Weakly collisional plasmas: simple, drift & gyrokinetic

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(see JPP for more details)

Typical steady state kinetic equations

Fokker-Planck equation is of the "big, beautiful" form

resonance + collisions = drive + nonlinearity

*To keep it linear & resolve singularity need collisions

*Nonlinearity enters for weak collisions due to ∇_v or ∇

Can never ignore collisions for a monochromatic wave

Plasma wave and/or lower hybrid current drive

Electron kinetic equation in $v_{||}, v_{\perp}$ variables

$$\frac{\partial f_1}{\partial t} + v_{||}\frac{\partial f_1}{\partial z} - \frac{eE_{||}}{m}\left(\frac{\partial f_0}{\partial v_{||}} + \frac{\partial f_1}{\partial v_{||}}\right) = C\{f_1\}$$

with $\vec{B} = 0$ for PW or $\vec{B} \neq 0$ for LHCD

$$f_0 = n(m/2\pi T)^{3/2} e^{-mv^2/2T}$$

f_1 = perturbed distribution, but allow $\partial f_1/\partial v_{||} \sim \partial f_0/\partial v_{||}$

$E_{||} = \tilde{E}_{||}\sin(\omega t - k_{||}z)$ is an applied monochromatic wave

$C\{f_1\}$ = collision operator for electrons $\sim \nu_e v_e^2\, \partial^2 f_1/\partial v_{||}^2$

Widths: island vs. collisional boundary layer

* Collisional boundary layer width = $(\Delta v_{||})_\nu$: balancing

$$(k_{||}v_{||} - \omega)f_1 = k_{||}\Delta v_{||} f_1 \sim \nu_e v_e^2\, \partial^2 f_1 / \partial v_{||}^2 \sim \nu_e v_e^2 f_1/(\Delta v_{||})^2$$

gives

$$(\Delta v_{||})_\nu / v_e \sim (\nu_e / k_{||} v_e)^{1/3} \quad \& \quad \nu_{eff} \sim \nu_e (k_{||} v_e / \nu_e)^{2/3} \gg \nu_e$$

* Velocity space island width $=(\Delta v_{||})_{is}$: nonlinearity allows

$$f_1 k_{||} \Delta v_{||} \sim (e\tilde{E}_{||}/m)\, \partial f_1 / \partial v_{||} \sim (e\tilde{E}_{||}/m) f_1 / \Delta v_{||}$$

giving

$$(\Delta v_{||})_{is} / v_e \sim (e\tilde{E}_{||}/m k_{||} v_e^2)^{1/2} \ll 1$$

Collisional boundary layer ≫ island width

* Usual quasilinear (QL) limit = resonant plateau (RP)

* Need collisions to resolve singularity!
  Results seem independent of collisions, but are not!

* QL/RP theory fails when

$$1 \sim \frac{\partial f_1 / \partial v_{||}}{\partial f_0 / \partial v_{||}} \sim \frac{f_1 v_e}{f_0 (\Delta v_{||})_\nu} \sim \frac{(eE_{||}/m)}{k_{||}(\Delta v_{||})_\nu^2} \sim \frac{(\Delta v_{||})_{is}^2}{(\Delta v_{||})_\nu^2} \sim \frac{1}{\Delta^{2/3}}$$

* What happens when $(\Delta v_{||})_{is} \gg (\Delta v_{||})_\nu$ or

$$(e\tilde{E}_{||}/mk_{||}v_e^2)^{3/2} \gg (\nu_e/k_{||}v_e)$$

Full nonlinear equation for PW or LHCD

* Define $\phi = \omega t - k_{||}z$ & $u = v_{||} - \omega/k_{||}$, consider

$$k_{||}u\frac{\partial f_1}{\partial\phi} - \frac{e\widetilde{E}_{||}}{m}\sin\phi\left(\frac{\partial f_0}{\partial v_{||}} + \frac{\partial f_1}{\partial u}\right) = \nu v_{\perp z}^2\frac{\partial^2 f_1}{\partial u^2}$$

with $\partial f_0/\partial v_{||} \approx$ constant, then $f_1 = f_1(\phi, u)$. Let

$$f_1 = g(u,\phi) - (u - \sigma\alpha)\,\partial f_0/\partial v_{||}$$

with α a constant to be determined & $\sigma = u/|u| = \pm 1$ or 0

* Need to solve

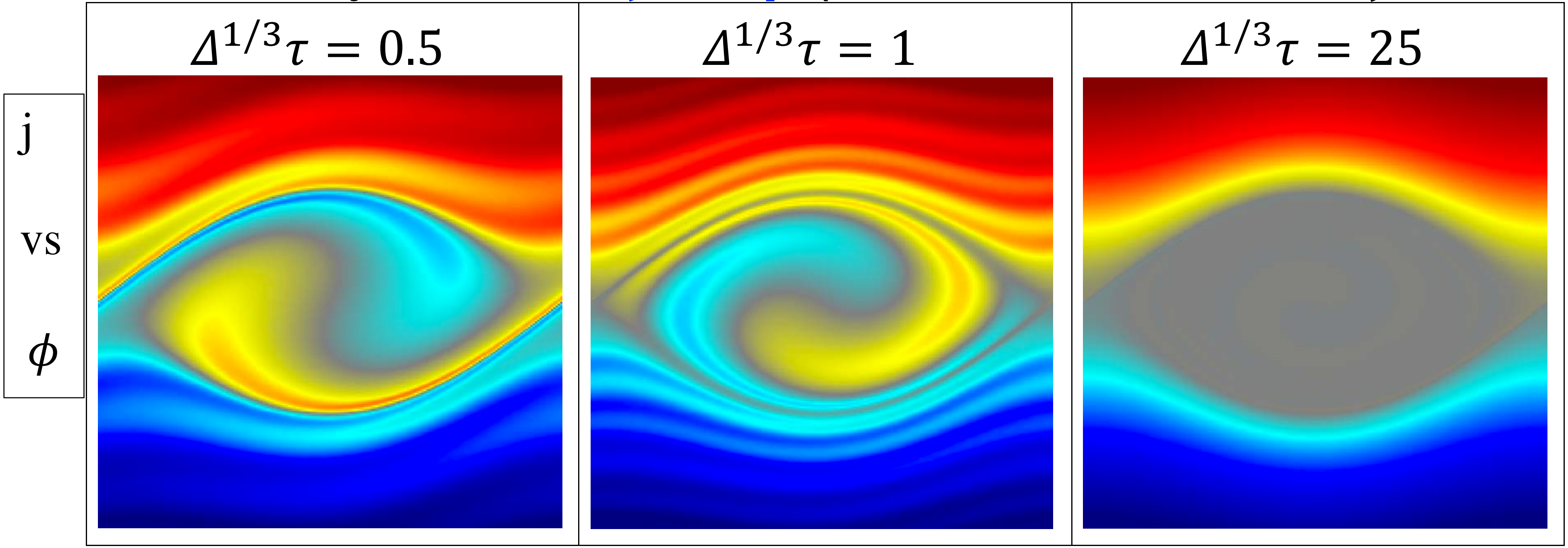
$$k_{||}u\frac{\partial g}{\partial\phi} - \frac{e\widetilde{E}_{||}}{m}\sin\phi\frac{\partial g}{\partial u} = \nu v_{\perp}^2\frac{\partial^2 g}{\partial u^2}$$

Nonlinear effects & collisional phase mixing

* Hamilton, Tolman, Arzamasskiy, Duarte (AJ 2023) solve

$$\partial g/\partial\tau + j\,\partial g/\partial\phi - \sin\phi\;\partial g/\partial j = \Delta\,\partial^2 g/\partial j^2$$

to find steady state for **j vs ϕ** (shown for $\Delta = 0.001$)



Collisionless vs collisional contrasted

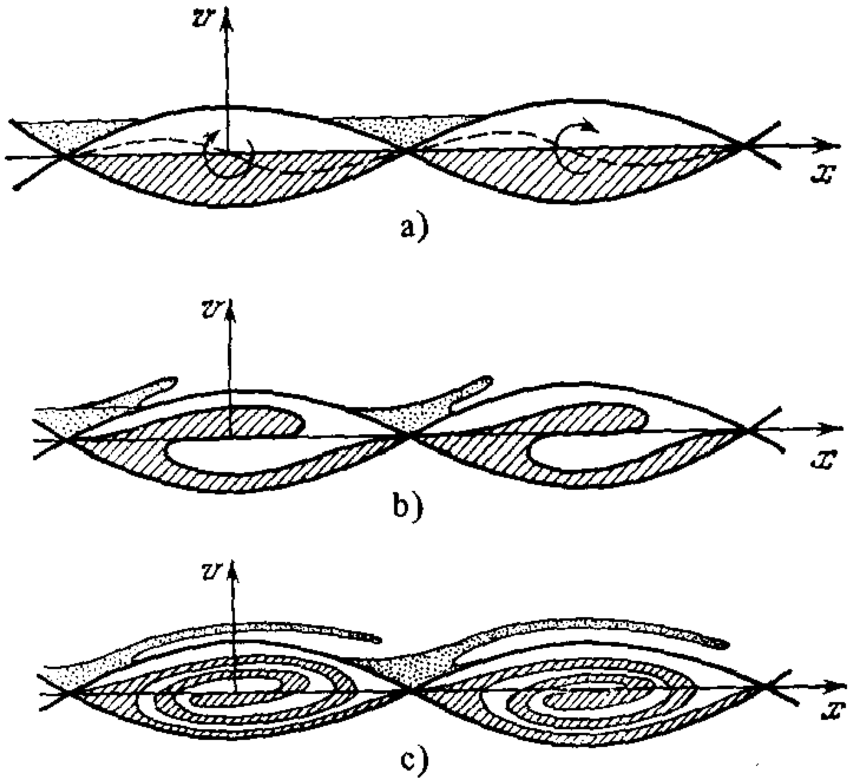
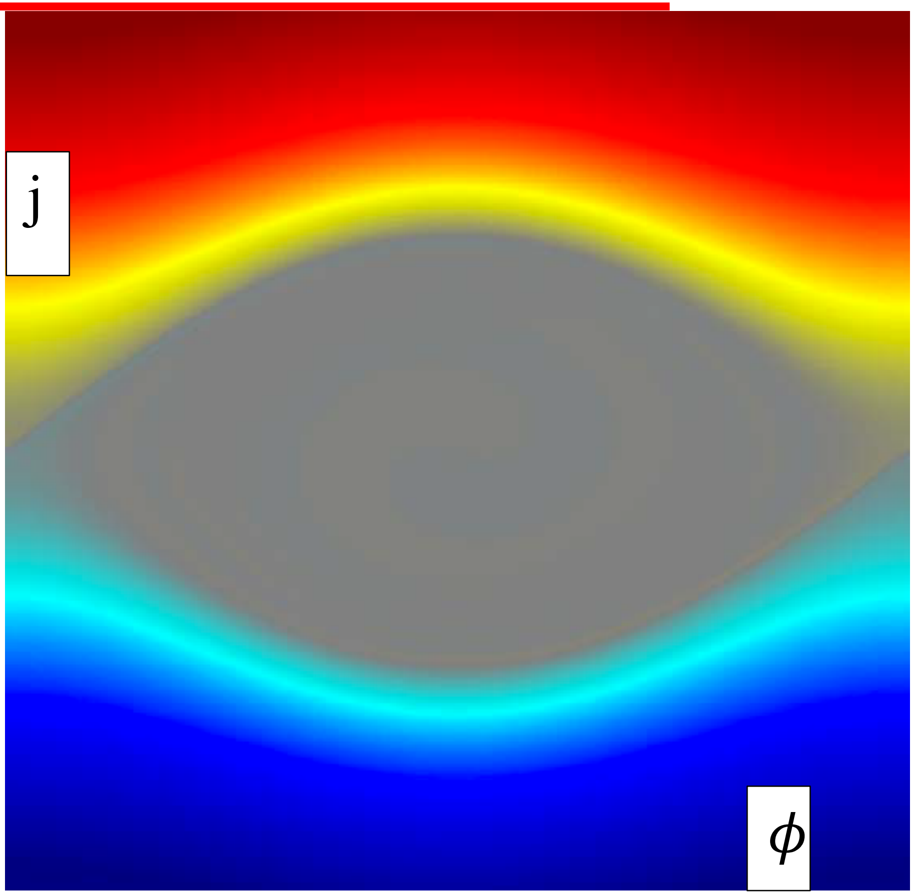


FIG. 3. Motion of resonant particles in a wave.



Kadomstev 1968 Sov. Phys. Usp. vs Hamilton *et al.* 2023 AJ

Nonlinear effects for $\Delta \ll 1$

* Can't solve $\Delta \sim (\Delta v_{||})_{\nu}^{3} / (\Delta v_{||})_{is}^{3} \sim 1$, but can solve $\Delta \ll 1$

* Normalizing gives Hamilton *et al.* steady state form

$$j \frac{\partial g}{\partial \phi} - \sin\phi \frac{\partial g}{\partial j} = \Delta \frac{\partial^2 g}{\partial j^2}$$

with $j \propto u$, island centered at $j = 0$ and

$$f_1 = g(u, \phi) - (u - \sigma\alpha)\, \partial f_0 / \partial v_{||}$$

where $\sigma = \pm 1$ for unbound and $\sigma = 0$ for bound

* Seek skew symmetric solution: $g(j, \phi) = -g(-j, -\phi)$

Reduced Hamiltonian

* Introduce reduced Hamilton *et al*. Hamiltonian

$$h = j^2/2 - \cos\phi$$

so that

$$j = \pm\sqrt{2(h + \cos\phi)} = j(h, \phi)$$

* Changing variables from j, ϕ to h, ϕ

$$\left.\frac{\partial g}{\partial \phi}\right|_h = \Delta \left.\frac{\partial}{\partial h}\right|_\phi \left(j \left.\frac{\partial g}{\partial h}\right|_\phi\right)$$

* Unlike Hamilton *et al*., interested in steady state

* Collisional boundary layer about separatrix at h = 1

Change variables

* In h, ϕ variables desire to solve for $\Delta \ll 1$

$$\left.\frac{\partial g}{\partial \phi}\right|_h = \Delta \left.\frac{\partial}{\partial h}\right|_\phi \left(j \left.\frac{\partial g}{\partial h}\right|_\phi\right)$$

* Lowest order motion is collisionless. Therefore

$$g = g_1(h, \sigma) + g_2(h, \varphi) + \ldots$$

* Desire skew symmetric solution: $g_1(j, \phi) = -g_1(-j, -\phi)$

* No need to solve next order, but must satisfy solubility

$$\left.\frac{\partial g_2}{\partial \phi}\right|_h = \Delta \left.\frac{\partial}{\partial h}\right|_\phi \left(j \left.\frac{\partial g_1}{\partial h}\right|_\phi\right)$$

Solubility & solution

* Integrate over a full bound period or full circulation

$$\frac{\partial}{\partial h}\Big|_{\phi}\left[\left(\oint_{h} d\phi\, j\right)\frac{\partial g_1}{\partial h}\Big|_{\phi}\right)\Big] = 0$$

* Bound: $g_1 = 0$ (no collisional flux across h surfaces)

* Unbound: $g_1 \propto \sigma \int_{\kappa}^{1} d\tau/\tau^2 E(\tau)$ & $\kappa = \sqrt{2/(h+1)}$

* Full solution

$$f_1 = \{\sigma \left|\frac{e\tilde{E}_{||}}{mk_{||}}\right|^{1/2}[\pi \int_{\kappa(j,\phi)}^{1} \frac{d\tau}{\tau^2 E(\tau)} + 1.379] - u\} \frac{\partial f_0}{\partial v_{||}}$$

Plasma wave (PW)

Power absorbed by electrons in the Landau (1946) limit is

$$P_0 = (\tilde{E}_{||}^2/8\pi)(2\pi^{1/2}\omega_p^2\omega^2/k_{||}^3 v_e^3)e^{-\omega^2/k_{||}^2 v_e^2}$$

Power absorbed in Zakharov & Karpman (1963) limit is

$$P \approx 0.144(Z+2)mnv_e^2\nu_{ee}|e\tilde{E}_{||}k_{||}/m\omega^2|^{1/2}e^{-\omega^2/k_{||}^2 v_e^2}$$

The ratio vanishes as $\Delta \rightarrow 0$

$$\frac{P}{P_0} \approx 0.081(Z+2)\frac{\nu_{ee}k_{||}^2 v_e^2}{\omega^3}\left|\frac{mk_{||}v_e^2}{e\tilde{E}_{||}}\right|^{3/2} \sim \Delta \ll 1$$

No collisionless ($\nu \equiv 0$) limit for finite $\tilde{E}_{||}$: Landau limit is a plateau regime with $1 \gg \nu_{ee}/\omega \gg (\omega/k_{||}v_e)^2|e\tilde{E}_{||}/k_{||}T_e|^{3/2}$

Intense applied LHCD vs. quasilinear

* Normalized quasilinear current drive efficiency of Fisch

$$\frac{J_{||}^{LH}/env_e}{P_{cd}^{LH}/mnv_e^2\nu_{ee}} = \frac{16\omega^2}{3\pi^{1/2}(Z+5)k_{||}^2 v_e^2}$$

* Normalized intense lower hybrid wave efficiency

$$\frac{\langle J_{||}\rangle_\phi/env_e}{P/mnv_e^2\nu_{ee}} = \frac{2.99\omega^2}{(Z+2)k_{||}^2 v_e^2}\left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right|$$

* Intense limit smaller by $\frac{(\Delta v_{||})_{is}^2}{v_e^2} \sim \left|\frac{e\tilde{E}_{||}}{mk_{||}v_e^2}\right| \ll 1$

Stellarator: trapped alpha drift resonance at $\bar{\omega}_\alpha = 0$

* $\bar{\omega}_\alpha$ reverses direction at some pitch angle⟹resonance

* Drift reversal results in collisional transport

* Superbanana or resonant plateau neglects nonlinear term

* Islands can form at resonances⟹small radial scale

* Island width ~ collisional boundary layer width when

$$\partial \tilde{f} / \partial r \sim \partial \bar{f} / \partial r$$

* Transport transitions from "plateau" to "linear" in ν

Form of alpha resonance

* At large aspect ratio $\overline{\omega}_\alpha$ reverses at $2E(\kappa_0) = K(\kappa_0)$

$$\kappa^2 = [1-(1-\epsilon)\lambda]/2\epsilon\lambda$$

* Trapped boundary depends on inverse aspect ratio ϵ

$$1-\epsilon < \lambda = 2\mu B_0/v^2 < 1+\epsilon$$

* Expand about $\kappa_0^2 = 0.83$: $\overline{\omega}_\alpha = -2(\kappa^2 - \kappa_0^2)\overline{\omega}'_\alpha \sim v^2/\Omega_0 R_0^2 \epsilon$

$$\overline{\omega}_\alpha = [\lambda - (1-0.66\epsilon)]\overline{\omega}'_\alpha/\epsilon$$

* Resonance depends on a different λ at each $\epsilon \Longrightarrow$ **pods**
  (same $\overline{\omega}_\alpha$ if $\Delta\lambda + 0.66\Delta\epsilon = 0$)

* Small departure from QS: $\epsilon \gg \delta$ = non-QS

Islands can help in a nearly quasiymmetric stellarator

Bounce averaged drift kinetic eq. for trapped alphas

$$\bar{\omega}_\alpha \frac{\partial \tilde{f}}{\partial \phi} - \bar{V}\sin\phi \frac{\partial(\bar{f} + \tilde{f})}{\partial r} = \bar{\nu}\epsilon \frac{\partial^2 \tilde{f}}{\partial \lambda^2}$$

with $\bar{f}$ = unperturbed slowing down tail distribution

$\tilde{f}$ = perturbed distribution ($\bar{f} \gg \tilde{f}$, trapped fraction $\sim\sqrt{\epsilon}$)

$\bar{\omega}_\alpha$ = bounce average trapped drift in a flux surface

$\bar{V}\sim\bar{\omega}_\alpha R_0 \delta$ = bounce average radial drift due to QS departure

ϕ = QS breaking helical angular variation

$\bar{\nu} = v_\lambda^3/v^3\tau_s$ pitch angle collision freq. of alphas by ions

Superbanana plateau or resonant plateau limit

* Linear eq., no pods, inhomogeneous Airy eq.

$$[\lambda - \lambda_0(\epsilon)]\frac{\bar{\omega}'_\alpha}{\epsilon}\frac{\partial\tilde{f}}{\partial\phi} - \frac{\bar{V}}{R_0}\sin\phi\,\frac{\partial\bar{f}}{\partial\epsilon} = \bar{\nu}\epsilon\frac{\partial^2\tilde{f}}{\partial\lambda^2}$$

* Drift ~ collisions ⟹ RP layer width $\Delta\lambda \sim (\epsilon^2\bar{\nu}/\bar{\omega}_\alpha)^{1/3}$

$$\bar{\nu}_{\mathrm{eff}} \sim \bar{\nu}\epsilon/(\Delta\lambda)^2 \sim \bar{\nu}\epsilon(\bar{\omega}_\alpha/\epsilon^2\bar{\nu})^{2/3}$$

* RP diffusivity independent of collisions: v_0 = birth speed

$$D_{rp} \sim (\Delta\lambda/\epsilon^{1/2})(\bar{V}/\bar{\nu}_{\mathrm{eff}})^2\bar{\nu}_{\mathrm{eff}} \sim \epsilon^{1/2}\bar{V}^2/\bar{\omega}_\alpha \sim q v_0^2\delta^2/\Omega_0\epsilon^{1/2}$$

When does the island width matter?

* Full nonlinear eq. allows pods since $\lambda = 1 - 0.66\epsilon$

$$[\lambda - (1 - 0.66\epsilon)]\frac{\bar{\omega}'_\alpha}{\epsilon}\frac{\partial\tilde{f}}{\partial\phi} - \frac{\bar{V}}{R_0}\sin\phi\,\frac{\partial(\bar{f}+\tilde{f})}{\partial\epsilon} = \bar{\nu}\epsilon\frac{\partial^2\tilde{f}}{\partial\lambda^2}$$

* Island width: nonlinear term ~ drift, $\bar{V}/R_0\Delta\epsilon \sim \bar{\omega}'_\alpha\Delta\epsilon/\epsilon$

$$\Delta\epsilon \sim (\bar{V}\epsilon/\bar{\omega}_\alpha R_0)^{1/2} \sim (\epsilon\delta)^{1/2} \ll \epsilon^{1/2}$$

* Collisional boundary layer larger than island width if

$$(\epsilon^2\bar{\nu}/\bar{\omega}_\alpha)^{1/3} \sim \Delta\lambda \gg \Delta\epsilon \sim (\epsilon\delta)^{1/2}$$

* Plateau limit assumes $\partial\tilde{f}/\partial\epsilon \sim \tilde{f}/\Delta\epsilon \ll \partial\bar{f}/\partial\epsilon \sim \bar{f}/\epsilon$ or

$$\tilde{f}/\bar{f} \ll \Delta\epsilon/\epsilon \sim (\delta/\epsilon)^{1/2} \ll \Delta\lambda/\epsilon \sim (\bar{\nu}/\bar{\omega}_\alpha\epsilon)^{1/3}$$

* What happens if $\Delta\lambda \sim \Delta\epsilon$ or $\Delta\lambda \ll \Delta\epsilon$?

Nonlinear effects for $\Delta \ll 1$

* Normalizing the trapped nonlinear drift kinetic equation

$$(x-\Lambda)\frac{\partial\tilde{f}}{\partial\phi} - \sin\phi\left(\bar{f}' + \frac{\partial\tilde{f}}{\partial x}\right) = \Delta\frac{\partial^2\tilde{f}}{\partial\Lambda^2}$$

with $x \propto \epsilon$, $\Lambda \propto (1-\lambda)$ and $\bar{f}' = \partial\bar{f}/\partial x = \text{constant}$

* Let $\tilde{f} = g - (x-\Lambda)\bar{f}'$ & $j = x - \Lambda = \sqrt{2(h+\cos\varphi)}$

$$j\frac{\partial g}{\partial\phi} - \sin\phi\frac{\partial g}{\partial j} = \Delta\frac{\partial^2 g}{\partial j^2}$$

* Pods because $j = j(x,\Lambda)$ depends on both ϵ & λ

* $D_\nu/D_{rp} \sim \Delta \ll 1 \Longrightarrow$ large islands reduce radial transport

What about gyrokinetics?

Keeping the magnetic $\vec{v}_M$ & perturbed $\langle\vec{v}_E\rangle_R$ drifts

$$\partial\bar{h}/\partial t + [v_{||}\vec{b} + \vec{v}_M + \langle\vec{v}_E\rangle_R]\cdot[\nabla_R\bar{h} - Ze\nabla_R\langle\Phi\rangle_R\,\partial\bar{h}/\partial\bar{E}]$$

$$- C_1\{\bar{h}\} = -(Zef_M/T)[(\partial\langle\tilde{\Phi}\rangle_R/\partial t) - \omega_*^T(\partial\tilde{\Phi}/\partial\zeta)]$$

with $\bar{E} = v^2/2$

$$\tilde{f} = h - (Ze\tilde{\Phi}/T)f_M - (Iv_{||}/\Omega)\,\partial f_M/\partial\psi$$

$$\langle\vec{v}_E\rangle_R = (c/B)\vec{b}\times\nabla_R\langle\Phi\rangle_R$$

and

$$\omega_*^T = \frac{cT}{Zef_M}\frac{\partial f_M}{\partial\psi} = \frac{cT}{Zep}\left[\frac{\partial p}{\partial\psi} + Zen\frac{\partial\bar{\Phi}}{\partial\psi} + \left(\frac{Mv^2}{2T} - \frac{5}{2}\right)n\frac{\partial T}{\partial\psi}\right]$$

Island forming nonlinearity normally neglected

Define

$$\langle\dot{\vec{R}}\rangle_R = v_{||}\vec{b} + \vec{v}_M + \langle\vec{v}_E\rangle_R$$

then usually assume

$$\frac{[Ze\,\langle\dot{\vec{R}}\rangle_R\cdot\nabla_R\langle\Phi\rangle_R]\,\partial\bar{h}/\,\partial\bar{E}}{\langle\dot{\vec{R}}\rangle_R\cdot\nabla_R\bar{h}} \sim \frac{Ze\langle\Phi\rangle_R}{\bar{h}}\frac{\partial\bar{h}}{\partial\bar{E}} \sim \frac{Ze\langle\Phi\rangle_R}{T} \ll 1$$

If there is velocity space structure due to $\tilde{\Phi}$ then

$$\frac{[Ze\,\langle\dot{\vec{R}}\rangle_R\cdot\nabla_R\langle\tilde{\Phi}\rangle_R]\,\partial\bar{h}/\,\partial\bar{E}}{\langle\dot{\vec{R}}\rangle_R\cdot\nabla_R\bar{h}} \sim \frac{Ze\langle\tilde{\Phi}\rangle_R}{\bar{h}}\frac{\partial\bar{h}}{\partial\bar{E}} \sim \frac{Ze\langle\tilde{\Phi}\rangle_R}{T}\frac{\sqrt{T/M}}{\Delta v_{||}} \sim 1$$

Velocity space structure

Resonances lead to velocity space structure so estimate

$$\partial\bar{h}/\partial t + \langle\dot{\vec{R}}\rangle_R \cdot [\nabla_R \bar{h} - Ze\nabla_R\langle\tilde{\Phi}\rangle_R\, \partial\bar{h}/\partial\bar{E}] \sim \bar{h}\Delta v_{||}/qR$$

As before, balancing with

$$C_1\{\bar{h}\} \sim (\nu T/M)\bar{h}/(\Delta v_{||})_\nu^2$$

and

$$Zev_{||}\vec{b}\cdot\nabla_R\langle\tilde{\Phi}\rangle_R\,\partial\bar{h}/\partial\bar{E} \sim \bar{h}ZeE_{||}/M(\Delta v_{||})_{is}$$

Might matter for "stronger" turbulence

$$\frac{(\Delta v_{||})_{is}^3}{(\Delta v_{||})_\nu^3} \sim \frac{(ZeE_{||}qR/T)^{3/2}}{(\nu qR/\sqrt{T/M})} \sim \frac{1}{\Delta} \sim 1$$

What about mode coupling term of "weak" turbulence?

Mode coupling $\langle\vec{v}_E\rangle_R \cdot \nabla_R \bar{h}$ drives a cascade to small scales but also alters the resonance (as in stellarator transport)

Resonance estimate must depend on tokamak geometry

$$\langle\vec{v}_E\rangle_R \cdot \nabla_R \bar{h} \sim \langle\vec{v}_E\rangle_{rad} \bar{h}/\Delta r$$

$$\partial\bar{h}/\partial t + \langle\dot{\vec{R}}\rangle_R \cdot \nabla_R \bar{h} \sim \bar{h} v \Delta r/qR^2$$

for crude guesstimate $\partial v_{||}/\partial r \sim v/R$ & $k_{||} \sim 1/qR$. Then

$$\Delta r/R \sim (q\langle\vec{v}_E\rangle_{rad}/v)^{1/2} \ll 1$$

and perhaps $\Delta r/R \gg (\Delta v_{||})_\nu/v$ implies pods matter?

Comments & crazy thought

There is no collisionless limit (Zakharov & Karpman)

Is phase mixing ever collisionless?

Islands become pods (and uglier) in confined plasmas

Collisions always matter, but do details in codes matter?
(they do matter for a monochromatic wave)

Do pods/islands and/or resonances need to be resolved?
(do these details of the cascade to small scales matter)

Will simulations go to the same saturated state?