Monge-Ampère gravitation

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16th Plasma Kinetics Working Meeting
WPI U-Wien, 21 Jul-1/01 Aug 2025

The classical gravitational Vlasov-Poisson system

$$\partial_t f(t,x,\xi) + \nabla_x \cdot (\xi f(t,x,\xi)) + \nabla_\xi \cdot (\nabla \varphi(t,x) f(t,x,\xi)) = 0, \quad (t,x,\xi) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}^d,$$

$$f(t,x,\xi) \geq 0, \quad \Delta \varphi(t,x) = \rho(t,x) - 1, \quad \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)\, d\xi, \quad \int_{\mathbb{T}^d} \rho(t,x)\, dx = 1$$

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admits a non-linear, "Monge-Ampère", correction

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as ρ concentrates: $|\nabla\varphi(t,x)| \leq \mathrm{diam}(\mathbb{T}^d)$ (Y. B., G. Loeper GAFA 2004).

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N.B. This is similar to Born-Infeld 1934 nonlinear Electromagnetism where any electrostatic force is unconditionaly bounded (see Y.B. ARMA 2004).

MONGE-AMPERE GRAVITATION: a 256^3 particle simulation of the early universe based on the 3D version of Mérigot's semi-discrete Monge-Ampère solver. Each "Laguerre cell" corresponds to a cluster of galaxies!

With B. Lévy (INRIA) and R. Mohayaee (Institut d'Astrophysique de Paris) 2024.

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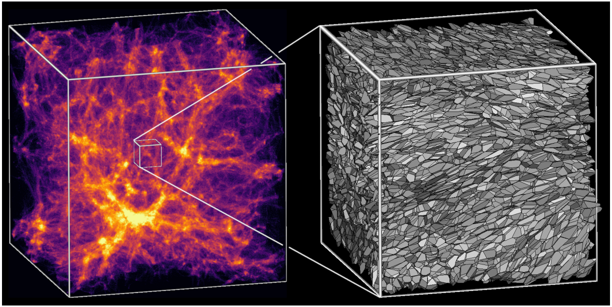

FIG. 5. Simulation of Monge-Ampère gravity (60 Mpc/h, 256^3 particles), and zoom on the Laguerre cells of the central region.

MONGE-AMPERE GRAVITATION (MAG)

(Y.B., G. Loeper GAFA '04, Y.B.Confl. Math '11, B. Lévy, Y.B., R. Mohayahee arXiv 24)

$$\rho(t,x) = \det(I + D^2\varphi(t,x)) \ \text{ instead of } \ \rho(t,x) = 1 + \triangle\varphi(t,x)$$

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iii) might be as good as the Poisson equation as an approximation to the Einstein equations (conjecture), based on the analogy

$$\frac{\text{Einstein equation}}{\text{Ricci curvature}} \sim \frac{\text{Monge}-\text{Ampere equation}}{\text{Gauss curvature}}$$

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v) enjoys a nice stochastic interpretation in terms of brownian clouds!

MONGE-AMPERE vs NEWTON (B. Lévy, Y.B., R. Mohayaee arxiv 2404.07697v2)

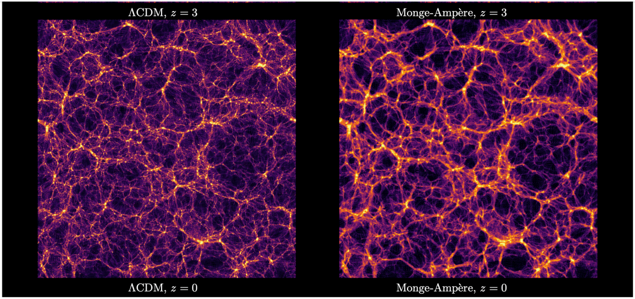

FIG. 8. Comparison between 3D simulations of ΛCDM (using an adaptive-mesh algorithm similar to [21]) and Monge-Ampère in a cube of 300 Mpc/h, 512^3 particles, z=5, 3 and 0. Projected integrated density in a 15 Mpc/h thick slab, using a logarithmic color scale. Large-scale similarity between the two models is striking, however MAG creates more abundant and diffuse filaments, whereas ΛCDM creates highly-clustered small haloes. There is weaker clustering because MAG does not diverge and is screened at short distances.

PURELY STOCHASTIC ORIGIN OF MONGE-AMPERE GRAVITATION FROM THE LARGE DEVIATIONS OF BROWNIAN CLOUDS

Ambrosio, Baradat, B., Analysis and PDEs '22, Léonard, Mohayaee arXiv 24
(picture taken from B. Lévy, Y.B., R. Mohayaee arxiv 2404.07697v2)

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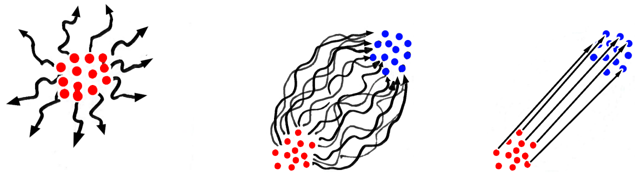

FIG. 1. Left panel: unconditioned motion of M independent Brownian particles; Center panel: motion of independent Brownian particles conditioned by their initial and final positions (in red and blue respectively); Right: conditioned Brownian motion with vanishing noise, all trajectories tend to geodesics.

BROWNIAN CLOUDS

We define a brownian cloud to be a finite set of N indistinguishable points in the euclidean space, initially located on a finite cubic lattice $\{A(\alpha) \in \mathbb{R}^d,\ \alpha = 1, \cdots, N\}$ and subject to N independent Brownian motions in $\mathbb{R}^d$, with uniform noise ν.

DIFFUSION EQUATION AND BROWNIAN CLOUDS

In PDE terms, we just consider the diffusion equation in $\mathbb{R}^{Nd}$:

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\nu}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^{N} \delta(X(\alpha) - A(\sigma(\alpha)))$$

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where the initial data take the relabeling symmetry into account so that $\rho(t,X)$ is just the probability density of finding the brownian cloud at position X (up to a permutation of the labels) at time t

$$\rho(t,X) = \frac{1}{N!}(2\pi\nu t)^{-Nd/2} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^{N} \exp\left(-\frac{|X(\alpha) - A(\sigma(\alpha)|^2}{2\nu t}\right)$$

L'ONDE PILOTE aka osmotic velocity or score (AI)

After solving the diffusion equation in the space of "clouds" $X \in \mathbb{R}^{Nd}$

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we may solve the companion ODE in the same space $\mathbb{R}^{Nd}$

$$\frac{dX_t}{dt} = v(t, X_t), \quad v(t, X) = -\frac{\nu}{2} \nabla(\log \rho)(t, X), \quad X_{t_0} = Y_0 \text{ given in } \mathbb{R}^{Nd}$$

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This is an adaptation of de Broglie's "onde pilote" idea. As a matter of fact, a similar calculation also works for the free Schrödinger equation.

"ONDE PILOTE" AND ZERO NOISE LIMIT

Setting $t = \exp(2\tau)$, we more explicitly get

(with abuse of notation $X_t \to X_\tau$):

$$\frac{dX_\tau}{d\tau} = -\nabla_X \Phi_{\nu,\theta}(X_\tau)\ , \quad \Phi_{\nu,\tau}(X) = \nu \exp(2\tau) \log \sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-||X - A_\sigma||^2}{2\nu \exp(2\tau)}\right)$$

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Surprisingly enough, we may easily pass to the limit $\nu \to 0$
in the class of maximal monotone operators (cf. Brezis' book)

$$\frac{d_+ X_\tau}{d\tau} = -\overline{\nabla}_X \Phi(X_\tau), \quad \Phi(X) = \lim_{\nu \to 0} \Phi_{\nu,\tau}(X) = -\inf_{\sigma \in \mathcal{S}_N} \ ||X - A_\sigma||^2/2$$

Indeed, $\Phi_{\nu,\tau}(X)$ reads $-\frac{||X||^2 + ||A||^2}{2} \ + \ $ a convex function of X.

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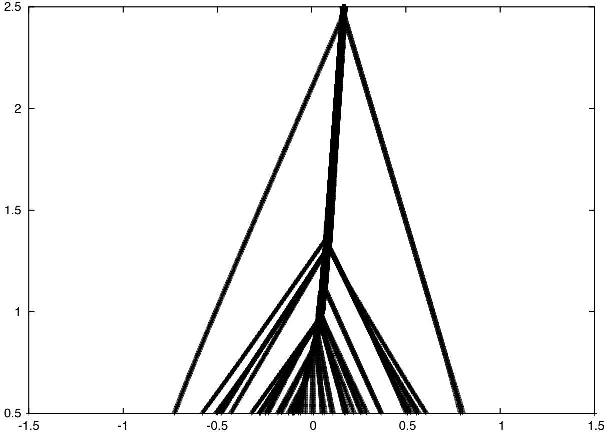
Indeed, $\Phi_{\nu,\tau}(X)$ reads $-\frac{||X||^2+||A||^2}{2}$ $+$ a convex function of X.
N.B. Through $\overline{\nabla}_X \Phi$, this equation includes sticky collisions in 1D.

In 1D this just reduces to "dust" with sticky collisions

horizontal : 51 *grid points in x* / *vertical* : 60 *grid points in t*



LARGE DEVIATIONS OF THE "ONDE PILOTE"

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$\eta \to 0$, WHILE $\nu > 0$ IS KEPT FIXED:

$$\mathrm{Prob}(X_{\tau_0} = Y_0,\ X_{\tau_1} = Y_1) \sim \exp\left(-\frac{\mathcal{A}}{2\eta}\right)$$

$$\boxed{\mathcal{A} = \inf_X \int_{\tau_0}^{\tau_1} ||\frac{dX_\tau}{d\tau} + \nabla_X \Phi_{\nu,\tau}(X_\tau)||^2 d\tau, \quad X_{\tau_0} = Y_0,\ X_{\tau_1} = Y_1}$$

ZERO-NOISE LIMIT OF THE ACTION FUNCTIONAL

THEOREM (L. Ambrosio, A. Baradat, Y.B. Analysis and PDE 2023)

$$\int_{\tau_0}^{\tau_1} ||\frac{dX_\tau}{d\tau} + \overline{\nabla}_X \Phi(X_\tau)||^2 d\tau, \quad \Phi(X) = -\inf_{\sigma \in \mathfrak{S}_N} \; ||X - A_\sigma||^2/2$$

(which -at least in 1D- handles sticky collisions thanks to $\overline{\nabla}_X \Phi$)
is the "$\Gamma-$limit", as $\nu \to 0$, of the Freidlin-Wentzell Action functional

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RECOVERY OF MONGE-AMPERE GRAVITATION!

Using the least action principle, we obtain

$$\frac{d^2 X_\tau(\alpha)}{d\tau^2} = X_\tau(\alpha) - A(\sigma_{opt}(\alpha)) \ , \quad X_\tau(\alpha) \in \mathbb{R}^d, \ \ \alpha = 1, \cdots, N$$

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Finally, using Optimal Transport tools, we find that, as $N \to \infty$

$$f_N(\tau, x, \xi) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(x - X_\tau(\alpha)) \delta\left(\xi - \frac{dX_\tau(\alpha)}{d\tau}\right)$$

asymptotically solves the Monge-Ampère gravitational model

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THANKS!

THE SEMI-NEWTONIAN GRAVITATIONAL MODEL OF THE EARLY UNIVERSE (Zeldovich, Peebles...)

The trajectory $t \in \mathbb{R}_+ \to X_t(a) \in \mathbb{R}^3$ of each "particle" labelled by $a \in \mathbb{R}^3$ (mod $\mathbb{Z}^3$ for simplicity) is driven by

$$\frac{2t}{3} \frac{d^2 X_t}{dt^2} + \frac{dX_t}{dt} + (\nabla\varphi)(t, X_t) = 0, \quad 1 + t \triangle \varphi = \rho = \int_{\mathbb{T}^3} \delta(x - X_t(a)) da$$

where $\rho(t, x)$ and $\varphi(t, x)$, $x \in \mathbb{T}^3$, respectively denote the density field (supposed to be of unit average) and the gravitational potential.

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cf. Uriel Frisch and coll. Nature 417 (2002), with a renewed interest after the launching of the James Webb Space Telescope 25/12/2021.

VLASOV-POISSON FORMULATION

The Peebles equations

$$\frac{2t}{3}\frac{d^2X_t}{dt^2}+\frac{dX_t}{dt}+(\nabla\varphi)(t,X_t)=0,\quad 1+t\triangle\varphi=\rho=\int_{\mathbb{T}^3}\delta(x-X_t(a))da$$

can be translated as the singular (at $t=0$), non-autonomous, Vlasov-Poisson system

$$\partial_t f+\nabla_x\cdot(\xi f)+\nabla_\xi\cdot\left(\frac{3}{2t}(\xi+\nabla\varphi)f\right)=0,\quad 1+t\triangle\varphi=\int_{\mathbb{R}^3}f(t,x,\xi)d\xi$$

just by setting

$$f(t,x,\xi)=\int_{\mathbb{T}^3}\delta\left(x-X_t(a)\right)\delta\left(\xi-\frac{dX_t}{dt}(a)\right)da,\quad (t,x,\xi)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3.$$

ZELDOVICH APPROXIMATION

A very simple approximate solution EXACT in 1D was proposed by Zeldovich in the 1970s for the semi-newtonian model

$$\frac{2t}{3}\frac{d^2X_t}{dt^2} + \frac{dX_t}{dt} + (\nabla\varphi)(t, X_t) = 0, \quad \rho = \int \delta(x - X_t(a))da = 1 + t \triangle \varphi$$

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$$\to: \quad X_t(a) = a - t\nabla\varphi_0(a), \quad \bigtriangleup\varphi_0(x) = \lim_{t\to 0}\frac{\rho(t,x) - 1}{t}$$

Each particle just travels with a constant velocity due to the initial density fluctuation, until a collision ocurs, which is somewhat reminiscent of Lucretius' (99-55 BC) "DE RERUM NATURA".

DE RERUM NATURA LIBER SECUNDUS 216 – 224

LUCRETIUS (99 – 55*BC*)

Quod nisi declinare solerent (corpora), omnia deorsum imbris uti guttae caderent per inane profundum ...Ita nihil umquam natura creasset.

But if (corpora) were not in the habit of deviating, they would all fall straight down through the depths of the void, like drops of rain... In that case, nature would never have produced anything.

MONGE-AMPERE GRAVITATION (MAG)

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iv) has a computational complexity similar to Poisson thanks to the Monge-Ampère solver by Quentin Mérigot (2D) and Bruno Lévy (3D).