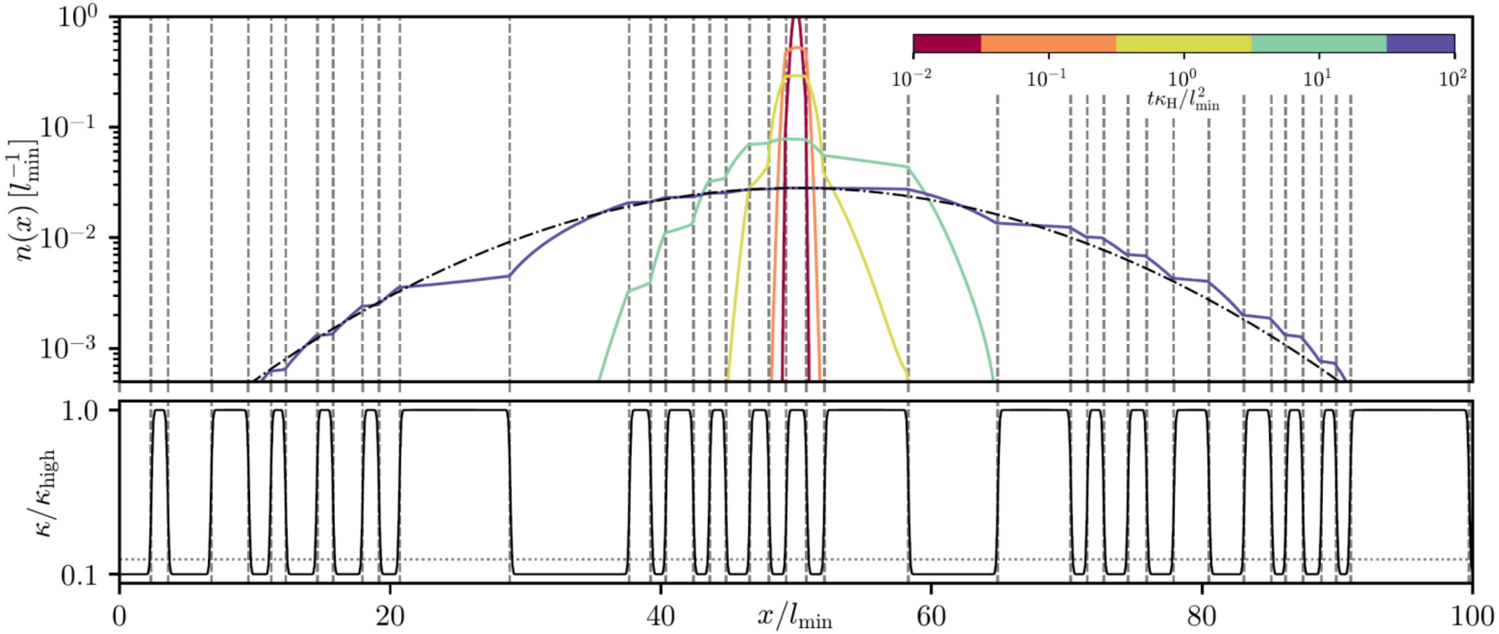
- “Solving” this model would mean having a Green’s function $n(x,t;x_0)$ (the solution with $n(x,0;x_0) = \delta(x - x_0)$) from which everything can be calculated.
- We cannot solve this.
- We should instead hope for useful asymptotics of $n(x,t;x_0)$ or for statements about moments of it. One of the most important such moments is the “running diffusion coefficient”.

$$\bar{\kappa}(t) = \frac{1}{2t}\left\langle \int \mathrm{d}x \, (x - x_0)^2 n(x,t;x_0) \right\rangle_{x_0}$$



What is a running diffusion coefficient?

- We solved for these distributions by inverting sparse matrices: is this the way forward?
- The analogy here with resistors and capacitors is exact: is this the way forward?



- $\bar{\kappa}(t)$ is a measure of width the cosmic rays have spread from a point source. There are two analytically tractable limits:

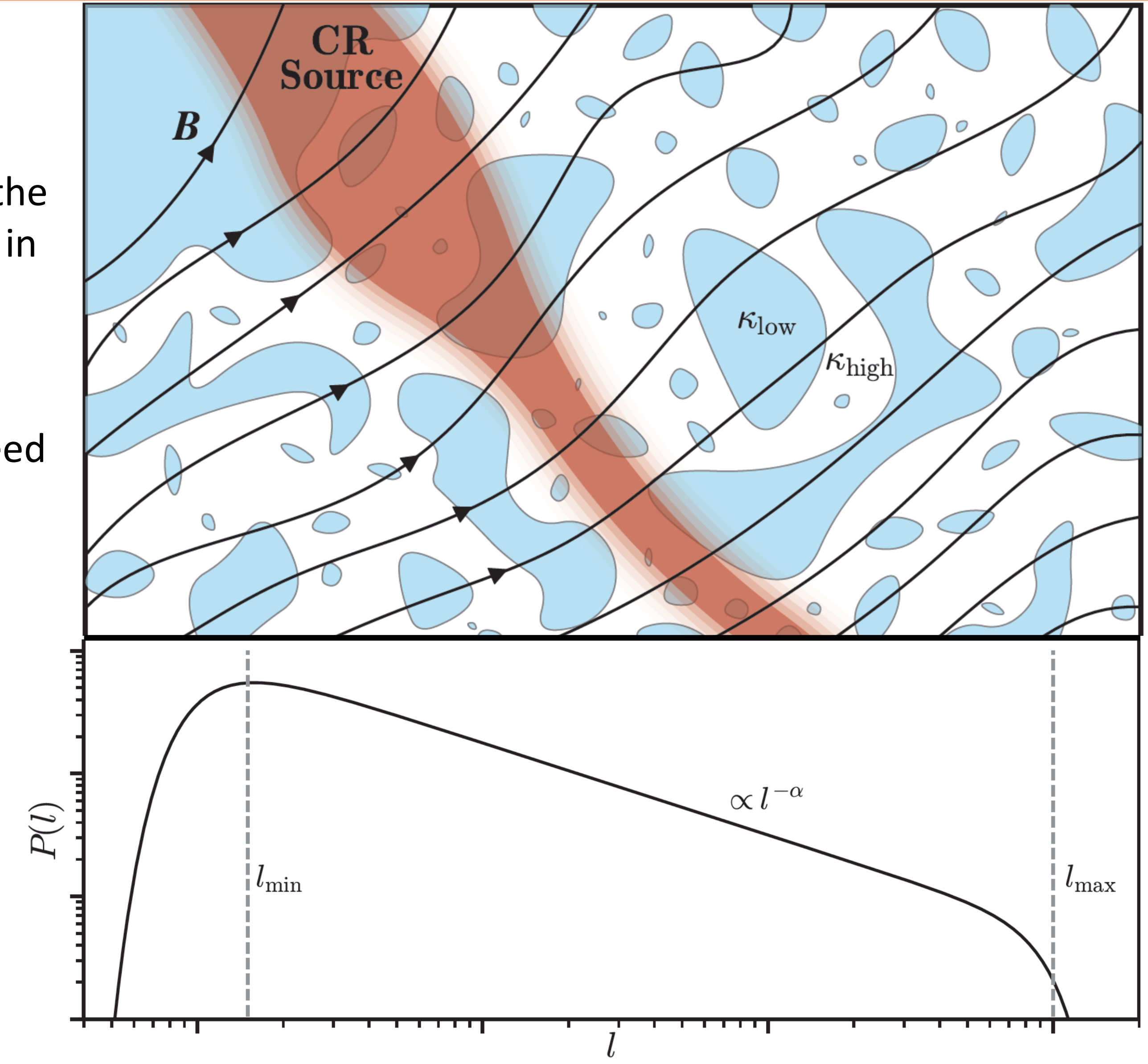
$$\lim_{t\to 0} \bar{\kappa}(t) = \kappa_A \equiv \lim_{X\to\infty} \frac{1}{X}\int_0^X \mathrm{d}X'\,\kappa(X') \implies \kappa_A = f\kappa_{\rm low} + (1-f)\kappa_{\rm high} \approx (1-f)\kappa_{\rm high}$$

$$\lim_{t\to\infty} \bar{\kappa}(t) = \kappa_H \equiv \lim_{X\to\infty} \frac{X}{\int_0^X \frac{\mathrm{d}X'}{\kappa(X')}} \implies \kappa_H = \frac{1}{\frac{f}{\kappa_{\rm low}} + \frac{1-f}{\kappa_{\rm high}}} \approx \frac{\kappa_{\rm low}}{f}$$

Transient sub-diffusivity

- The long-time average diffusive behaviour κ_H is always slower than the short-time average behaviour κ_A .
- The short-term high diffusivity behaviour is controlled by the number of cosmic rays that were lucky enough to be born in high-diffusion patches. After a time t, only those born in patches larger than $\sim\sqrt{\kappa_{high} t}$ will contribute to the large diffusion coefficient.
- To know how many such particles there need to be, we need pdfs of sizes of patches (along field lines), P_{high} and P_{low} .

$$\bar{\kappa}(t) \sim \kappa_{high} \frac{\int_{\sqrt{\kappa_{high} t}}^{l_{max}} \mathrm{d}l \, P_{high}(l) l}{\int_{l_{min}}^{l_{max}} \mathrm{d}l \, \left[P_{high}(l) + P_{low}(l)\right] l}$$

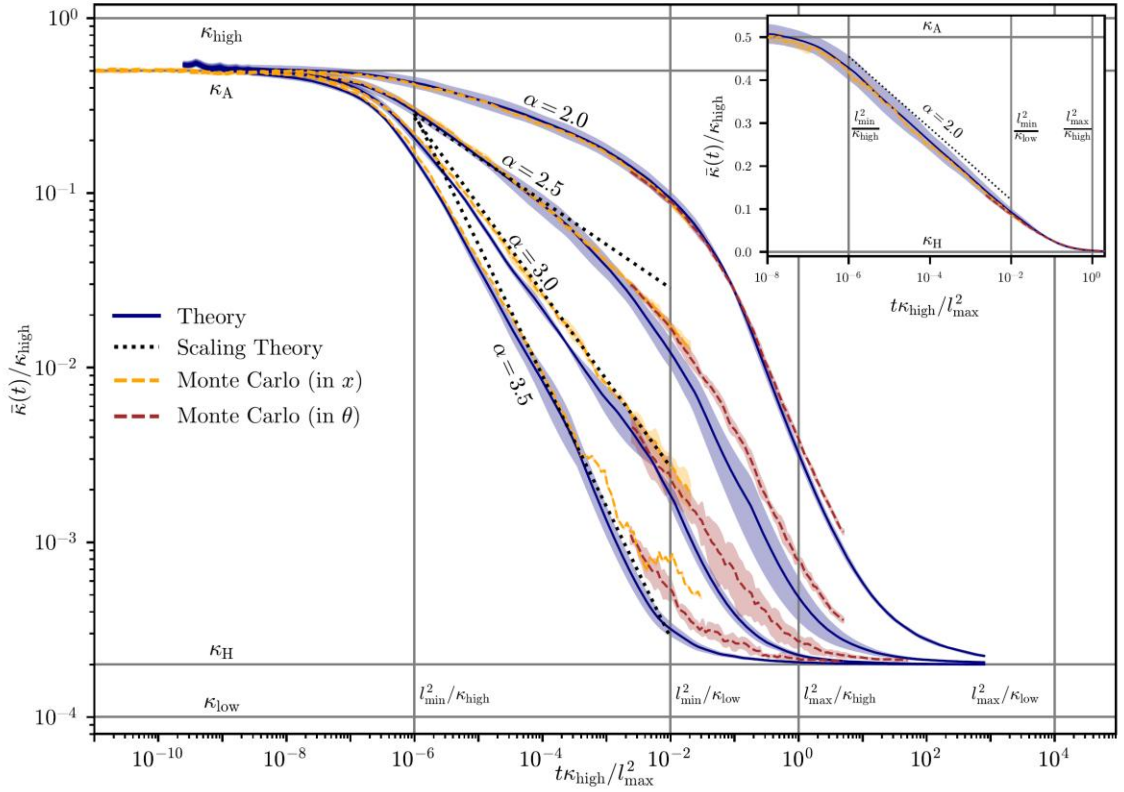


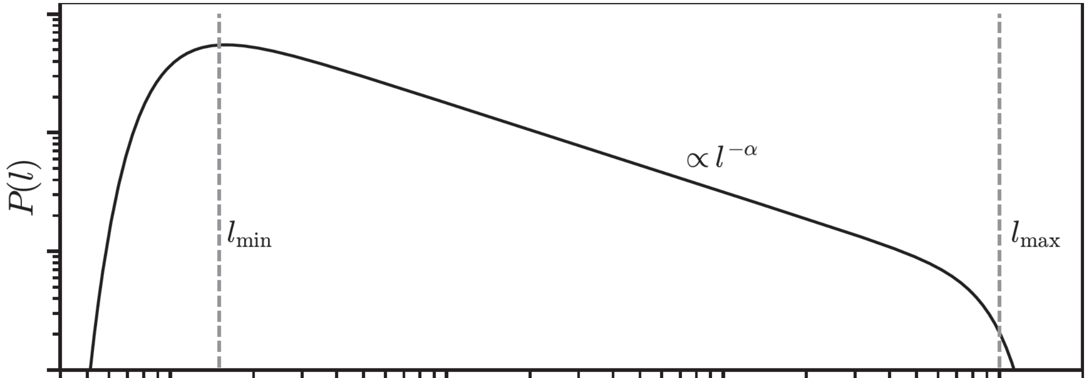
Transient sub-diffusivity

- Monte Carlo simulations are very inefficient, and the time stepping is limited by the inter-patch smoothing.

- If we assume that the patches are multi-scale then it is reasonable to assume $P(l) \sim l^{-\alpha}$ between some smallest patch size l_{min} and some largest patch size l_{max}

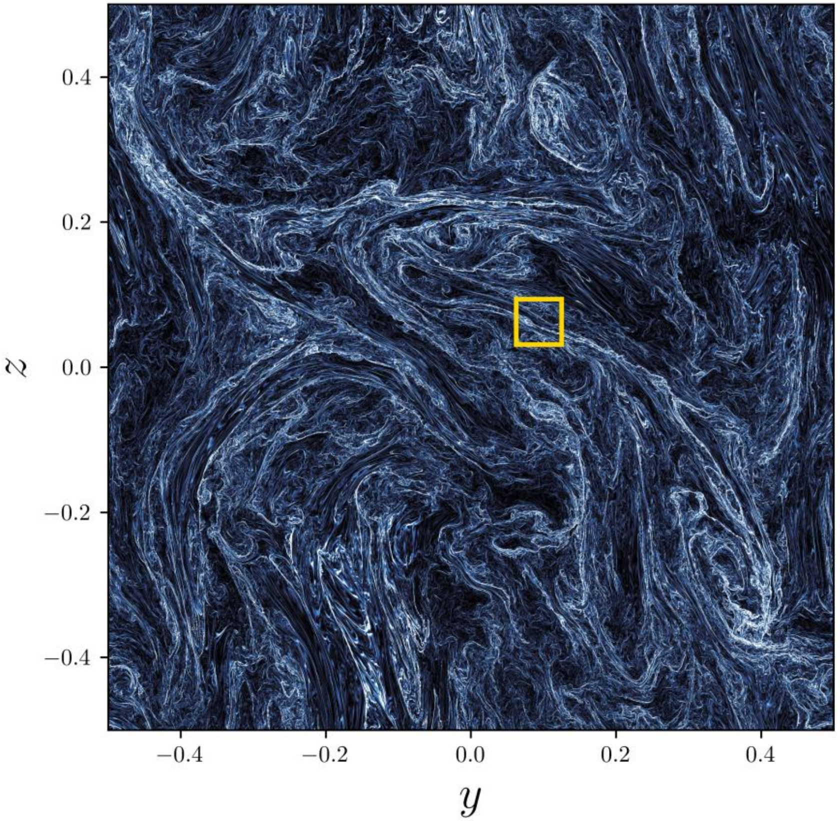
$$\bar{\kappa}(t) \sim (1-f)\kappa_{\text{high}} \begin{cases} 1 - \left(\dfrac{\kappa_{\text{high}} t}{l_{\text{max}}^2}\right)^{(2-\alpha)/2} & \text{for } \alpha < 2, \\ \ln\left(\dfrac{l_{\text{max}}^2}{t\kappa_{\text{high}}}\right) \Big/ \ln\left(\dfrac{l_{\text{max}}^2}{l_{\text{min}}^2}\right) & \text{for } \alpha = 2, \\ \left(\dfrac{t\kappa_{\text{high}}}{l_{\text{min}}^2}\right)^{(2-\alpha)/2} & \text{for } 4 \geq \alpha > 2, \\ \left(\dfrac{t\kappa_{\text{high}}}{l_{\text{min}}^2}\right)^{-1} & \text{for } \alpha > 4. \end{cases}$$

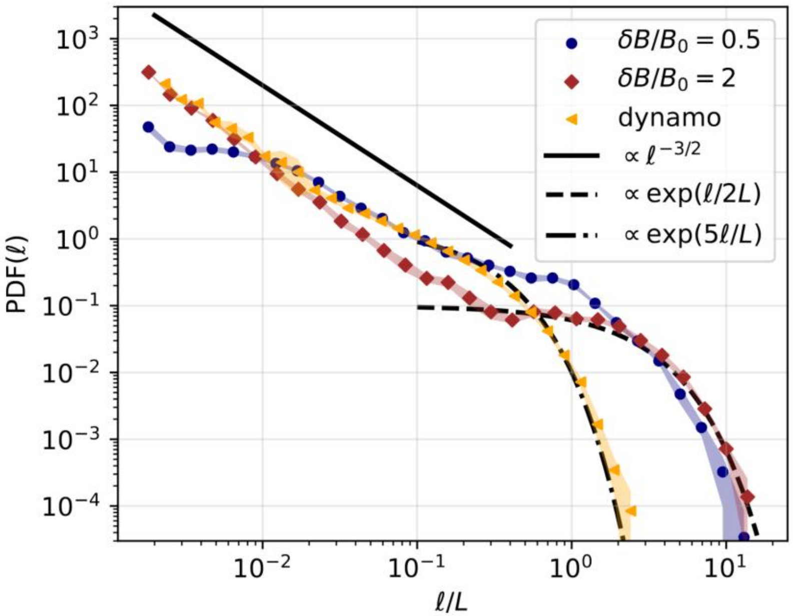


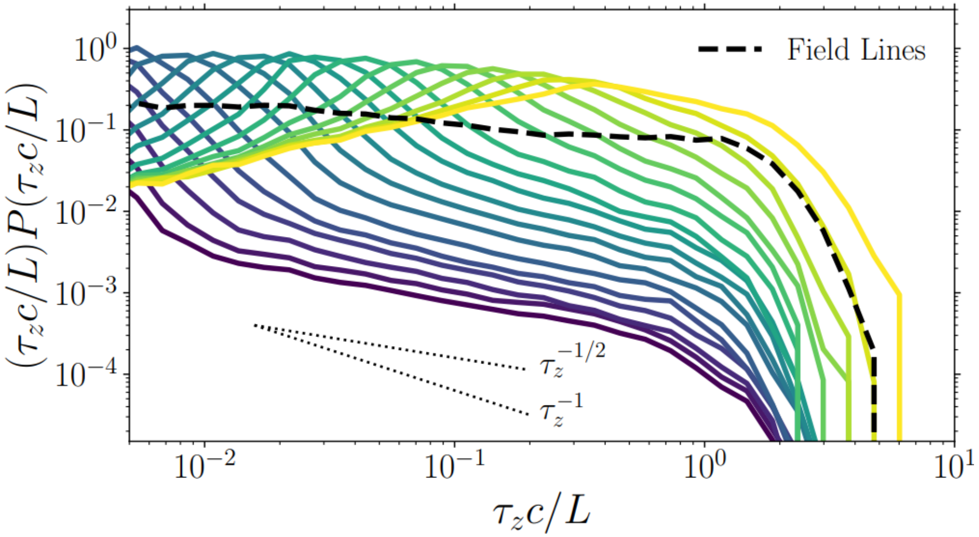


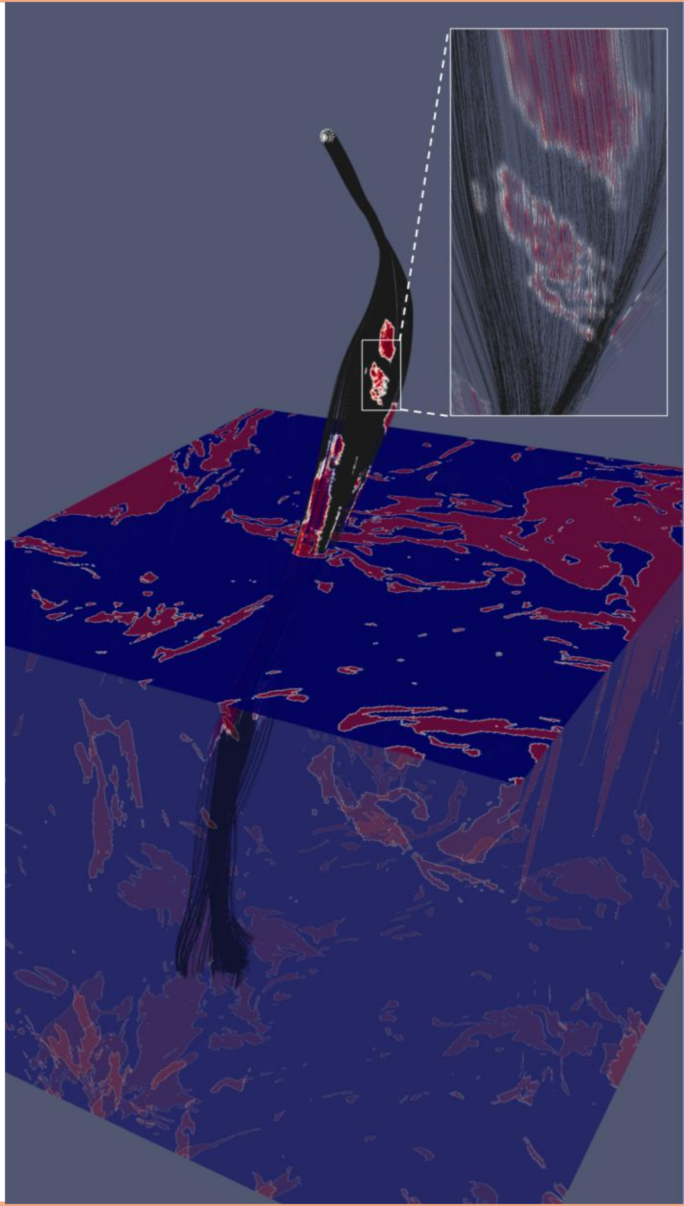
This is all (sort of) real

- These pdfs can be measured in simulation.
- For the Majeski fields we find $f \sim 5\%$ and $\alpha \sim -3/2$.
- Similar features can also be found in other simulations.



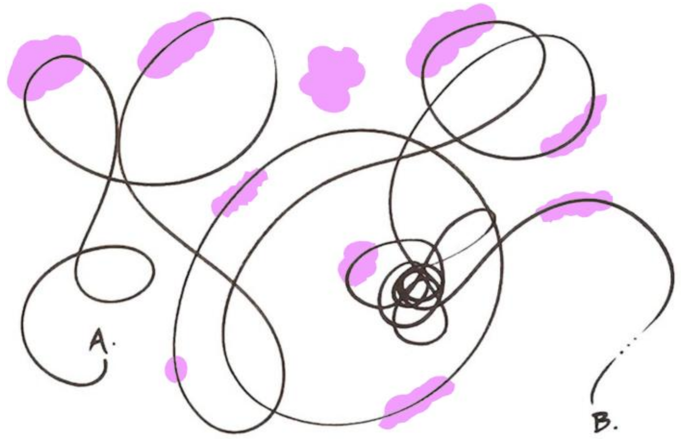






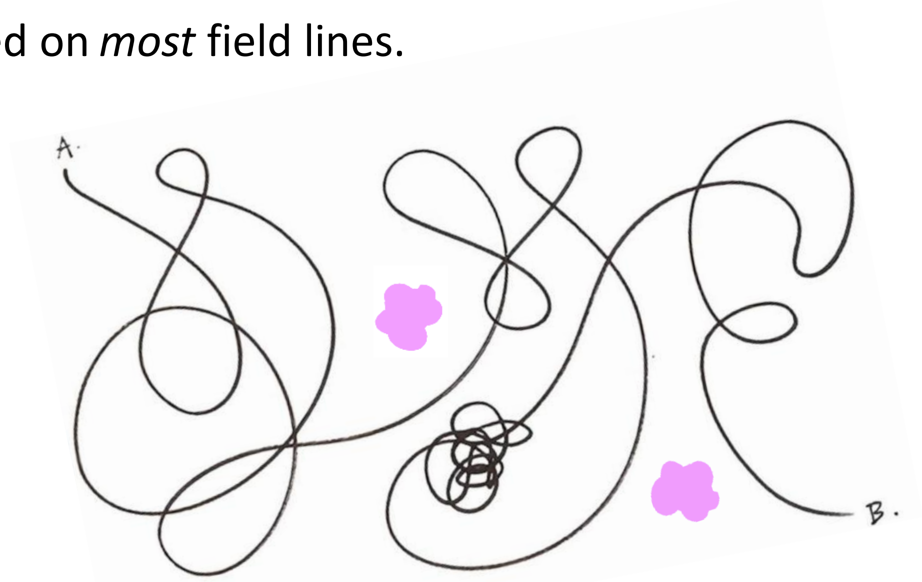
It's also totally irrelevant.

- You should not care about interesting transient behaviour. You should care about escape rates (because we only see cosmic-rays at the end of their journey).
- Consider the following case in which cosmic rays have to travel from A to B. In the first case the patches are sufficiently frequent that in the distance from A to B, a patch is encountered on *most* field lines. In the second patches are sufficiently rare that patches are not encountered on *most* field lines.



Frequent patches/long escape distance

$$t_{\rm esc}^{-1} \sim \kappa_{\rm H}/L_{\rm esc}^2$$



Infrequent patches/short escape distance

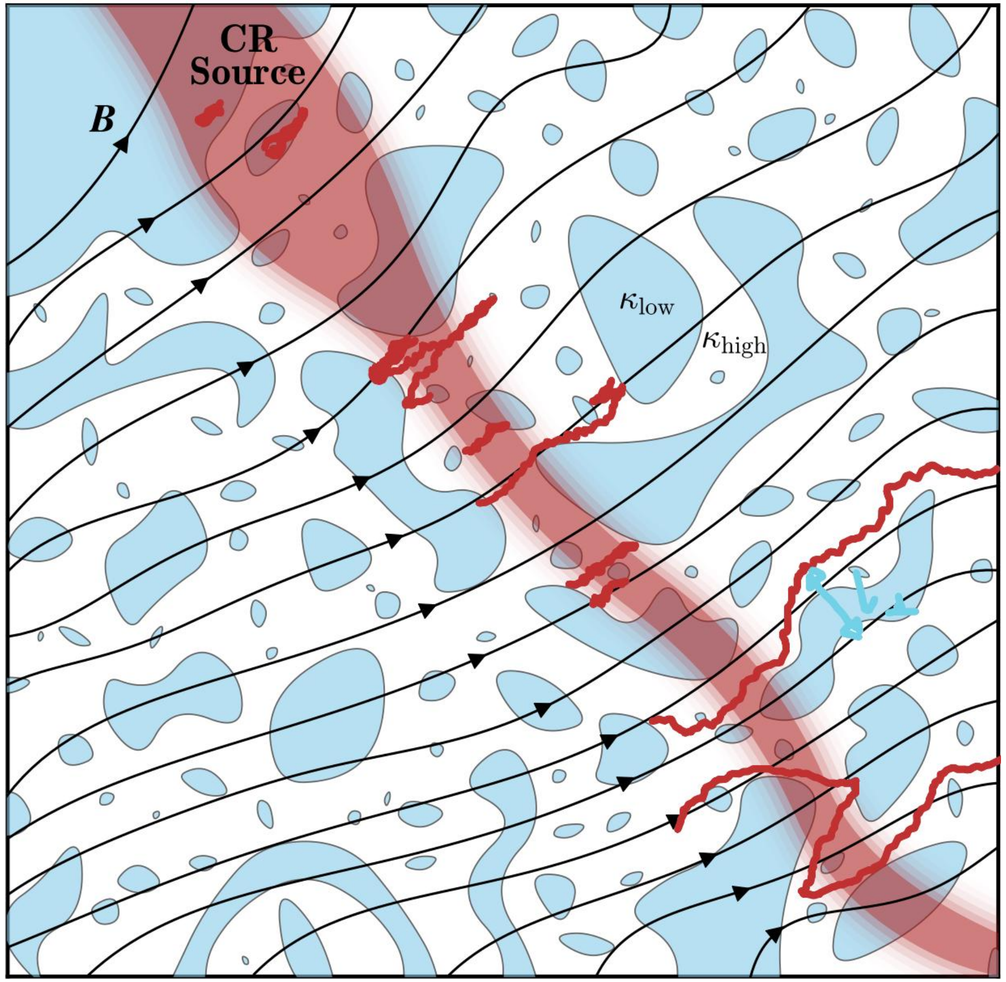
$$t_{\rm esc}^{-1} \sim \kappa_{\rm A}/L_{\rm esc}^2$$

No information about α has been inherited from the distribution of patches: **the system is *precisely* the sum of its parts.**

What about perpendicular transport

- This story, was due to 1D diffusion being incredibly constraining: every cosmic ray had to experience every patch between A and B.
- When cosmic rays can diffuse *around* patches, different energies will have different perpendicular diffusivities so will be more capable of dodging low-diffusion patches.
- Suppose that the patches have a typical perpendicular correlation length $l_\perp$.
- The time to cross that correlation length will be roughly $\tau_\perp \sim l_\perp^2/\kappa_\perp$.
- In that time the typical cosmic ray will have travelled a distance $\sqrt{\tau_\perp \bar{\kappa}(\tau_\perp)}$ set by the 1D running diffusion coefficient.
- Thus, the lazy estimate for the parallel diffusion coefficient is

$$\bar{\kappa}_{2D} \sim \bar{\kappa}(t \sim l_\perp^2/\kappa_\perp)$$



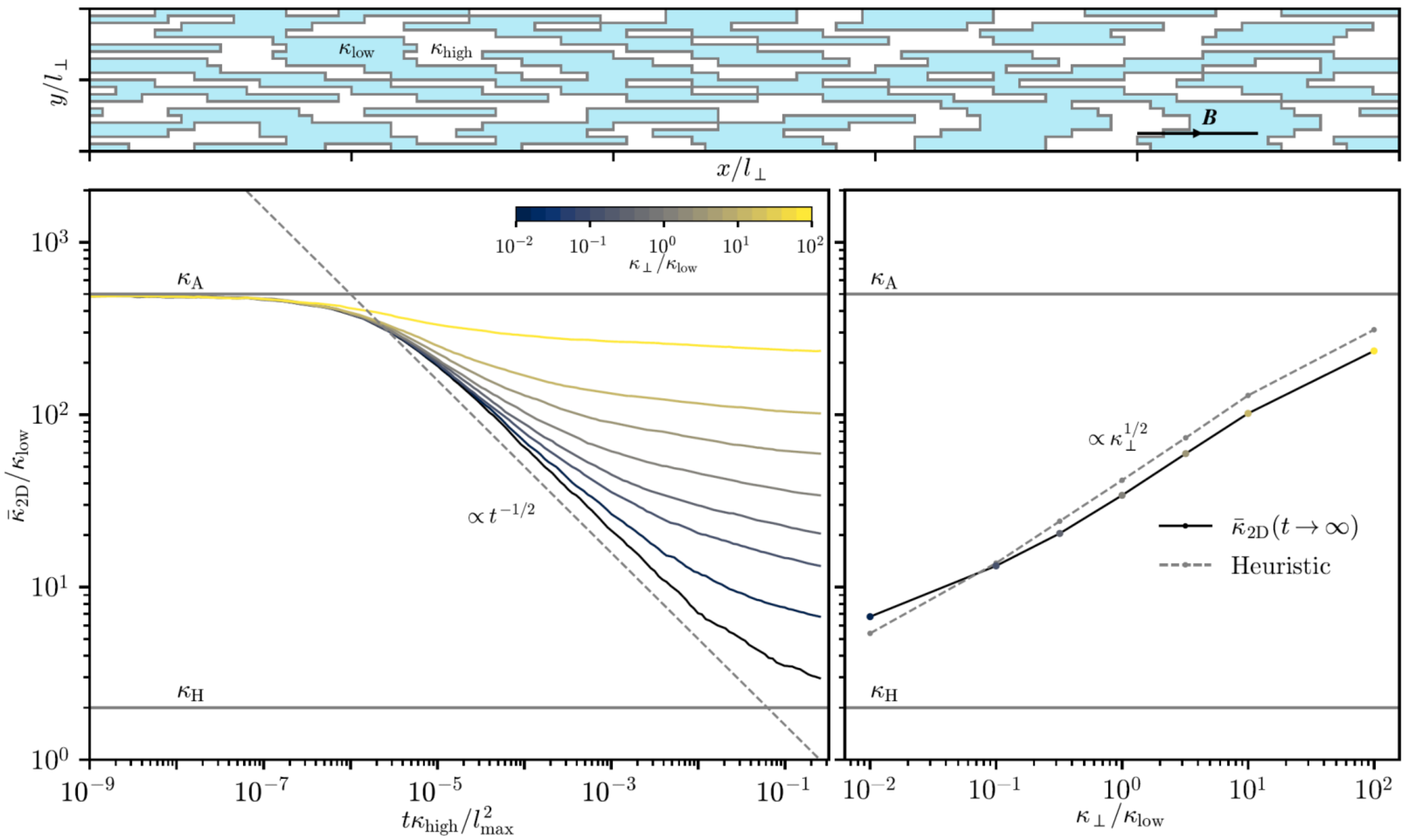
- This is exciting because the running diffusion coefficient *had* information about the structure of the turbulence. The medium is more than the sum of its parts.

And we(Patrick) did the homework

$$\bar{\kappa}_{2D} \sim \bar{\kappa}(t \sim l_\perp^2/\kappa_\perp)$$

$$\bar{\kappa}_{2D} \sim (1-f)\kappa_{high} \begin{cases} 1 - \left(\frac{\kappa_{high} l_\perp^2}{\kappa_\perp l_{max}^2}\right)^{(2-\alpha)/2} & \text{for } \alpha < 2, \\ \ln\left(\frac{\kappa_\perp l_{max}^2}{\kappa_{high} l_\perp^2}\right) \Big/ \ln\left(\frac{l_{max}^2}{l_{min}^2}\right) & \text{for } \alpha = 2, \\ \left(\frac{\kappa_{high} l_\perp^2}{\kappa_\perp l_{min}^2}\right)^{(2-\alpha)/2} & \text{for } 4 \geq \alpha > 2, \\ \left(\frac{\kappa_{high} l_\perp^2}{\kappa_\perp l_{min}^2}\right)^{-1} & \text{for } \alpha > 4, \end{cases}$$

- In principle, if you tell me the energy-dependence of $\kappa_\perp$ and κ_{high} you can now dial up a medium that will fake the galactic propagation of cosmic rays for you.



Conclusion and loose ends

- We understand diffusion in “simple” inhomogeneous media. It boils down to the filling fraction and the spectral index of the probability distribution function of lengths high diffusion patches.
- The structure of the medium can imprint itself, via α on the energy dependence of diffusion, opening up the possibility that the diffusion coefficient we see for galactic cosmic rays is an emergent property of a multi-phase media.

- Was there a better way we could have done all this? Monte-Carlo simulations are expensive.
- The mapping of the system onto Kirchoff’s equations is precise. You can treat this as a system of resistors and capacitors. Does this help?
- The matrix equation for diffusion (appropriately discretised) is sparse (actually tri-diagonal in 1D). You can invert for eigenvectors and eigenvalues and solve for arbitrary time much quicker than Monte-Carlo simulations.
- What about the case of a **truly** 2D/3D system, in which patches are multiscale in all directions. This problem has been studied extensively in condensed matter, roughly under the guise of percolation/large derviation theory/“ants in labyrinths”.
- It might boil down to cosmic rays minimising some action (cost function…) as they search for optimal paths through the medium. It’s understood what this action should be, but it’s not obvious (at least to me), how to efficiently search for it numerically/relate it to properties of the turbulence/whether it even matters.