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FÜR PLASMAPHYSIK

Tightening energetic bounds on linear gyrokinetic instabilities

Paul Costello & Gabe Plunk
With thanks to L. Podavini, A. Zocco & P. Helander

[1] P. Helander and G. G. Plunk, *Journal of Plasma Physics,* **88**, (2), 905880207, (2022).
[2] G. G. Plunk and P. Helander, *Journal of Plasma Physics*, **88**, (3), 905880313, (2022).
[3] G. G. Plunk and P. Helander, *Journal of Plasma Physics*, **89**, (4), 905890419, (2023).
[4] P. J. Costello and G. G. Plunk, *Journal of Plasma Physics*, **91**, (1), E12, (2025).
[5] L. Podavini, P. Helander, G. G. Plunk, and A. Zocco, *Journal of Plasma Physics*, **91**, (3), p. E79, (2025).
[6] P. J. Costello and G. G. Plunk, *Submitted to Journal of Plasma Physics*, arXiv:2505.17757, (2025)

Outline

1. **Introduction and Background**
    a. Turbulence, instabilities and gyrokinetics
    b. Energetic upper bounds and optimal modes: a brief primer
2. **Tightening energetic bounds on linear gyrokinetic instabilities**
    a. Tightest possible energetic bounds
    b. Constrained optimal modes
3. **Conclusions and future/ongoing work**

Introduction and background

Turbulence, instability and gyrokinetics

- **Turbulent transport *largely* dictates the size of a tokamak/stellarator fusion power plant**
  - Turbulence is initiated by *microinstabilities*, on the scale of the gyroradius
  - These tap into the free energy in the radial gradients of temperature and density of the plasma

Turbulence, instability and gyrokinetics

- **Turbulent transport *largely* dictates the size of a tokamak/stellarator fusion power plant**
  - Turbulence is initiated by *microinstabilities*, on the scale of the gyroradius
  - These tap into the free energy in the radial gradients of temperature and density of the plasma
- **The foundation of our understanding is *gyrokinetics***
  - Local, electrostatic gyrokinetic equation for species a and Fourier mode k:

$$\frac{\partial g_{a,k}}{\partial t} + v_{\|} \frac{\partial g_{a,k}}{\partial l} + i\omega_{da} g_{a,k} + [\delta\phi_k, g_{a,k}] = \frac{e_a F_{a0}}{T_a} \left(\frac{\partial}{\partial t} + i\omega_{*a}^{T} \right) \delta\phi_k J_{0a}$$

  - Simulated by *many* codes: GENE, STELLA, GX, GS2 *etc.*

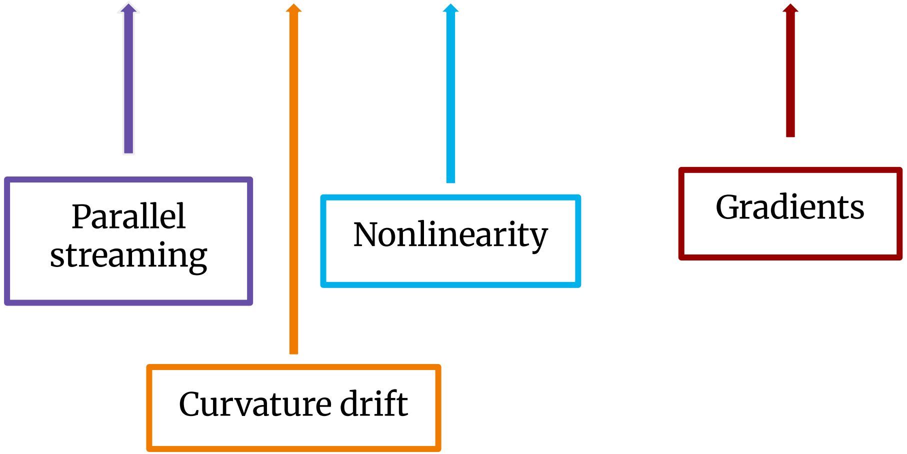
Electrostatic, collisionless gyrokinetics

$$\frac{\partial g_{a,k}}{\partial t} + \underbrace{v_{\|} \frac{\partial g_{a,k}}{\partial l}}_{\text{Parallel streaming}} + \underbrace{i\omega_{da} g_{a,k}}_{\text{Curvature drift}} + \underbrace{[\delta\phi_k, g_{a,k}]}_{\text{Nonlinearity}} = \frac{e_a F_{a0}}{T_a} \left(\frac{\partial}{\partial t} + \underbrace{i\omega_{*a}^T}_{\text{Gradients}} \right) \delta\phi_k J_{0a}$$

$$\left\{ \begin{aligned} \omega_{*a}^T &= \omega_{*a} \left[1 + \eta_a \left(\frac{v^2}{v_{Ta}^2} - \frac{3}{2} \right) \right] \\ \omega_{*a} &= (k_\alpha T_a / e_a) \frac{\mathrm{d} \ln n_a}{\mathrm{d}\psi} \\ \eta_a &= \frac{\mathrm{d} \ln T_a}{\mathrm{d} \ln n_a} \end{aligned} \right.$$

Electrostatic, collisionless gyrokinetics

$$\frac{\partial g_{a,k}}{\partial t} + v_\parallel \frac{\partial g_{a,k}}{\partial l} + i\omega_{da} g_{a,k} + [\delta\phi_k, g_{a,k}] = \frac{e_a F_{a0}}{T_a}\left(\frac{\partial}{\partial t} + i\omega_{*a}^T\right)\delta\phi_k J_{0a}$$



Parallel streaming

Nonlinearity

Gradients

Curvature drift

$$\left\{\begin{aligned} \omega_{*a}^T &= \omega_{*a}\left[1 + \eta_a\left(\frac{v^2}{v_{Ta}^2} - \frac{3}{2}\right)\right] \\ \omega_{*a} &= (k_\alpha T_a/e_a)\frac{\mathrm{d}\ln n_a}{\mathrm{d}\psi} \\ \eta_a &= \frac{\mathrm{d}\ln T_a}{\mathrm{d}\ln n_a} \end{aligned}\right.$$

$\delta\phi_k$ given by $\sum_a \frac{e_a^2 n_a}{T_a}\delta\phi_k = \sum_a e_a \int g_a J_{0a} \mathrm{d}^3 v$

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Compact form

$$\frac{\partial g_{a,k}}{\partial t} = \mathcal{L} g_{a,k} + \mathcal{N} g_{a,k}$$

Linear gyrokinetics

Linear instability theory is a cornerstone of our understanding

$$\frac{\partial g_{a,k}}{\partial t} = \mathcal{L} g_{a,k} + \cancel{\mathcal{N} g_{a,k}}$$

Neglect nonlinearity

- Look for "normal" modes of the form $g_{a,k} \sim \exp\left(-i\omega t\right)$ where $\omega = \omega_r + i\gamma$
- Solve the eigenvalue problem $\boxed{-i\omega g_{a,k} = \mathcal{L} g_{a,k}}$ to find instabilities.

J. W. Connor, R. J. Hastie, and J. B. Taylor, *Plasma Physics*, vol. 22, no. 7, p. 757, (1980)

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- Solve the eigenvalue problem $\boxed{-i\omega g_{a,k} = \mathcal{L} g_{a,k}}$ to find instabilities.
  - Understood in some limits -> “zoo” of modes and mechanisms: ITG, TEM, KBM, MTM etc.
  - This problem is usually *hard*, particularly in non-uniform magnetic fields (e.g., CHT)
  - *Simulations often necessary*

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Energetic upper bounds and optimal modes

(avoiding the zoo...)

Energetic bounds: a different approach to instability

$$(\mathbf{g}_1, \mathbf{g}_2) = \sum_{a,k} \left\langle T_a \int \frac{g^*_{a,1} g_{a,2}}{F_{a0}} \mathrm{d}^3 v \right\rangle$$

$$\langle \ldots \rangle = \lim_{L \to \infty} \int_{-L}^{L} (\ldots) \frac{\mathrm{d}l}{B} \Big/ \int_{-L}^{L} \frac{\mathrm{d}l}{B}$$

Instead of solving $-i\omega g_{a,k} = \mathcal{L} g_{a,k}$

- Consider **energy balance**:

$$\frac{\partial g_{a,k}}{\partial t} = \mathcal{L} g_{a,k} + \mathcal{N} g_{a,k} \xrightarrow{(\mathbf{g}, \tilde{\mathcal{H}}(\ldots))} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{H} = 2\tilde{D}$$

Energy $\tilde{H}$ grows via the energy source $\tilde{D}$

  - Can be chosen to ‘annihilate’ the nonlinearity (*nonlinear invariant*)
  - $\tilde{H}$ must be a *positive definite* quadratic ‘norm’

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- Instantaneous growth rate of $\tilde{H}$

$$\Lambda \equiv \frac{1}{2\tilde{H}} \frac{\mathrm{d}\tilde{H}}{\mathrm{d}t}$$

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$$\Lambda \equiv \frac{1}{2\tilde{H}} \frac{\mathrm{d}\tilde{H}}{\mathrm{d}t} = \frac{\tilde{D}}{\tilde{H}} \longrightarrow \Lambda = \frac{(\mathbf{g}, \tilde{\mathcal{D}}\mathbf{g})}{(\mathbf{g}, \tilde{\mathcal{H}}\mathbf{g})}$$

Optimal modes and energetic upper bounds

- Ask the question, “*Which* $\mathbf{g}$ *gives the fastest possible energy growth?*”

$$\Lambda \equiv \frac{(\mathbf{g}, \tilde{\mathcal{D}}\mathbf{g})}{(\mathbf{g}, \tilde{\mathcal{H}}\mathbf{g})} \xrightarrow{\frac{\delta\Lambda}{\delta\mathbf{g}} = 0} \boxed{\Lambda\tilde{\mathcal{H}}\mathbf{g} = \tilde{\mathcal{D}}\mathbf{g}}$$

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- The solutions to this problem are the *optimal modes* $(\Lambda, \mathbf{g})$
- The largest $\Lambda_{\max}$ is an **upper bound on *any* instability growth** in the system
  - *Bounds linear instability growth* at each $\mathbf{k}_\perp$: $\boxed{\gamma_L \leq \Lambda_{\max}}$
  - Largest $\Lambda_{\max}$ over all $\mathbf{k}_\perp$ values *bounds nonlinear energy growth*
  - Low-dimensionality $\longrightarrow$ a system of gyrofluid-like equations
  - Freedom in the choice of 'energy norm'

Helmholtz free energy: a bound on all geometries

- Helmholtz free energy is a simple *nonlinear invariant*

$$\frac{\mathrm{d}}{\mathrm{d}t} H = 2D$$

$$H = \sum_{k,a} \left\langle T_a \int \frac{|g_{a,k}|^2}{F_{a0}} \mathrm{d}^3\mathbf{v} - \frac{n_a e_a^2}{T_a} |\delta\phi_k|^2 \right\rangle$$

$$D = \mathrm{Re} \sum_{k,a} \left\langle e_a \int i\omega_{*a}^T g_{a,k}^* \delta\phi_k J_{0a} \mathrm{d}^3 v \right\rangle$$

Only depends on gradients, no geometry!

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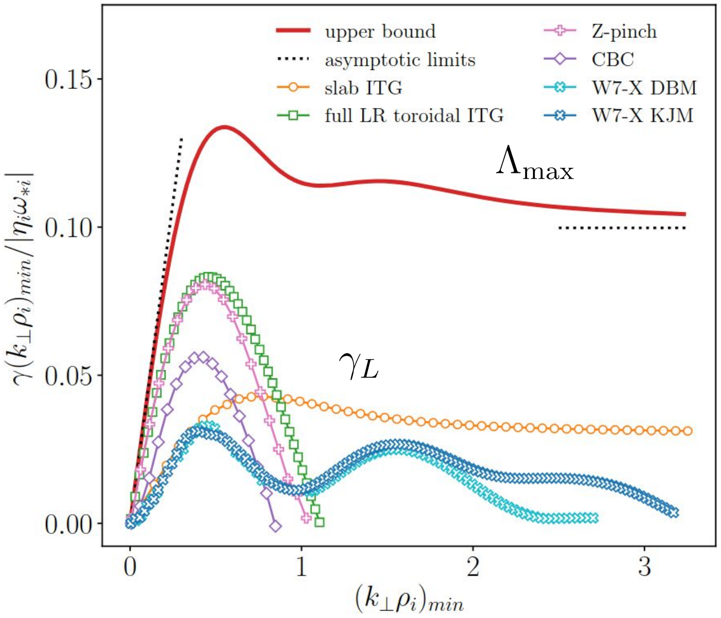
$$\Lambda\mathcal{H}\mathrm{g} = \mathcal{D}\mathrm{g}$$ (optimal mode problem 2 × 2 system of fluid moments)



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Generalised free energy: geometry dependent bound

- Electrostatic energy depends explicitly on geometry

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2K$$

$$E = \sum_k \frac{n e_a^2}{T_a} \left\langle [1 - \Gamma_0(b_{ak})] |\delta\phi_k|^2 \right\rangle$$

$$K = -\mathrm{Re} \sum_{a,k} \left\langle e_a \int \left(\underbrace{v_\parallel \frac{\partial g_{a,k}}{\partial l} + i\omega_{da} g_{ak}}_{\uparrow \atop \text{Geometry dependent}} \right) \delta\phi_k^* J_{0ak} \mathrm{d}^3 v \right\rangle$$

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- But E is not *positive definite*: not suitable for optimal mode analysis.
- Instead consider “**generalised free energy**”:

$$\tilde{H} = H - \Delta E \qquad \frac{\mathrm{d}\tilde{H}}{\mathrm{d}t} = 2(D - \Delta K)$$

- Aware of both gradients (D) and geometry (K).
- Tightest bound found by minimizing over Δ.

Ion Temperature Gradient (ITG)

- The optimal mode equation (5 × 5 system of fluid moments)

$$\Lambda\tilde{\mathcal{H}}g = (\mathcal{D} - \Delta\mathcal{K})g$$

- For adiabatic electrons, a geometry-dependent bound

  on ITG modes
  - Can be solved analytically in simple (local) limits

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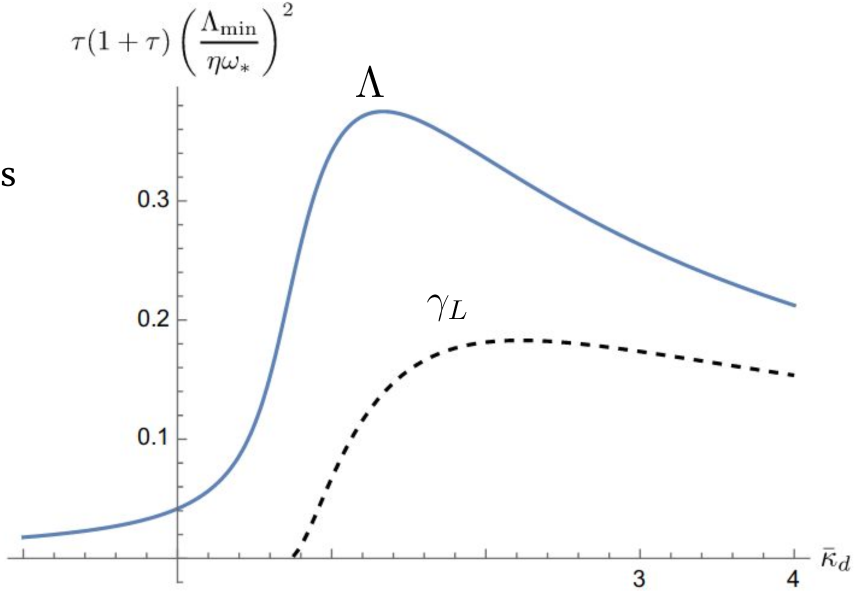
  - Can be solved analytically in simple (local) limits
- **Toroidal ITG**: *curvature driven*

$$\kappa_d = \omega_{*i}\eta_i/(\hat{\omega}_{di}) \sim R/L_T$$

  - Captures key features of linear growth



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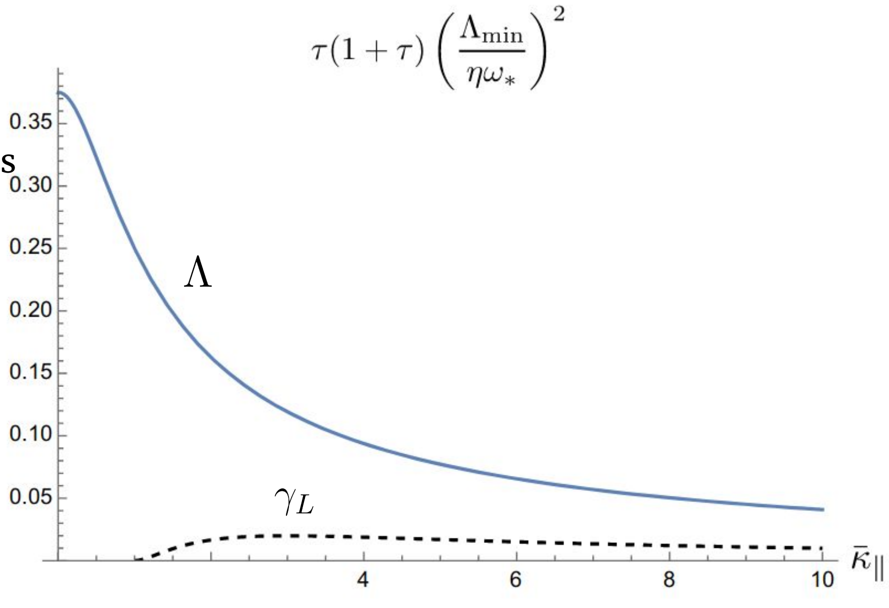
    - Can be solved analytically in simple (local) limits
- **Slab ITG:** driven by streaming along B

$$\kappa_{\parallel} = \omega_{*i}\eta_i/(v_{Ti}k_{\parallel})$$

    - Difference in behaviour at low-drive
    - Optimal modes and linear eigenmodes in *disagreement...*



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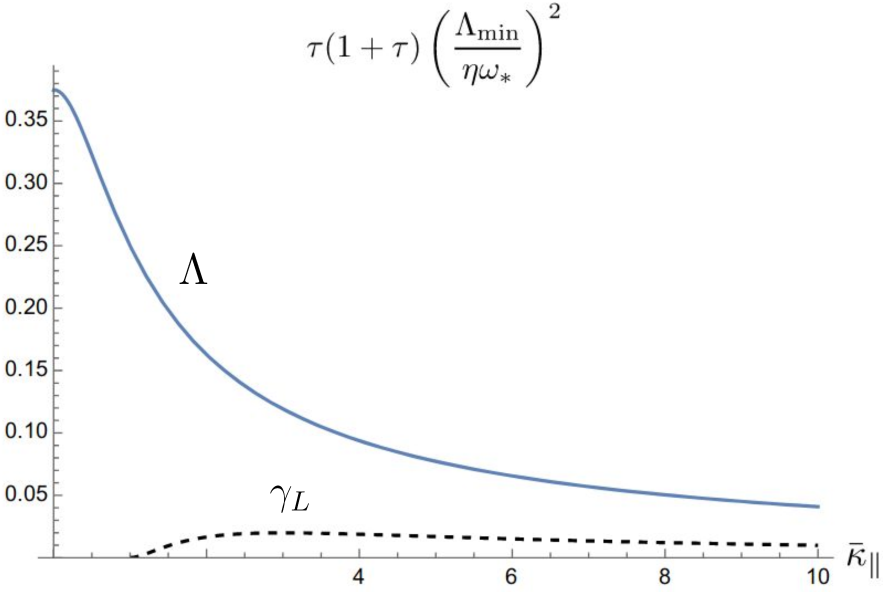
Optimal modes and linear eigenmodes: bridging the gap

Previous work has focused on bounds that are *linear and nonlinear*

- If we **focus on linear bounds** can we capture the behaviour γ_L?
    1. How **tightly** can we bound linear growth with optimal modes?
    2. Can we develop **general methods** that can tighten the upper bound on normal mode growth?



Linear gyrokinetic equation: slab ITG

- In the slab geometry, $\omega_{di} = 0$ and $\partial/\partial l \to ik_\parallel$
- Focusing on the linear dynamics

Linear gyrokinetic equation

$$\frac{\partial g}{\partial t} + iv_\parallel k_\parallel g = \frac{eF_{i0}}{T_i}\left(\frac{\partial}{\partial t} + i\omega_*^T\right)\phi J_0$$

Quasineutrality

$$\frac{e^2 n}{T_i}(1+\tau)\phi = e\int gJ_0 \mathrm{d}^3v$$

- The $v_\perp$ coordinate is ignorable here, we can integrate it out

$$\bar{g} = 2\pi\int_0^\infty \mathrm{d}v_\perp v_\perp g J_0 \quad\longrightarrow\quad \frac{\partial \bar{g}}{\partial t} + iv_\parallel k_\parallel \bar{g} = \frac{e\bar{F}_0}{T_i}\left[G_{\perp 0}\left(\frac{\partial}{\partial t} + i\omega_*(1+\eta(x_\parallel^2 - 3/2))\right) + i\omega_*\eta G_{\perp 2}\right]\phi$$

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Linear eigenmodes

- It is useful to write $f = \bar{g} - e\bar{F}_0 G_{\perp 0}\phi/T_i$

$$\frac{\partial f}{\partial t} + iv_\parallel k_\parallel f = -ik_\parallel S(v_\parallel, k_\parallel, b) \int f \mathrm{d}v_\parallel$$

- The **linear eigenmodes** are given by:

$$(\omega - v_\parallel k_\parallel) f = k_\parallel S(v_\parallel) \int f(v'_\parallel) \mathrm{d}v'_\parallel$$

Linear eigenmodes

- It is useful to write $f = \bar{g} - e\bar{F}_0 G_{\perp 0}\phi/T_i$

$$\frac{\partial f}{\partial t} + i v_\| k_\| f = -i k_\| S(v_\|, k_\|, b) \int f \mathrm{d}v_\|$$

- The **linear eigenmodes** are given by:

$$(\omega - v_\| k_\|) f = k_\| S(v_\|) \int f(v'_\|) \mathrm{d}v'_\|$$

For complex ω: **discrete modes**

$$f_n = \frac{S(v_\|)}{\omega_n/k_\| - v_\|}$$

For real ω: **continuum modes**

$$f_\omega = P\left[\frac{S(v_\|)}{(\omega/k_\| - v_\|)}\right] + \lambda(\omega)\delta(\omega/k_\| - v_\|)$$

Completeness and “Case–Van Kampen energy”

The work of Case and Van Kampen $\longrightarrow$ **completeness** of eigenmodes.

- For any $f(v_\parallel, t)$

$$f(v_\parallel, t) = \sum_n a_n(t) f_n(v_\parallel) + \int A(\omega, t) f_\omega(v_\parallel)\, \mathrm{d}\omega$$

- The amplitudes $a_n(t)$ and $A(\omega, t)$ can be found using orthogonal *adjoint eigenmodes*

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- The amplitudes $a_n(t)$ and $A(\omega, t)$ can be found using orthogonal *adjoint eigenmodes*
- Now consider an “energy”: *Case–Van Kampen energy*

$$E_C = \sum_n |a_n(t)|^2 + \int |A(\omega, t)|^2 \mathrm{d}\omega \qquad \frac{\mathrm{d}E_C}{\mathrm{d}t} = 2\sum_n \gamma_n |a_n|^2$$

  - E_C is **positive definite** by the completeness
  - Grows/damps due to projection onto discrete modes

Optimal modes and tightest possible bounds

- Instantaneous growth rate of E_C

$$\Lambda = \frac{1}{2E_C}\frac{\mathrm{d}E_C}{\mathrm{d}t} \qquad \Lambda = \frac{\sum_n \gamma_n |a_n|^2}{\sum_n |a_n|^2 + \int |A(\omega)|^2 \mathrm{d}\omega}$$

- **Optimal modes**

$$\frac{\delta\Lambda}{\delta f} = 0 \longrightarrow \Lambda \mathcal{E} f = \mathcal{K} f$$

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- **Optimal modes**

$$\frac{\delta\Lambda}{\delta f} = 0 \longrightarrow \Lambda\mathcal{E}f = \mathcal{K}f$$

- Project with eigenmodes to solve

$$(f_1, f_2) = \int \frac{f_1^* f_2}{\bar{F}_0} \mathrm{d}^3 v$$

Continuum mode projection

$$\Lambda(f_\omega, \mathcal{E}f) = (f_\omega, \mathcal{K}f)$$

$$\Lambda = 0$$

Discrete mode projection

$$\Lambda(f_m, \mathcal{E}f) = (f_m, \mathcal{K}f)$$

$$\boxed{\Lambda = \gamma_m}$$

Key takeaways from the tightest bounds

1. There is no fundamental limitation on the tightness of energetic bounds on linear eigenmode growth
2. The linear eigenmodes *are* the optimal modes of the Case–Van Kampen energy $\Lambda = \gamma_m$
    - Equality of $\gamma_L \leq \Lambda$
3. Constructed with **complete knowledge** of the linear spectrum... can we get away with less?

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Aside

Systems where E_C is a nonlinear invariant are special [†], they:

- Are *free from subcritical turbulence*
- have *energetically isolated* eigenmodes

† Plunk, G. G. "On the nonlinear stability of a quasi-two-dimensional drift kinetic model for ion temperature gradient turbulence". Physics of Plasmas 22 (4), 042305. (2015)

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Simple but still tight?

- From Case–Van Kampen: encoding information on linear modes should tighten the bounds
- Can we encode this into a simple energy norm to get *simple but tight* bounds?
    - **Helmholtz energy balance** for this system

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 2D \quad \begin{cases} H = T_i \displaystyle\int \frac{|\bar{g}|^2}{\bar{F}_0} \mathrm{d}v_\parallel - \frac{e^2 n}{T_i}(1+\tau) G_{\perp 0} |\phi|^2 \\ D = \mathrm{Re}\left\{ i\omega_* \eta \, G_{\perp 0} e\phi \displaystyle\int \bar{g}^* x_\parallel^2 \, \mathrm{d}v_\parallel \right\} \end{cases}$$

- D depends only on simple fluid moments of $\bar{g}$ ...
    - **Can we constrain these moments to behave like linear modes?**

Gyrofluid equations

Take moments of the linear GK equation

$$\frac{\partial \bar{g}}{\partial t} + i v_\parallel k_\parallel \bar{g} = \frac{e\bar{F}_0}{T_i}\left[G_{\perp 0}\left(\frac{\partial}{\partial t} + i\omega_*(1+\eta(x_\parallel^2 - 3/2))\right) + i\omega_*\eta G_{\perp 2}\right]\phi$$

- Density moment: κ_1

$$\frac{\partial \kappa_1}{\partial t} + i v_T k_\parallel \kappa_2 = \frac{1}{1+\tau}\left[G_{\perp 0}\left(\frac{\partial}{\partial t} + i\omega_*(1-\eta)\right) + i\omega_*\eta G_{\perp 2}\right]\kappa_1$$

$$\kappa_1 = \frac{1}{n}\int \bar{g}\mathrm{d}v_\parallel$$

$$\kappa_2 = \frac{1}{n}\int \left(\frac{v_\parallel}{v_T}\right)\bar{g}\,\mathrm{d}v_\parallel$$

$$\kappa_3 = \frac{1}{n}\int \left(\frac{v_\parallel^2}{v_T^2}\right)\bar{g}\,\mathrm{d}v_\parallel$$

Gyrofluid equations

Take moments of the linear GK equation

$$\frac{\partial \bar{g}}{\partial t} + i v_\parallel k_\parallel \bar{g} = \frac{e\bar{F}_0}{T_i}\left[G_{\perp 0}\left(\frac{\partial}{\partial t} + i\omega_*(1+\eta(x_\parallel^2 - 3/2))\right) + i\omega_*\eta G_{\perp 2}\right]\phi$$

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$$\kappa_1 = \frac{1}{n}\int \bar{g}\mathrm{d}v_\parallel$$

- Parallel flow moment: κ_2

$$\frac{\partial \kappa_2}{\partial t} + i v_T k_\parallel \kappa_3 = 0$$

$$\kappa_2 = \frac{1}{n}\int \left(\frac{v_\parallel}{v_T}\right)\bar{g}\,\mathrm{d}v_\parallel$$

$$\kappa_3 = \frac{1}{n}\int \left(\frac{v_\parallel^2}{v_T^2}\right)\bar{g}\,\mathrm{d}v_\parallel$$

Constraining the moments

- We would like these moments to behave **like eigenmodes**

$$\partial\kappa_{1,2}/\partial t = -i\omega'\kappa_{1,2}$$

- Where $\omega' = \omega_r' + i\gamma'$ is similar to an eigenvalue

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$$\alpha = \omega' - \frac{1}{1+\tau}\left(G_{\perp 0}\left[\omega' - \omega_*(1-\eta)\right] - \omega_* \eta \, G_{\perp 2} \right)$$

$$\frac{\partial \kappa_1}{\partial t} + i v_T k_\parallel \kappa_2 = \frac{1}{1+\tau}\left[G_{\perp 0}\left(\frac{\partial}{\partial t} + i\omega_*(1-\eta) \right) + i\omega_* \eta G_{\perp 2} \right] \kappa_1 \longrightarrow \alpha \kappa_1 - v_T k_\parallel \kappa_2 = 0$$

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$$\frac{\partial\kappa_2}{\partial t} + iv_T k_\parallel \kappa_3 = 0 \longrightarrow \omega'\kappa_2 - v_T k_\parallel \kappa_3 = 0$$

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$$\frac{\partial \kappa_2}{\partial t} + i v_T k_\parallel \kappa_3 = 0 \longrightarrow \omega' \kappa_2 - v_T k_\parallel \kappa_3 = 0$$

- These equations are obeyed by linear modes for $\omega' \to \omega$
- Can we use these as *constraints*?

Problem statement

We seek distributions which:

- **Maximise Helmholtz free energy growth subject to:**
  1. Density moment constraint: $\alpha\kappa_1 - v_T k_{\|}\kappa_2 = 0$
  2. Flow moment constraint: $\omega'\kappa_2 - v_T k_{\|}\kappa_3 = 0$
  3. Consistent free energy balance: $\gamma' H = D$
- The solution to this problem gives an **upper bound on the growth of linear eigenmodes**
  - Guaranteed to at least as good as the unconstrained Helmholtz bound

Constrained optimal mode problem

- Problem can be formulated with a Lagrangian and multipliers

$$L \equiv D - \Lambda(H - H_0) - \lambda_1(\gamma' H - D)$$

$$-\lambda_2^*(\alpha^* \kappa_1^* - v_T k_\| \kappa_2^*) - \lambda_2(\alpha \kappa_1 - v_T k_\| \kappa_2)$$

$$-\lambda_3^*(\omega'^* \kappa_2^* - v_T k_\| \kappa_3^*) - \lambda_3(\omega' \kappa_2 - v_T k_\| \kappa_3)$$

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- **Optimal modes:** $\delta L / \delta \bar{g} = 0 \, , \;\; \delta L / \delta \lambda_n = 0$
  - Kinetic eigenvalue problem yields $\Lambda = \gamma'$

$$\Lambda \left(\bar{g} - \frac{\bar{F}_0 G_{\perp 0}}{(1+\tau)} \kappa_1 \right) = \frac{i \omega_* \eta \bar{F}_0 G_{\perp 0}}{2(1+\tau)} (x_\|^2 \kappa_1 - \kappa_3) - \lambda_2^* \bar{F}_0 (\alpha^* - x_\| v_T k_\|) - \lambda_3^* \bar{F}_0 (\omega'^* x_\| - x_\|^2 v_T k_\|)$$

  - Closes into a 5 × 5 linear system of moments and lagrange multipliers

Energetic upper bound

Upon solving the system of equations we are left with

$$P\tilde{\Lambda}^4 + Q\tilde{\Lambda}^2 + R = 0$$

- ω_r' remains as a free parameter.
    - **Upper bound on linear eigenmode growth:** $\Lambda_{\max} \equiv \max_{\omega_r'} \Lambda$ (best done numerically)

Low-$k_\perp$ limit $\eta \to \infty$:

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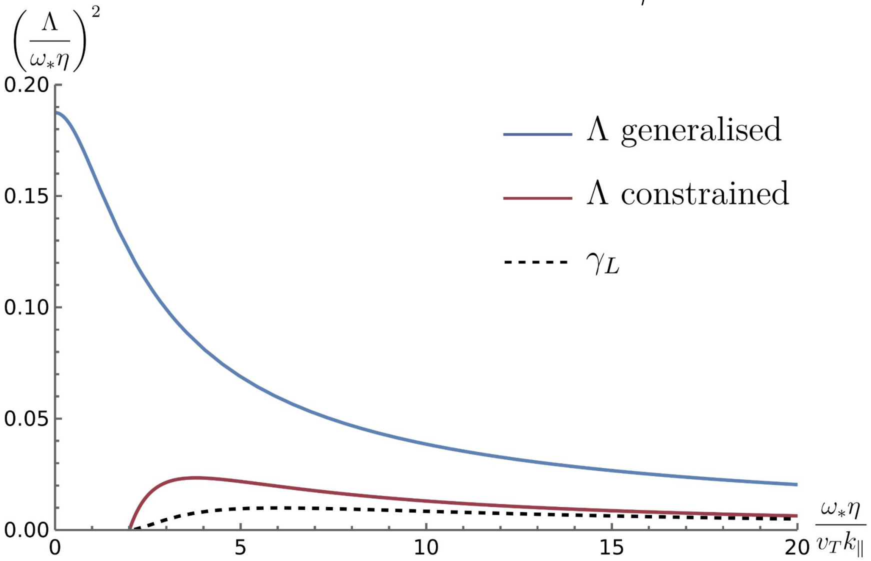
Low-$k_\perp$ limit $\eta \to \infty$:

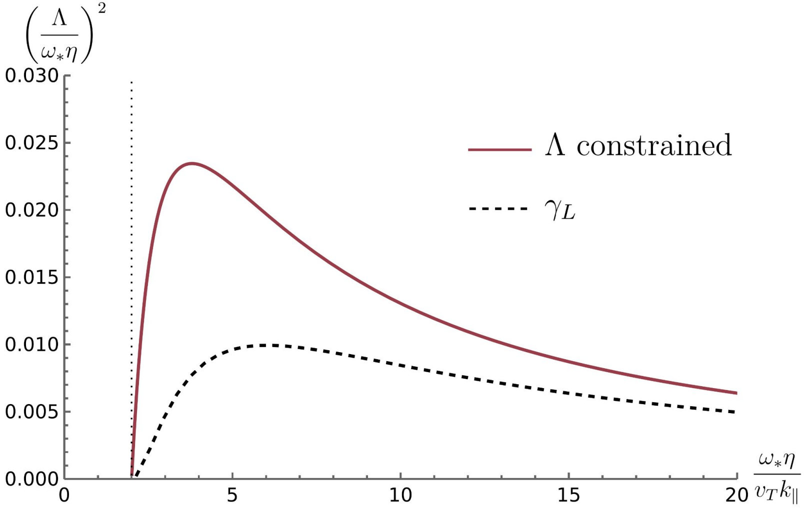


Constrained optimals capture “resonant” behaviour

In the low-$k_\perp$ limit

- Constrained optimal modes capture:

Critical gradient $\kappa_\| = \omega_* \eta / (v_{Ti} k_\|)$



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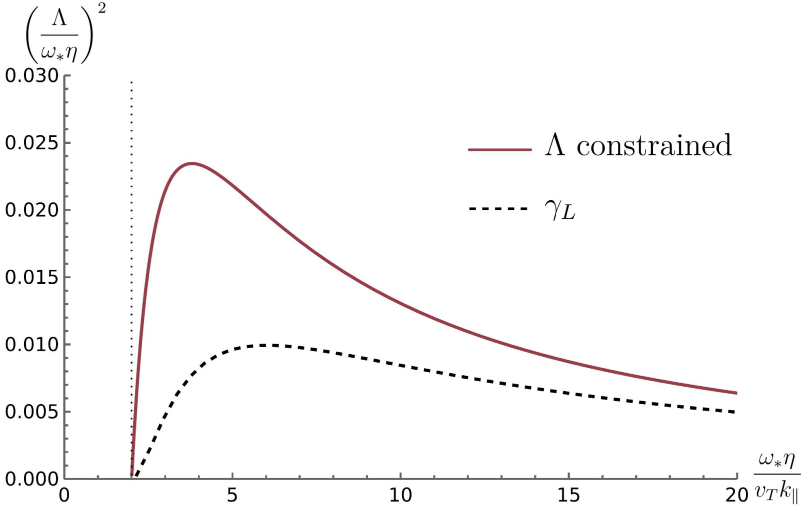
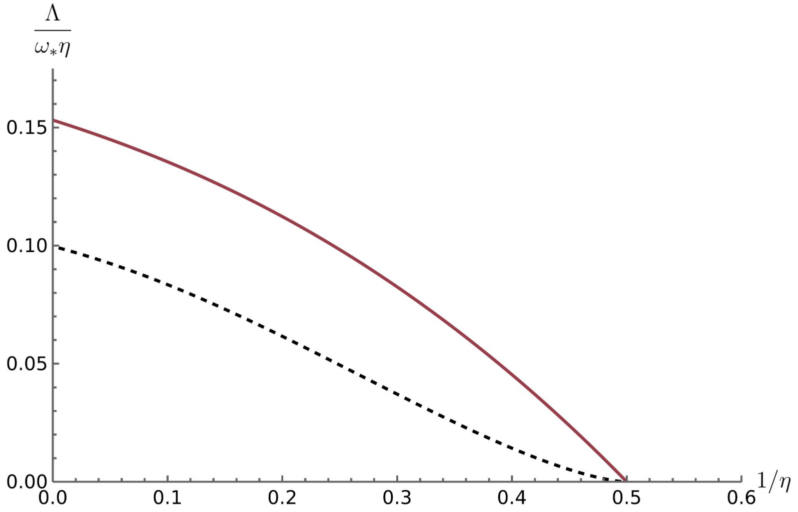


Density gradient stabilisation with $\frac{1}{\eta_a} = \frac{\mathrm{d} \ln T_i}{\mathrm{d} \ln n}$



Key takeaways from the constrained optimal modes

1. Gives a rigorous, **tighter** upper bound on linear eigenmode growth
2. Qualitative behaviour of linear instability can be captured across parameter space
3. Despite **low-dimensionality of the system**, critical gradient and density gradient stabilisation can be captured due to the inclusion of a "real frequency"
4. Provides a **general method** for tightening upper bounds to linear growth rates

Main results and ongoing/future work

Main results of this work:

- Tightest possible energetic upper bounds: *Case–Van Kampen energy*
- Simple bounds to capture behaviour of linear modes: *constrained optimal modes*

Next steps

- Constrained optimal modes in general geometry — stellarator optimisation?
- Optimal modes could be applied to lots of different problems
    - Subcritical turbulence, collisional optimals, ‘global’ optimals etc…
- Side projects: Van Kampen formulation of linear ZF problem (GAMs, residuals etc.)

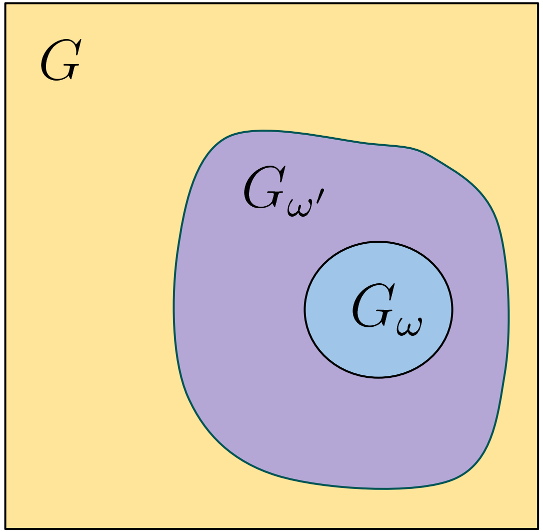
Case–Van Kampen optimal modes

$$\Lambda \mathcal{E} f = \mathcal{K} f$$

$$\mathcal{E}\, f = \bar{F}_0 \left(\sum_n \frac{\tilde{f}_n^*}{|C_n|^2} \int \tilde{f}_n' f' \,\mathrm{d}\, v_\parallel' + \int \mathrm{d}\omega \frac{\tilde{f}_\omega^*}{|C_\omega|^2} \int \tilde{f}_\omega' f' \,\mathrm{d}\, v_\parallel' \right)$$

$$\mathcal{K}\, f = \bar{F}_0 \sum_n \gamma_n \frac{\tilde{f}_n^*}{|C_n|^2} \int \tilde{f}_n' f' \,\mathrm{d}\, v_\parallel'$$

In essence: constrained space

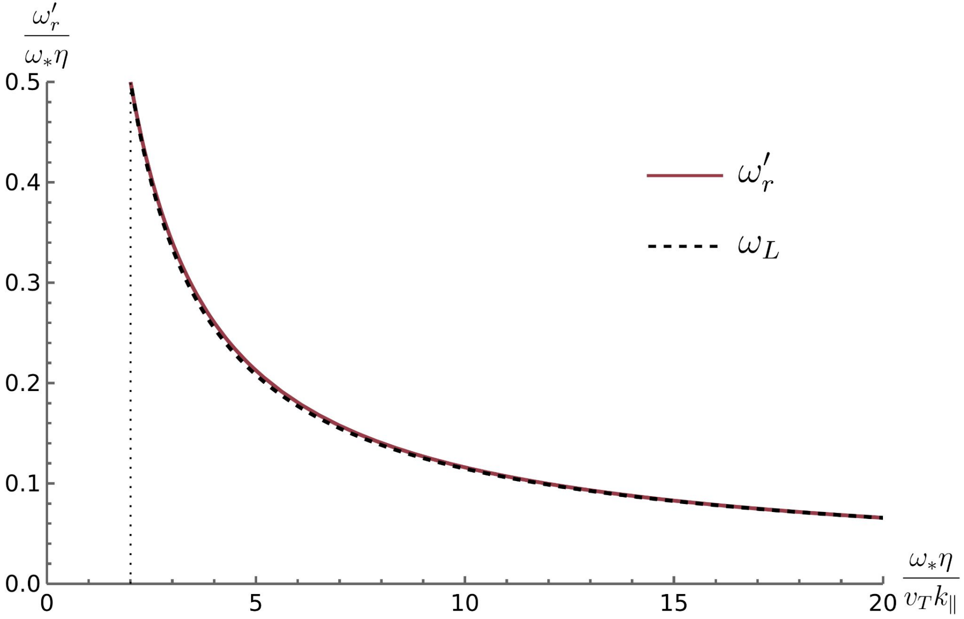


G : entire space of distributions

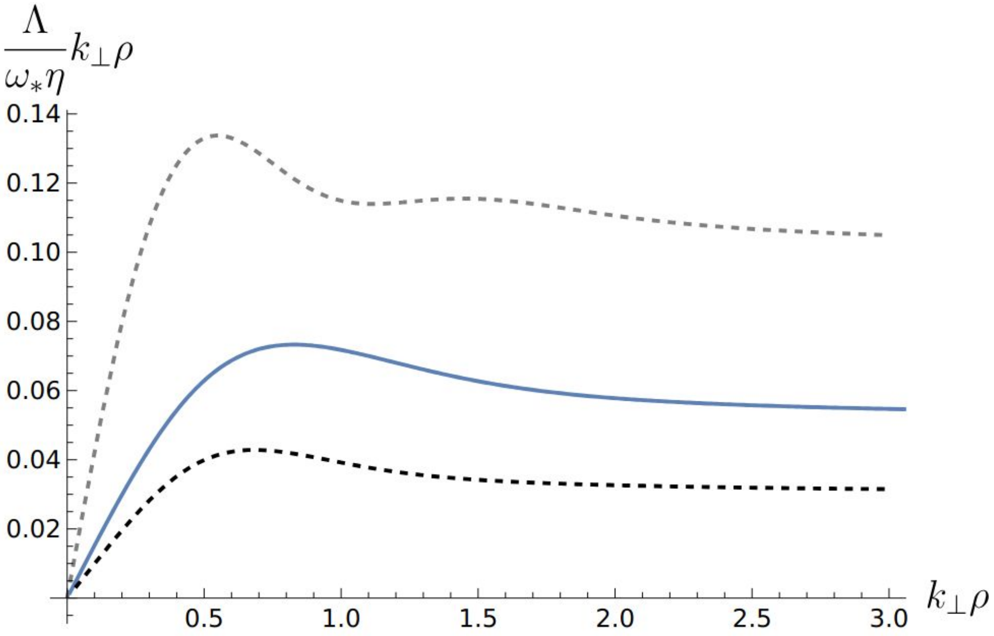
$G_{\omega'}$: distributions with
$$\begin{cases} \alpha\kappa_1 - v_T k_\| \kappa_2 = 0 \\ \omega'\kappa_2 - v_T k_\| \kappa_3 = 0 \end{cases}$$

G_ω : Set of linear eigenmodes

Frequency plot



Finite Larmor radius



Constrained optimal system of equations

$$\tilde{\Lambda}\left(1-\frac{G_{\perp 0}}{(1+\tau)}\right)\kappa_1=\frac{i}{2}\frac{G_{\perp 0}}{1+\tau}\left(\frac{\kappa_1}{2}-\kappa_3\right)-\tilde{\lambda}_2^*\tilde{\alpha}^*+\tilde{\lambda}_3^*\frac{\kappa_\parallel^{-1}}{2}$$

$$\tilde{\Lambda}\left(\kappa_3-\frac{G_{\perp 0}}{2(1+\tau)}\kappa_1\right)=\frac{i}{2}\frac{G_{\perp 0}}{1+\tau}\left(\frac{3\kappa_1}{4}-\frac{\kappa_3}{2}\right)-\tilde{\lambda}_2^*\frac{\tilde{\alpha}^*}{2}+\tilde{\lambda}_3^*\frac{3\kappa_\parallel^{-1}}{4}$$

$$2\tilde{\Lambda}\kappa_2=\tilde{\lambda}_2^*\kappa_\parallel^{-1}-\tilde{\lambda}_3^*\tilde{\omega}'^*$$

$$\tilde{\omega}'\kappa_2-\kappa_\parallel^{-1}\kappa_3=0$$

$$\tilde{\alpha}\kappa_1-\kappa_\parallel^{-1}\kappa_2=0$$