



UKAEA

Modelling EM turbulence in STEP

Some problems, ideas, and solutions(*)

Dan Kennedy *et al.*

United Kingdom Atomic Energy Authority

(*) Most of the solutions will be in the talk by M. Giacomin

STEP plasma turbulence

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The confinement challenge for STEP plasmas

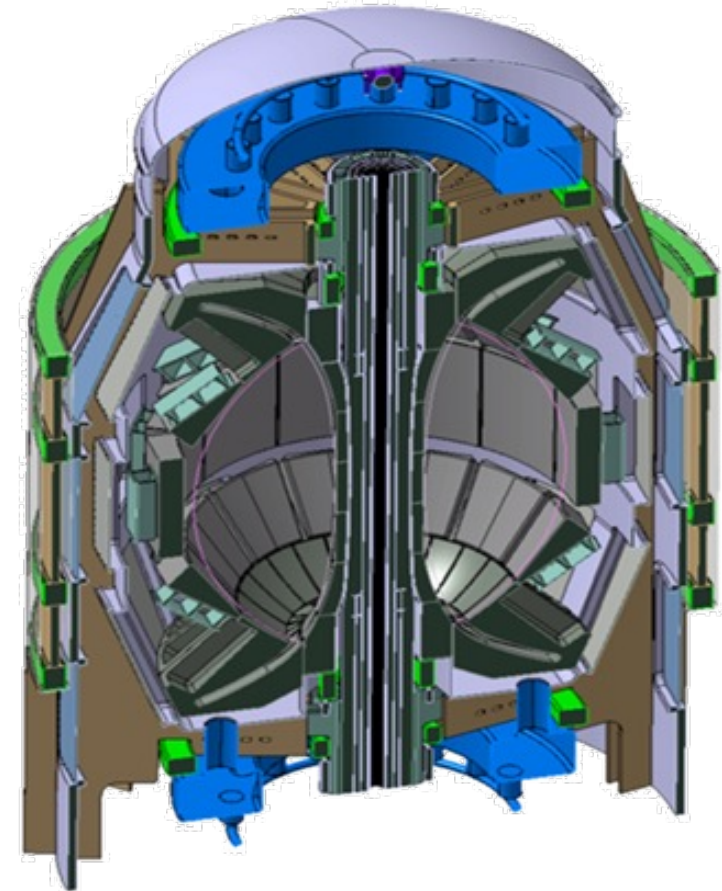
(the) Spherical Tokamak for Energy Production is a UK programme, aiming to develop a compact **prototype reactor** that aims to deliver **net electric power** to the national grid.

Fusion performance is dependent on **large core pressure** (or β) and **low turbulent transport**:

- Aiming to achieve high core pressure.
- Spherical tokamak (ST): small radius, steep gradients.
- ST necessitates both large β and large β' .
- The pressure and profiles attainable depend crucially on transport and confinement.
- **Increasing importance of electromagnetic (EM) instabilities.**

$$\beta = \frac{\text{thermal pressure}}{\text{magnetic pressure}}$$

$$\beta' = \frac{d\beta}{dr}$$

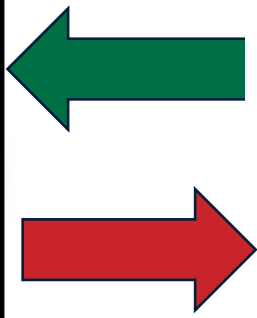
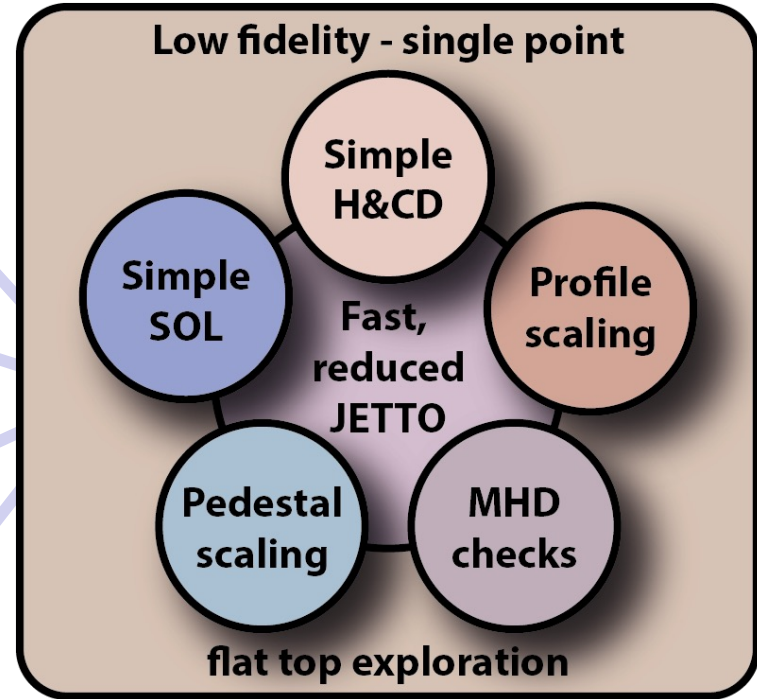


How are STEP plasma equilibria designed?

- 0D systems code PROCESS → 1D JETTO used as an assumption integrator.

- Fusion gain $Q > 11$
- Fusion power $P_{\text{fus}} > 1.5 \text{ GW}$
- Vertical stability
- MHD consistency

...



	SPR-45
R_{geo}	3.60
A	1.8
$B_T (R_{\text{geo}})$	3.2
I_p [MA]	20.9
n_{e0} [10^{20} m^{-3}]	2.05
T_{e0} [keV]	18.0
κ	2.93
δ	0.59
P_{fus} [GW]	1.76
P_{ECCD} [MW]	150
P_{rad} [MW]	338
Q	11.8
β_N	4.4

- **TAKEWAY:** multiple equilibria from low-fidelity modelling and assumptions
- Confinement assumed (relative to H98 scaling).
- Core transport based on an Bohm-gyro-Bohm model tuned on MAST to give dominant e^- heat transport and desired β_N (INPUT).

How are STEP plasma equilibria designed?

STEP has designed sets of plasmas based on sets of assumptions.

Do such plasmas exist?

Are the transport assumptions consistent with the predictions of local gyrokinetics (GK)?

PART I: D. Kennedy

- GK analysis of one STEP plasma.
- Progress understanding the transition to the ultra-high-flux state.

PART II: M. Giacomin

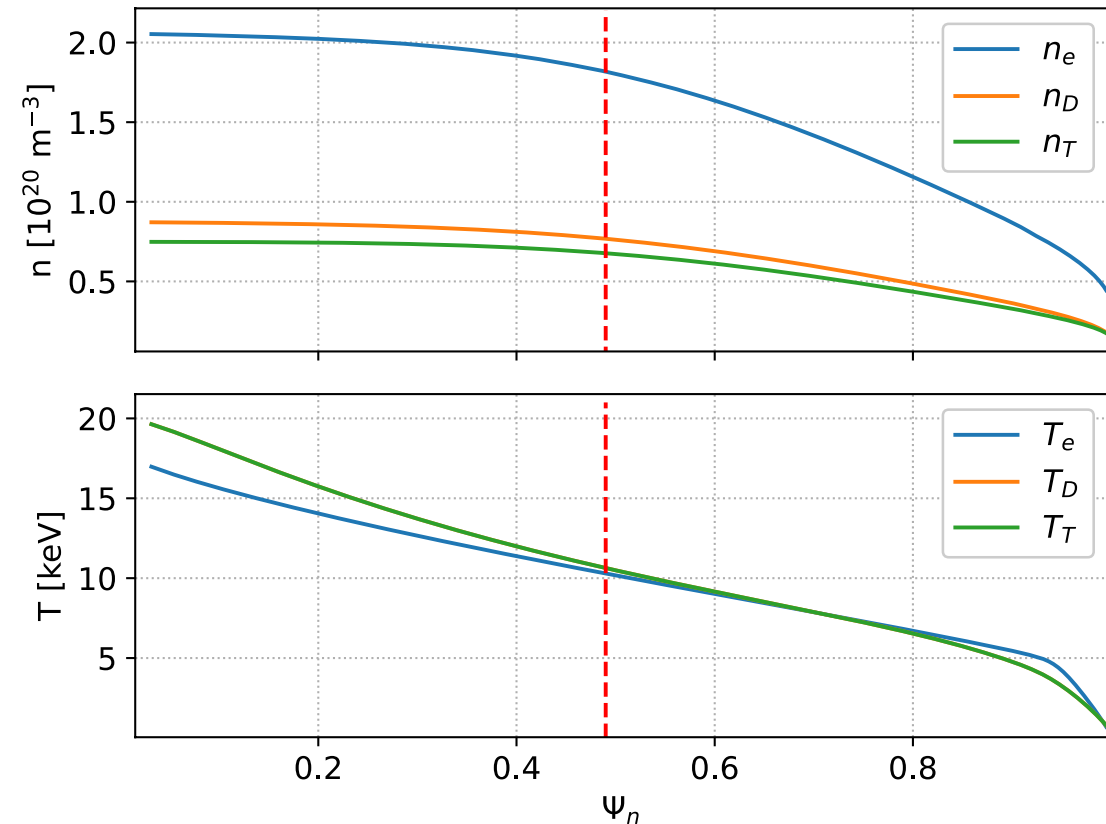
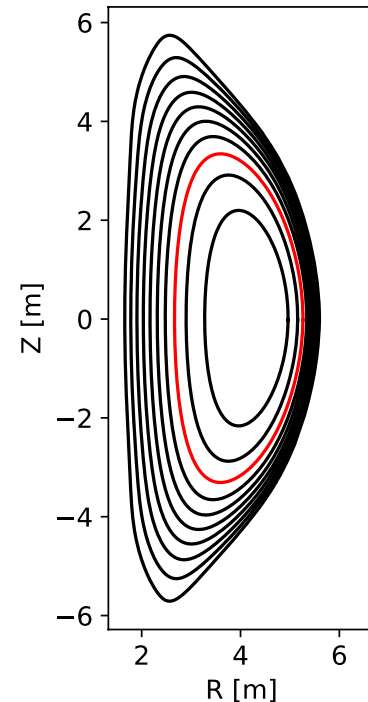
- Reduced transport model for STEP.
- Towards flux-driven STEP simulations.

Outline of Talk

Today: I will discuss **one** of the preferred scenarios, the STEP High Density Electron Cyclotron (HD-EC) flat-top (FTOP).

- **PART I:** local GK analysis on a single mid-radius flux surface*.
- **PART II:** Global GK analysis of a single step FTOP.
- **PART III:** Saturation (or lack thereof) of electromagnetic turbulence in local δf GK (stress).

Parameter	Value	Parameter	Value
Ψ_n	0.49	B_0 [T]	2.8
q	3.5	n_e [10^{20}m^{-3}]	1.81
\hat{s}	1.2	T_e [keV]	10.3
κ	2.56	ρ_s [mm]	5.2
κ'	0.06	n_D/n_e	0.53
δ	0.29	n_T/n_e	0.47
δ'	0.46	T_D/T_e	1.03
Δ'	-0.40	T_T/T_e	1.03
β_e	0.09	a/L_{n_e}	1.03
β'	-0.48	a/L_{n_D}	1.06
r [m]	1.3	a/L_{n_T}	0.99
R [m]	4.0	a/L_{T_e}	1.58
A_{surf} [m^2]	370	a/L_{T_D}	1.82
P_{surf} [MW]	500	a/L_{T_T}	1.82



PART I: Local Gyrokinetics

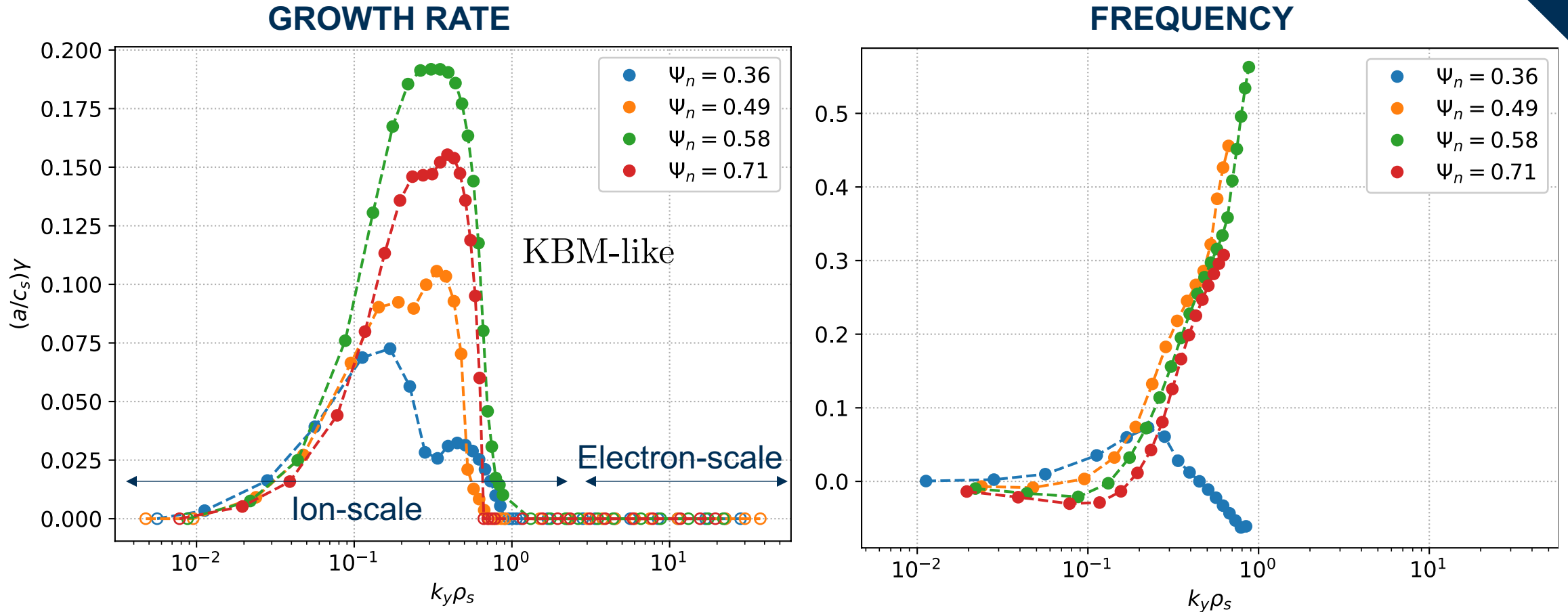
Results in this section:

D. Kennedy et al 2023 Nucl. Fusion 63 126061

M Giacomini et al 2024 Plasma Phys. Control. Fusion 66 055010

D. Kennedy et al 2024 Nucl. Fusion 64 086049

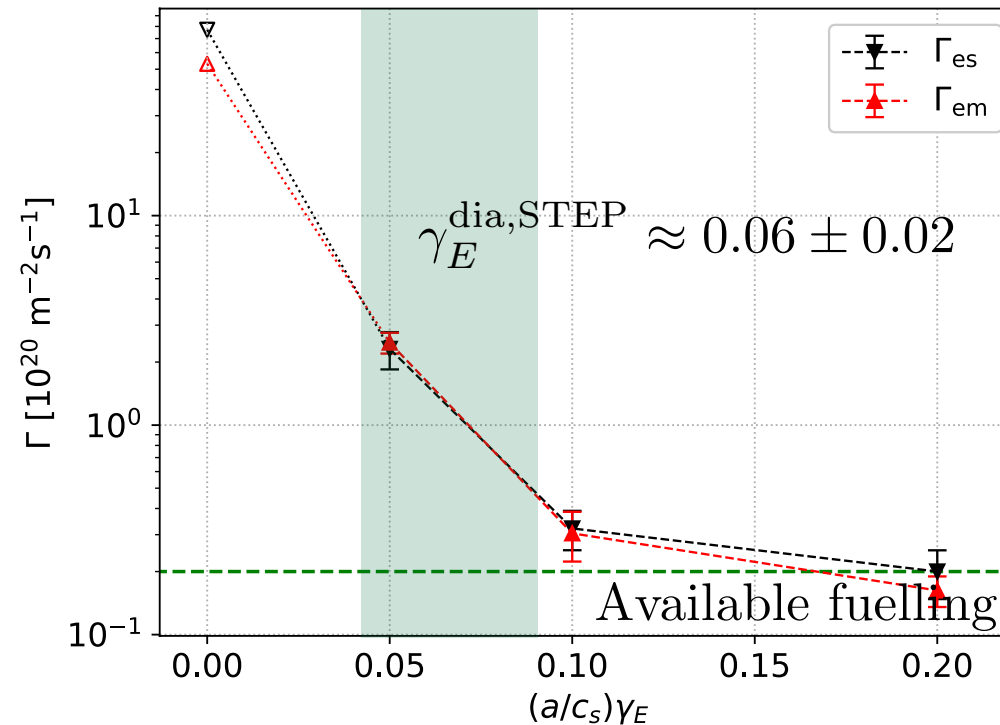
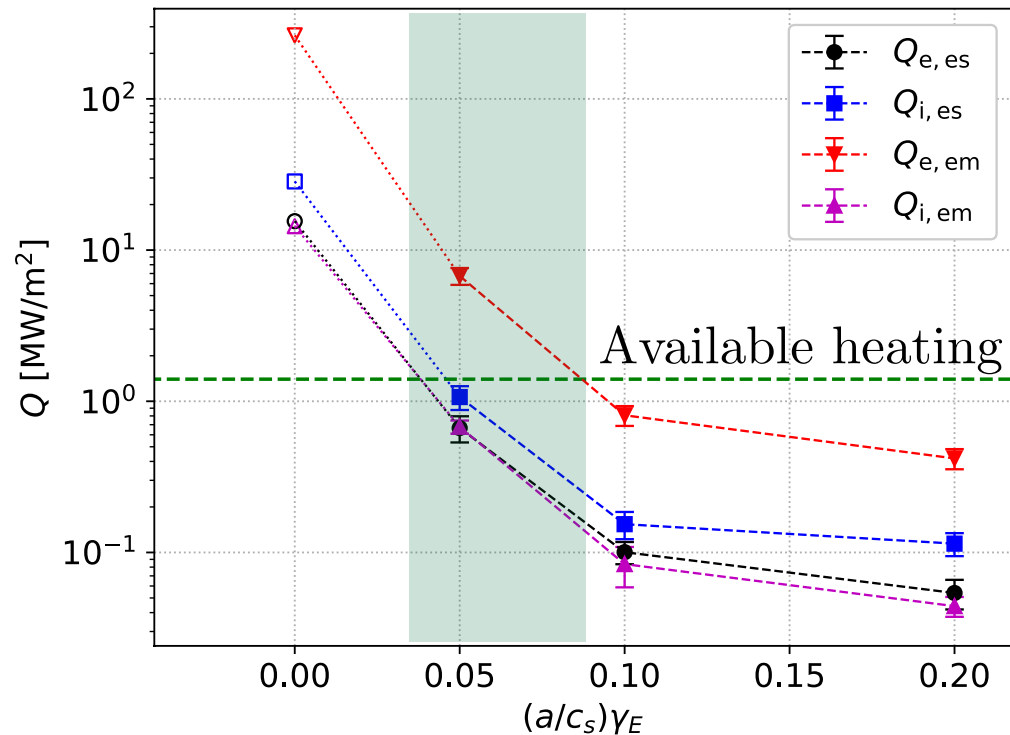
LINEAR stability analysis (most unstable mode)



- Ion binormal scale dominated by **KBM-like** mode. δB_{\parallel} essential for STEP, but not for the physics.
- **No** unstable electron scale modes.
- Subdominant ion scale MTM.

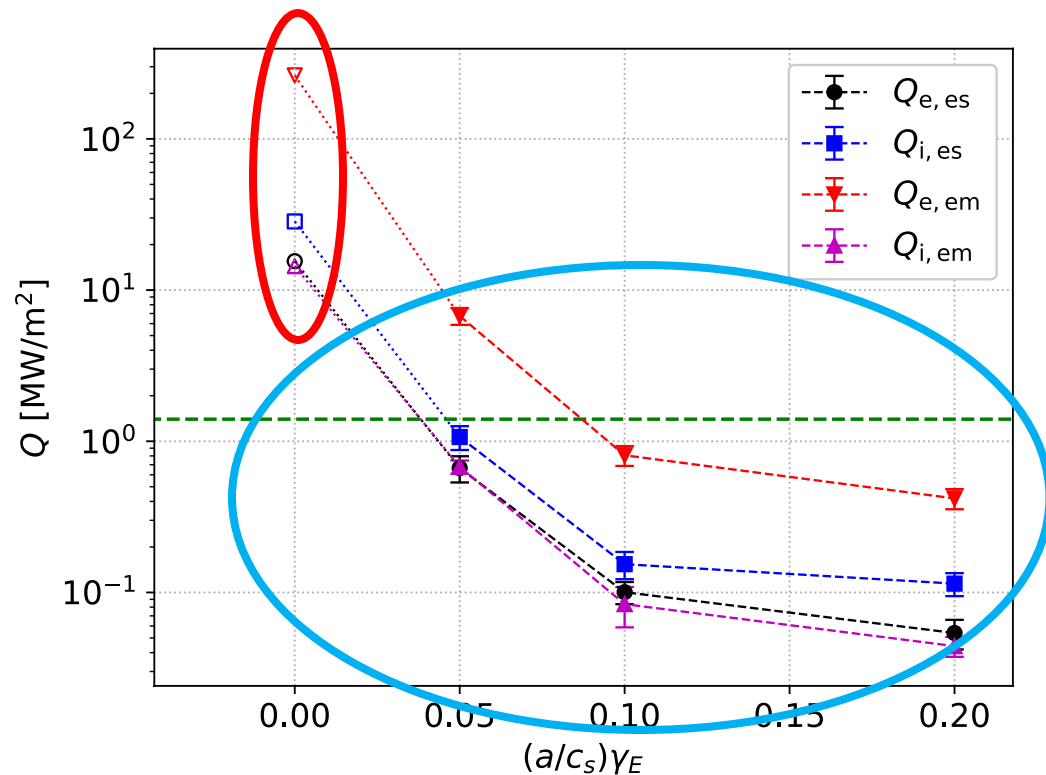
NONLINEAR local gyrokinetic calculations – the *hybrid-KBM*

- CGYRO simulations of *hybrid-KBM*-driven turbulence with diamagnetic flow shear (no PVG).
- Electromagnetic electron heat flux largely dominates.
- Strong effect of equilibrium flow shear on the predicted fluxes.



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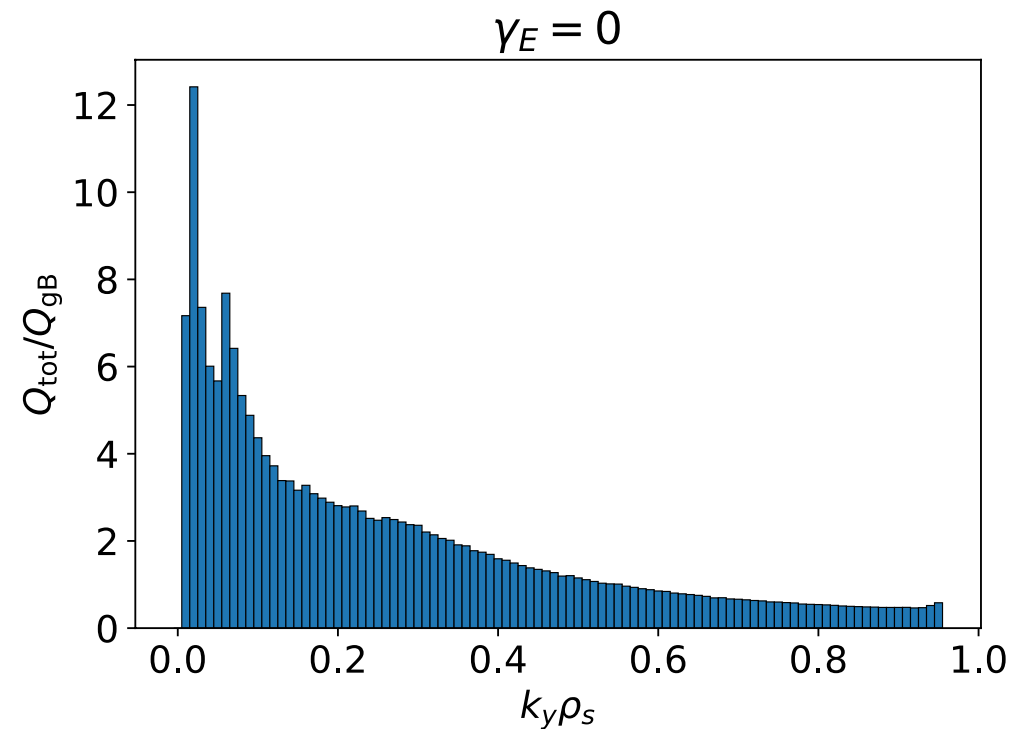
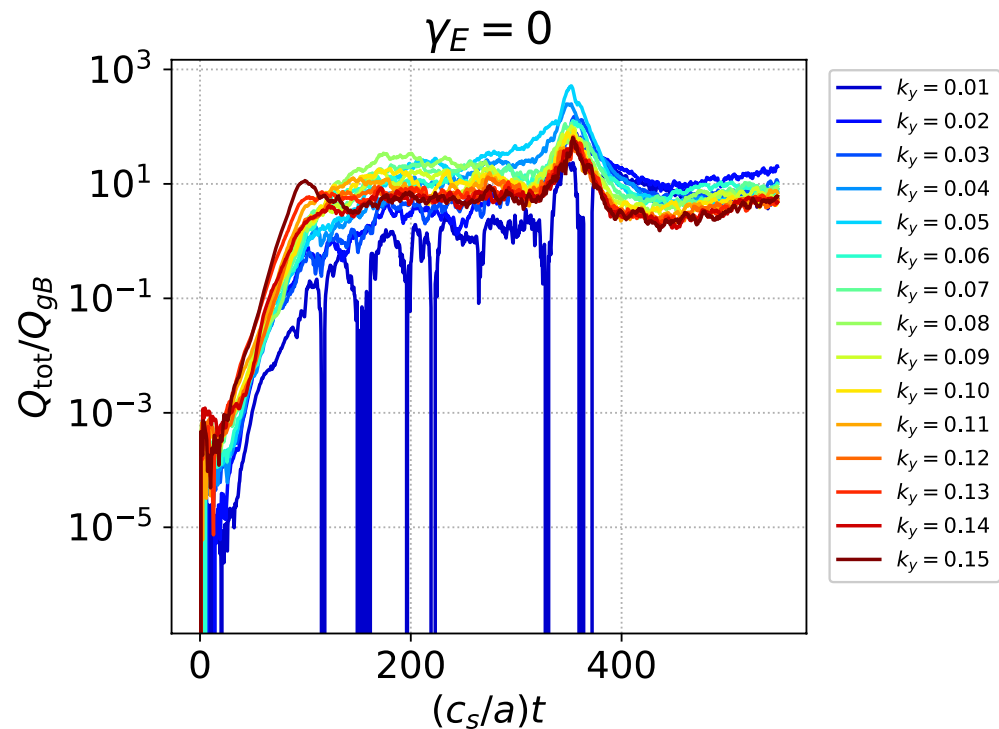
TWO classes of simulation:

- Simulations **with $\gamma_E > 0$**
 - Broadly compatible with available fueling and heating at shearing rates which are **difficult to achieve in STEP.**
- Simulations **with $\gamma_E = 0$**
 - Transition to an ultra-large-flux state.

Focus on simulations with no equilibrium flows.

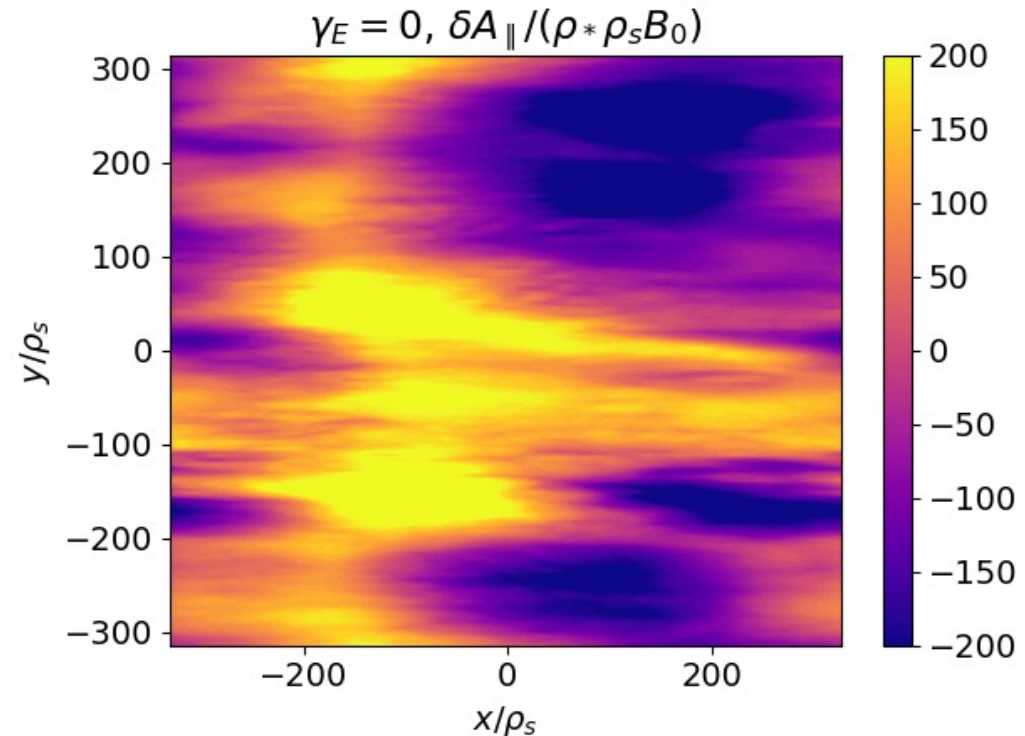
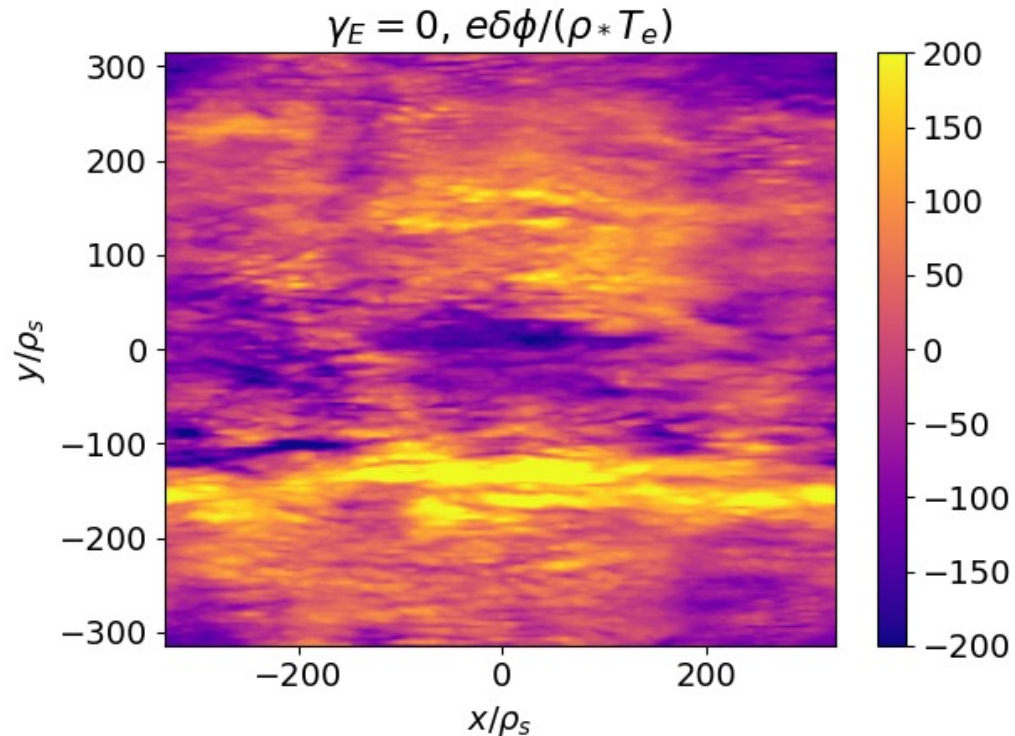
NONLINEAR local gyrokinetic calculations – the *hybrid-KBM* – simulations **without** equilibrium flow shear.

- Simulations **without** equilibrium flow shear are challenging to saturate.
- Modes at **long-wavelength** ($k_y \rho_s \ll 1$) grow slowly but reach very large values.



NONLINEAR local gyrokinetic calculations – the *hybrid-KBM* – simulations **without** equilibrium flow shear.

- Simulations **without** equilibrium flow shear are challenging to saturate.
- Modes at **long-wavelength** ($k_y \rho_s \ll 1$) grow slowly but reach very large values.
- Turbulence is characterised by **radially extended** eddies.



PART II: Global Gyrokinetics

Results in this section:

D. Kennedy, F. Sheffield, M. Giacomin, T. Görler, C. M. Roach, *et al.* ...

NONLINEAR δf global gyrokinetic calculations

- Some people are very worried about the “size” of boxes in the context of the validity of the local approximation.

“All of your simulations require “a large radial box width”. If I plug in the value of ρ_* then it looks like the size of the computational domain is larger than the minor radius of STEP. How can such a computation be “local”. I am very upset by this. Global simulations appear mandatory”

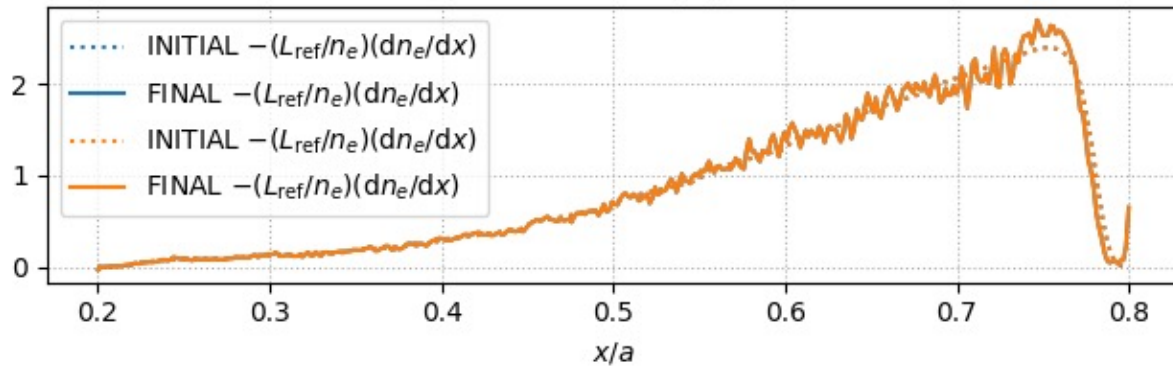
-- Some people

- This isn't how asymptotic analysis works.

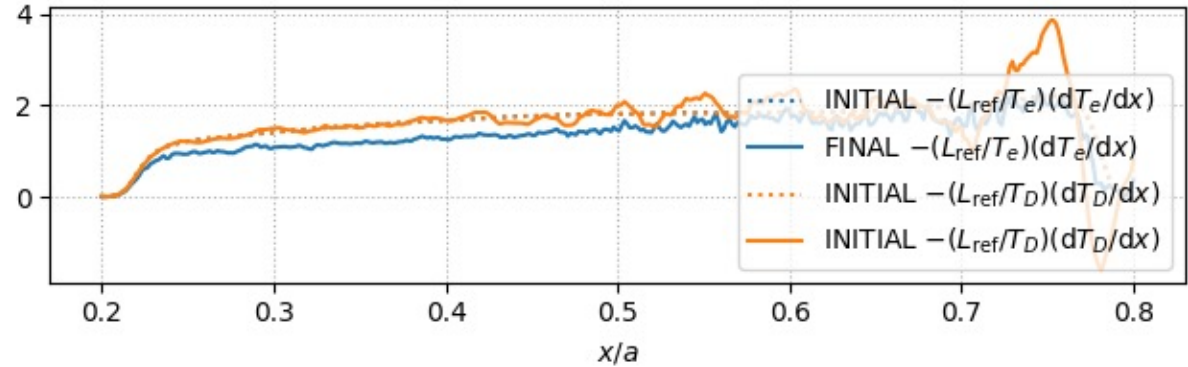
NONLINEAR δf global gyrokinetic calculations

- To assuage the masses, we are looking at doing global* calculations of STEP.

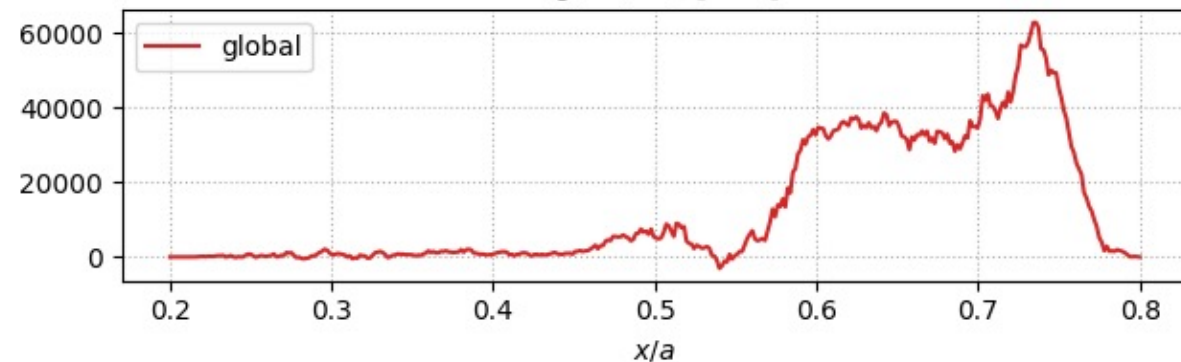
Normalised density gradient



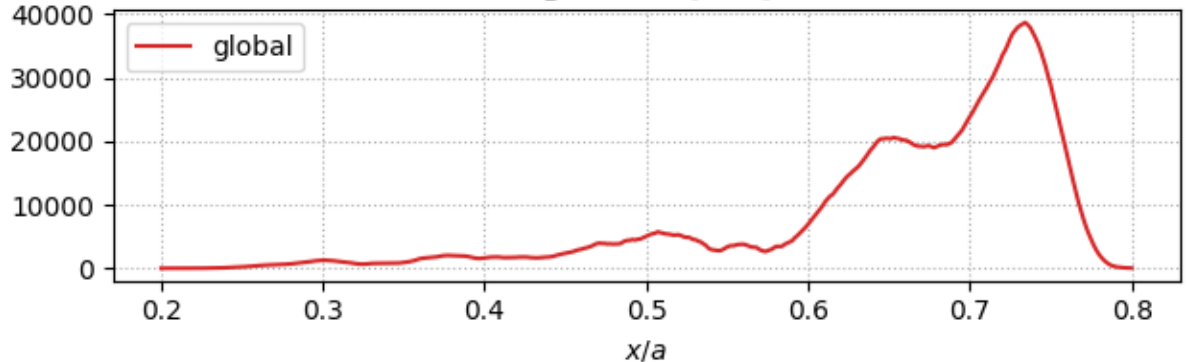
Normalised temperature gradient



$Q_e(dV/dx)$ [MW]



$Q_D(dV/dx)$ [MW]



NONLINEAR δf global gyrokinetic calculations

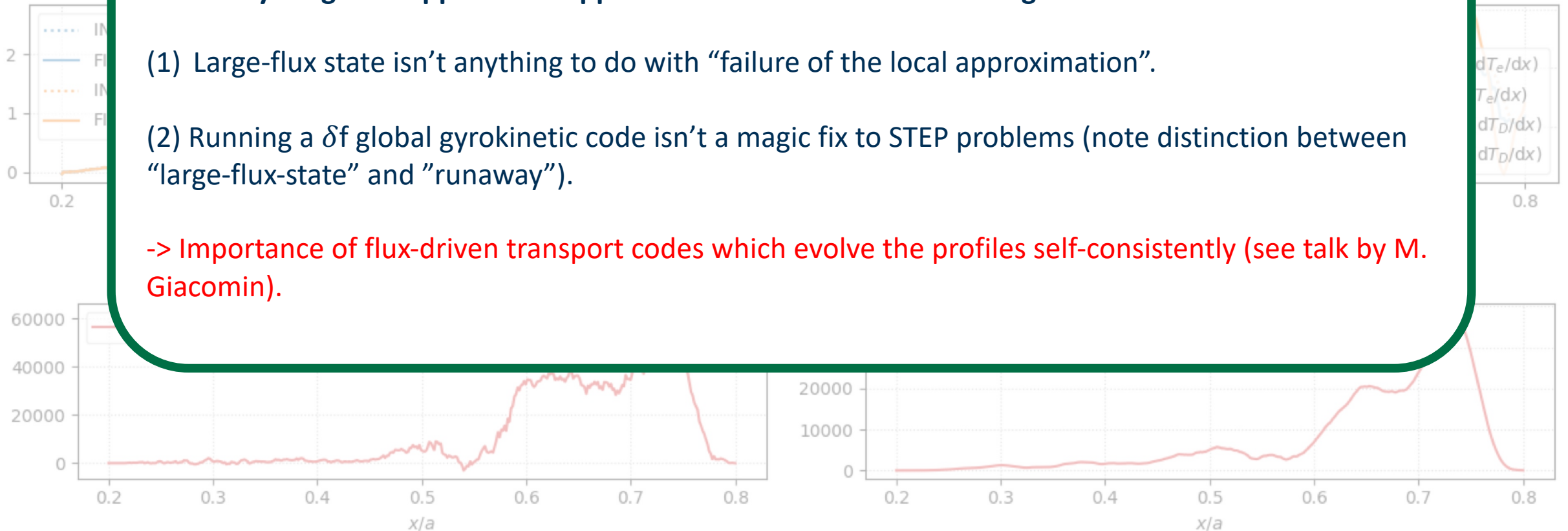
• To

Takeaway: δf global appears to support the existence of the ultra-large-flux state.

(1) Large-flux state isn't anything to do with "failure of the local approximation".

(2) Running a δf global gyrokinetic code isn't a magic fix to STEP problems (note distinction between "large-flux-state" and "runaway").

-> Importance of flux-driven transport codes which evolve the profiles self-consistently (see talk by M. Giacomini).



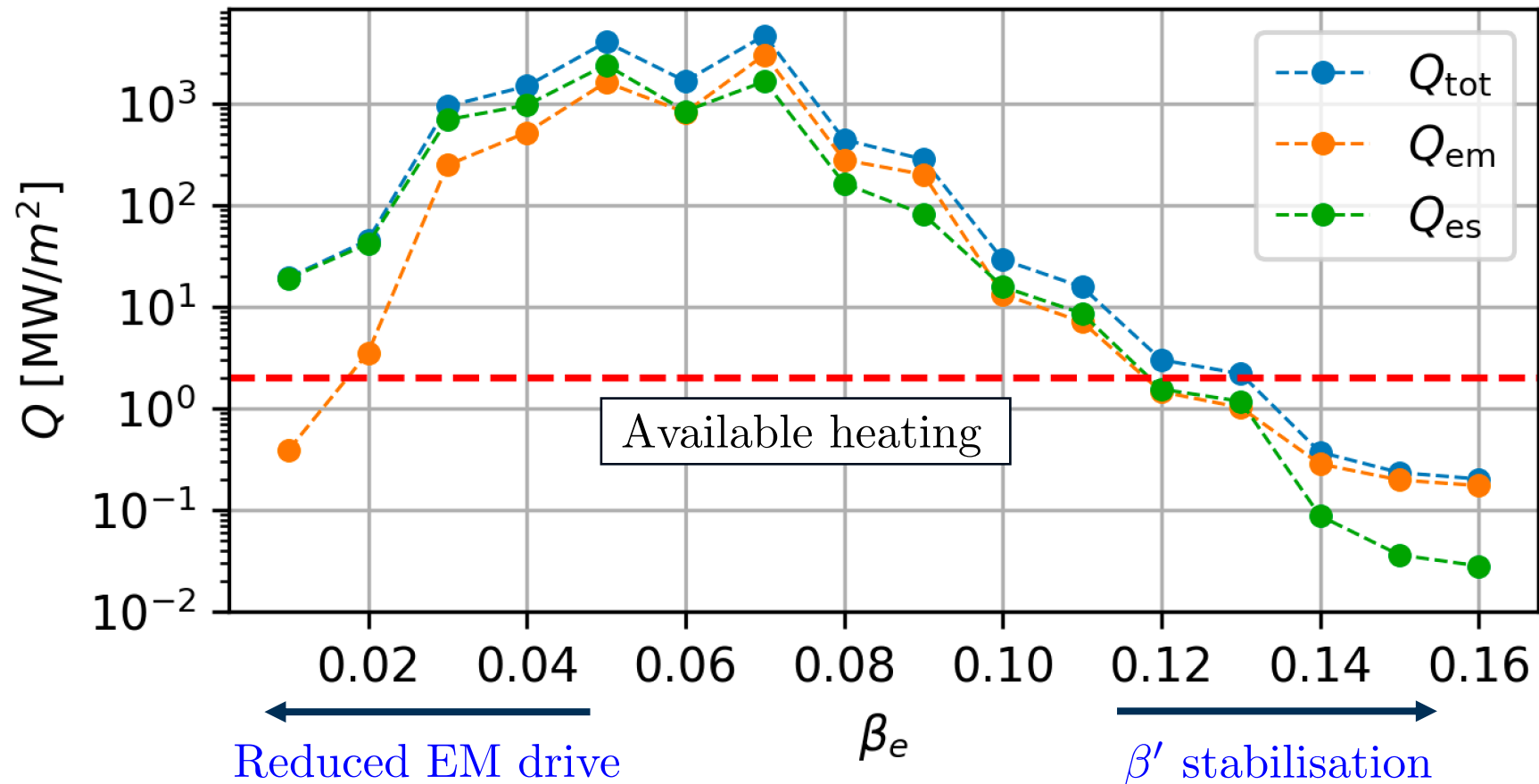
PART III: Non-zonal transition

Results in this section:

D. Kennedy, Y. Zhang, T. Adkins, M. Giacomini, P. Ivanov, G. Merlo, *et al.* ...

Non-zonal transition – electromagnetic threshold

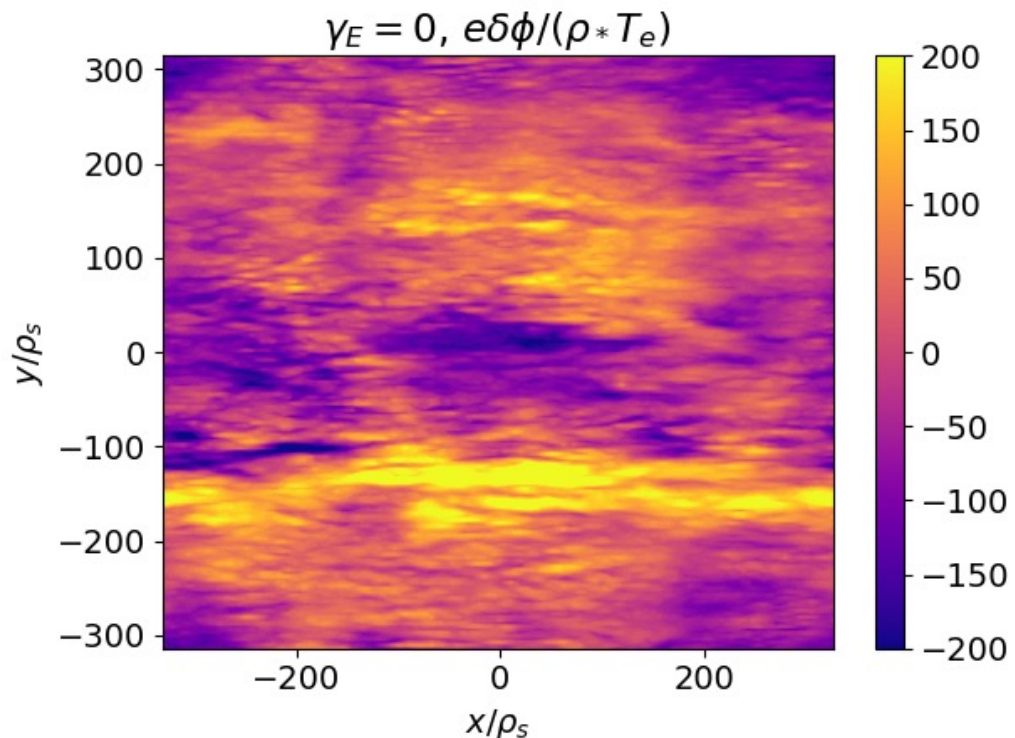
- We want to understand the threshold condition for runaway fluxes.
- Heat flux as a function of β (where β' is varied consistently).



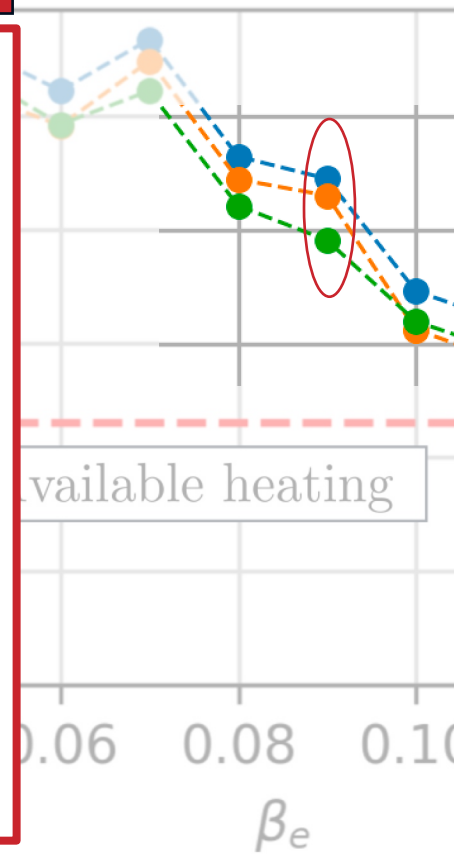
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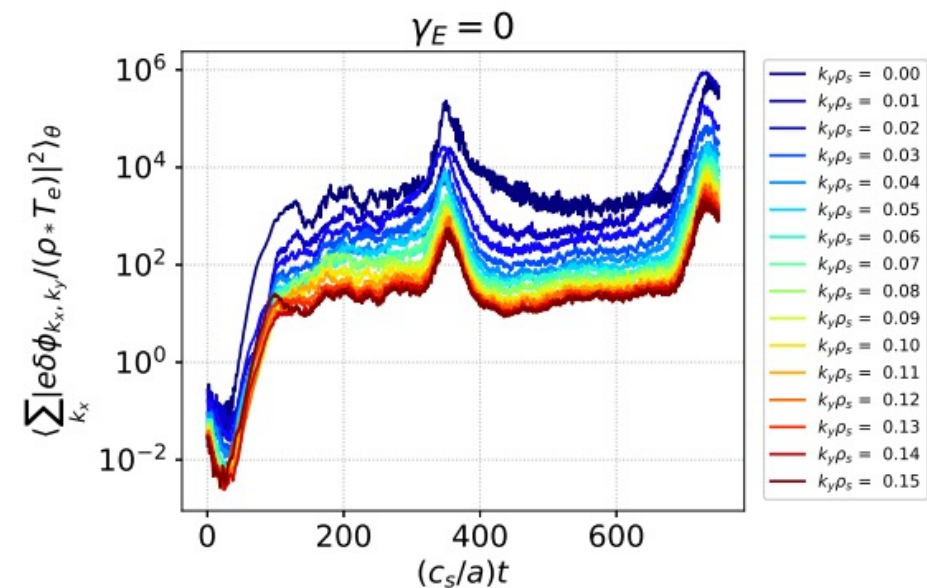
1. Linear instabilities (hybrid-KBM) form radially extended turbulent structures. Large transport.



(consistently).



2. Turbulence feeds zonal flows. Sufficiently strong zonal modes shear apart radially-extended structures.



Non-zonal transition – electromagnetic threshold

- **Conjecture** that the saturation, or otherwise, of electromagnetic δf -gyrokinetic turbulence is determined by the ability of the system (or lack thereof) to form sufficiently strong zonal perturbations to shear apart any streamer structures that attempt to be established by the linear instabilities present.

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$$\begin{aligned}
 & \frac{\partial}{\partial t} \left\langle \sum_{k_\psi} e^{ik_\psi \psi} \left[\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} (1 - \Gamma_{0s}) \phi_{\mathbf{k}_\perp} - \sum_s q_s n_{0s} \Gamma_{1s} \frac{\delta B_{\parallel \mathbf{k}_\perp}}{B} \right] \right\rangle_\psi \\
 & + \left\langle \sum_s q_s \int d^3 \mathbf{v} (\mathbf{v}_{ds} \cdot \nabla \psi) \left\langle \frac{\partial h_s}{\partial \psi} \right\rangle_{\mathbf{r}} \right\rangle_\psi + \left\langle \sum_s q_s \int d^3 \mathbf{v} \langle \mathbf{v}_\chi \cdot \nabla h_s \rangle_{\mathbf{r}} \right\rangle_\psi \\
 & = \left\langle \sum_{s,s'} q_s \int d^3 \mathbf{v} \left\langle \left\langle C_{ss'}^{(\ell)} [h_s] \right\rangle_{\mathbf{R}_s} \right\rangle_{\mathbf{r}} \right\rangle_\psi,
 \end{aligned}$$

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d/dt (zonal perturbation)

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LINEAR TERM
 Inhomogeneity
 of the magnetic
 field



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LINEAR TERM
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$$+ \left\langle \sum_s q_s \int d^3 \mathbf{v} (\mathbf{v}_{ds} \cdot \nabla \psi) \left\langle \frac{\partial h_s}{\partial \psi} \right\rangle_{\mathbf{r}} \right\rangle_\psi + \left\langle \sum_s q_s \int d^3 \mathbf{v} \langle \mathbf{v}_\chi \cdot \nabla h_s \rangle_{\mathbf{r}} \right\rangle_\psi$$

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NONLINEAR TERM

“Momentum flux” or

$$\Pi_\chi \equiv \left\langle \sum_s q_s \int d^3 \mathbf{v} \langle \mathbf{v}_\chi \cdot \nabla h_s \rangle_{\mathbf{r}} \right\rangle_\psi = \Pi_\phi + \Pi_{A_\parallel} + \Pi_{\delta B_\parallel}$$

Non-zonal transition – electromagnetic threshold

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COLLISIONS

Non-zonal transition – electromagnetic threshold

- **Conjecture** that the saturation, or otherwise, of electromagnetic δf -gyrokinetic turbulence is determined by the ability of the system (or lack thereof) to form sufficiently strong zonal perturbations to shear apart any streamer structures that attempt to be established by the linear instabilities present.

$$\frac{\partial}{\partial t}(\text{zonal flows}) = \Pi_{\chi}$$

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$$\frac{\partial}{\partial t}(\text{zonal flows}) = \Pi_{\chi}$$

Conjecture that the sign of Π_{χ} :

- Determines whether zonal flows grow or diminish.
- (HYPOTHESIS) Controls whether the simulations saturates.
- Electrostatic Dimits transition in ITG [P.G. Ivanov *et al*, *Journal of Plasma Physics*, vol. 86, no. (2020)].

$$\Pi_{\chi} \equiv \left\langle \sum_s q_s \int d^3\mathbf{v} \langle \mathbf{v}_{\chi} \cdot \nabla h_s \rangle_{\mathbf{r}} \right\rangle_{\psi} = \Pi_{\phi} + \Pi_{A_{\parallel}} + \Pi_{\delta B_{\parallel}}$$

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Diagnostics which compute these momentum fluxes $\Pi_f^s(k_x, k_y, z, t)$ are now in:

- GENE (D. Kennedy, G. Merlo)
- STELLA (Y. Zhang, M. Hardman)
- GKW (Rath *et al.* – [see previous talk](#))

$$\Pi_\chi \equiv \left\langle \sum_s q_s \int d^3\mathbf{v} \langle \mathbf{v}_\chi \cdot \nabla h_s \rangle_{\mathbf{r}} \right\rangle_\psi = \Pi_\phi + \Pi_{A_\parallel} + \Pi_{\delta B_\parallel}$$

- **Very** expensive to use these diagnostics.

Non-zonal transition – electromagnetic threshold

Introduce a turbulent zonal-flow viscosity

$$\nu_{\delta\phi} = -\frac{\int dx \Pi_{\delta\phi} \omega_{E \times B}}{\int dx \omega_{E \times B}^2} \quad \nu_{\delta A_{\parallel}} = -\frac{\int dx \Pi_{\delta A_{\parallel}} \omega_{E \times B}}{\int dx \omega_{E \times B}^2} \quad \nu_{\delta B_{\parallel}} = -\frac{\int dx \Pi_{\delta B_{\parallel}} \omega_{E \times B}}{\int dx \omega_{E \times B}^2}$$

$$\nu = \nu_{\delta\phi} + \nu_{\delta A_{\parallel}} + \nu_{\delta B_{\parallel}}$$

- GKW (Rath *et al.* – see previous talk)

- $\nu > 0$

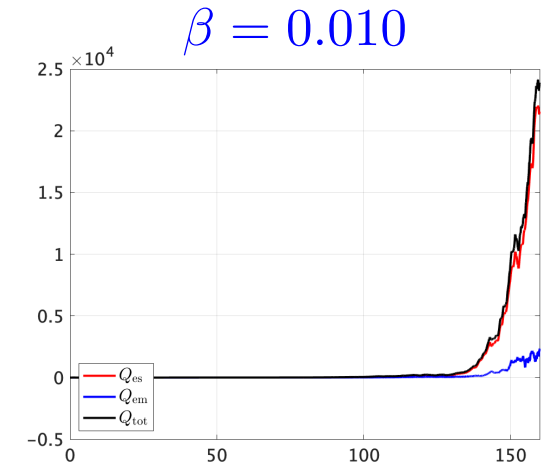
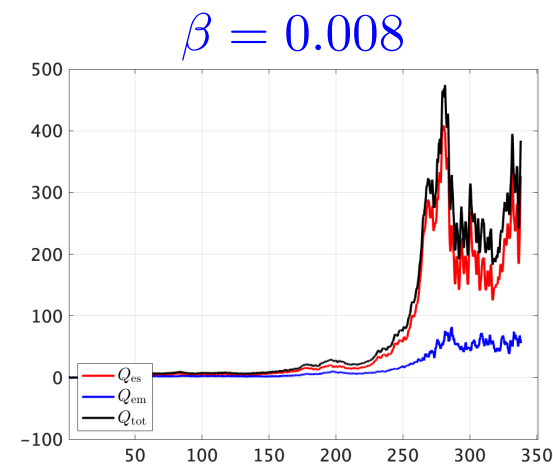
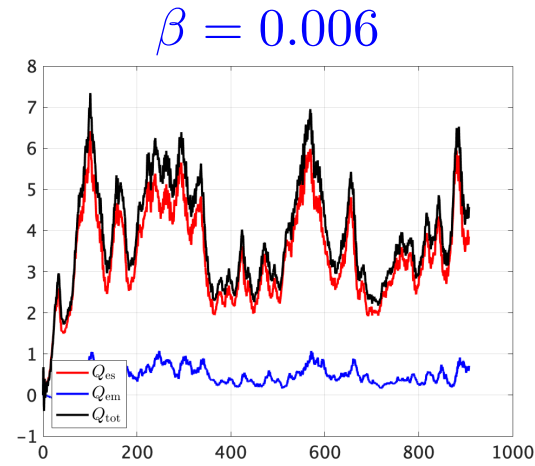
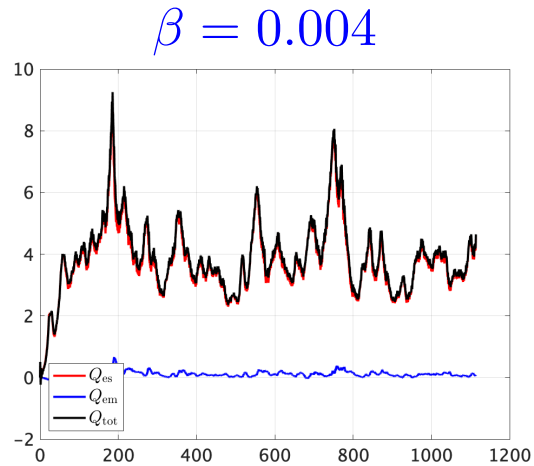
Energy is being extracted **from the zonal flow**. Allows turbulent structures to grow unchecked. High-transport.

- $\nu < 0$

Energy is being put **into the zonal flow**. Zonal perturbations can shear apart turbulent structures. Saturation.

Non-zonal transition – CBC simulations

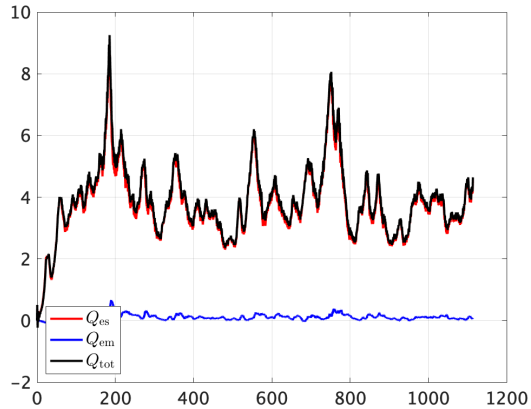
- **TEST case:** consider simulations of CBC where we increase β with all other parameters held fixed.



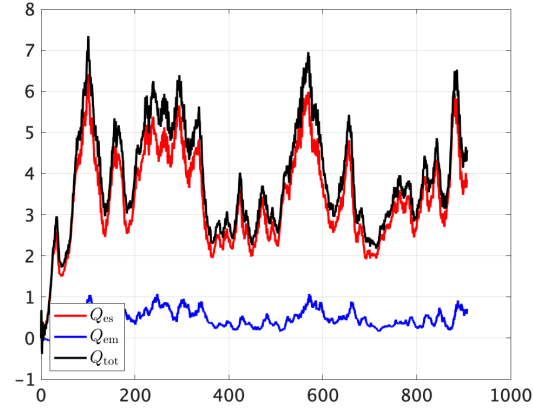
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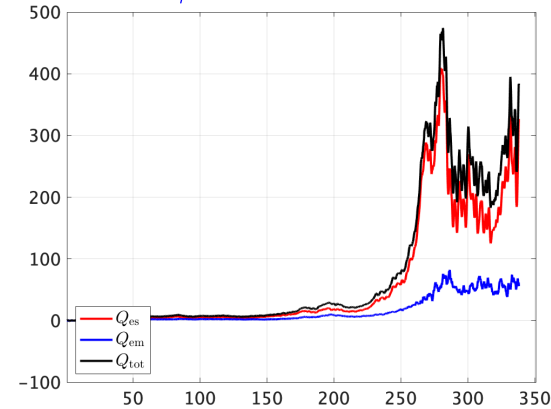
$\beta = 0.004$



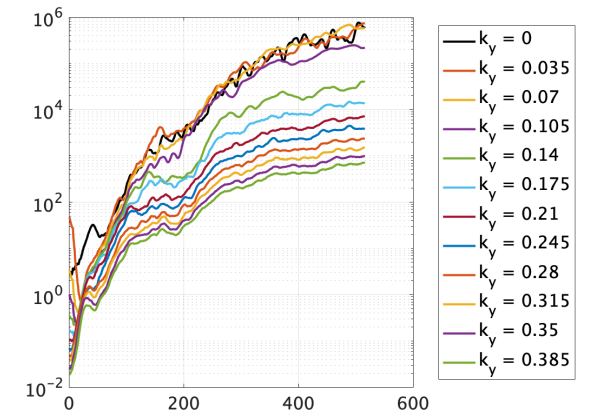
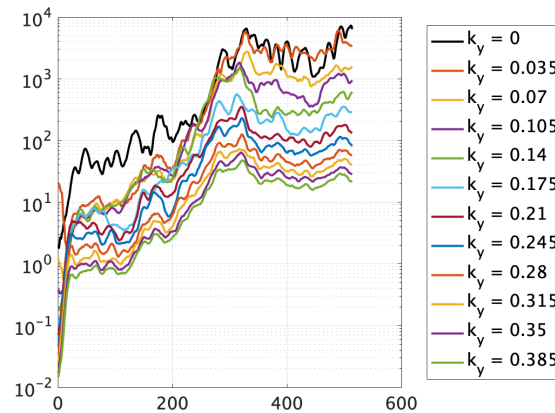
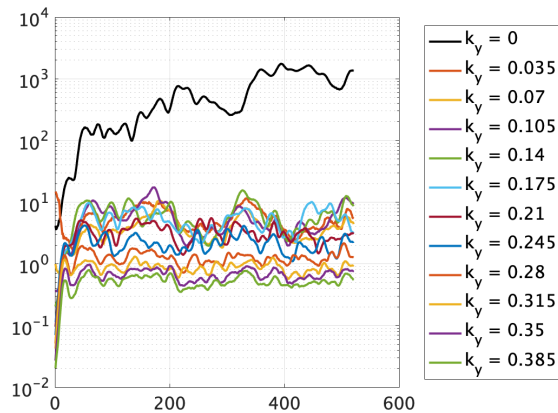
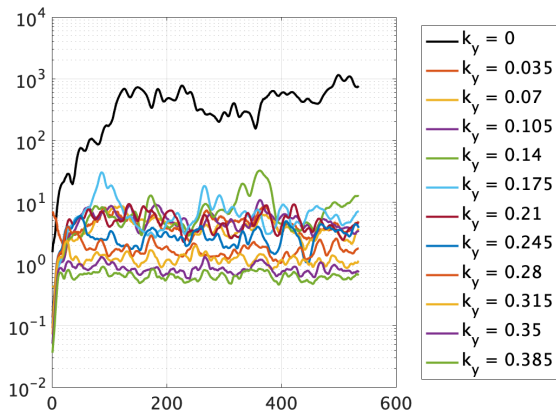
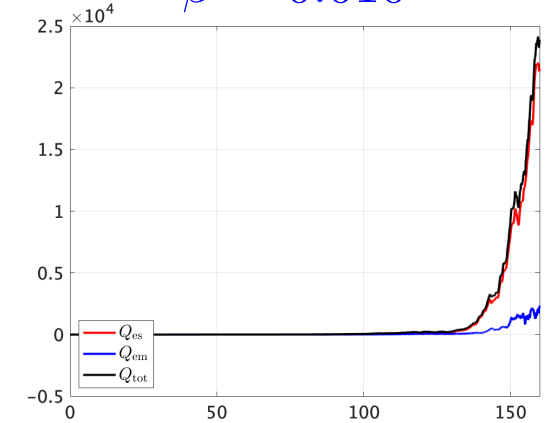
$\beta = 0.006$



$\beta = 0.008$



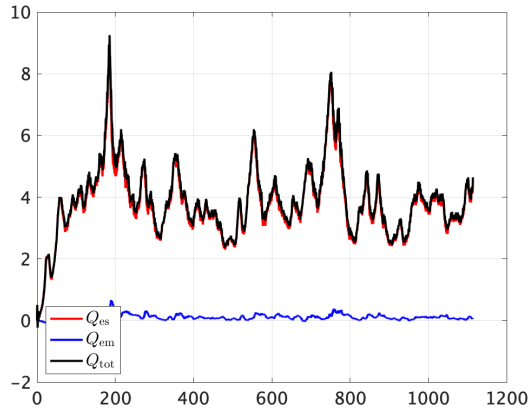
$\beta = 0.010$



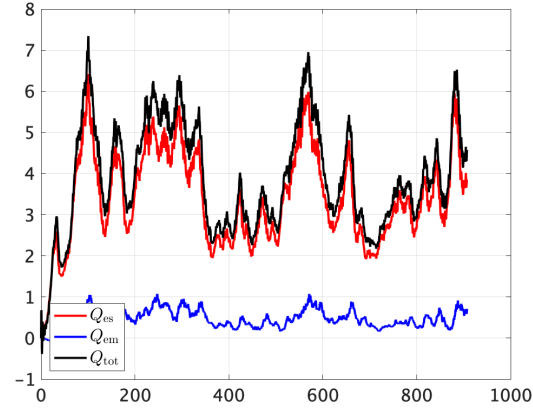
Non-zonal transition – CBC simulations

- **TEST case:** consider simulations of CBC where we increase β with all other parameters held fixed.

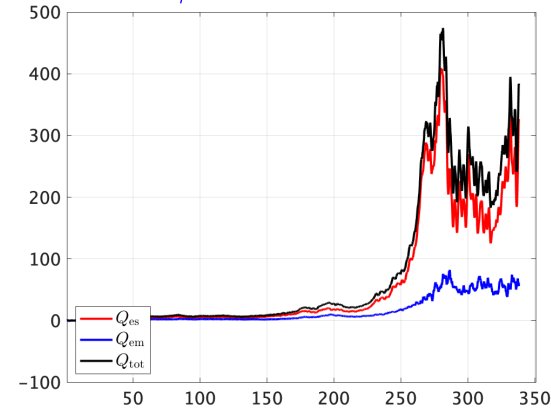
$\beta = 0.004$



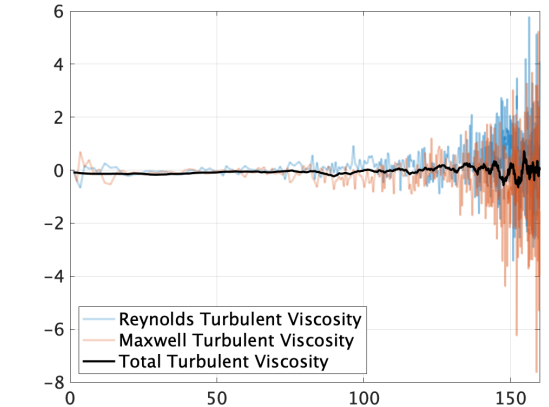
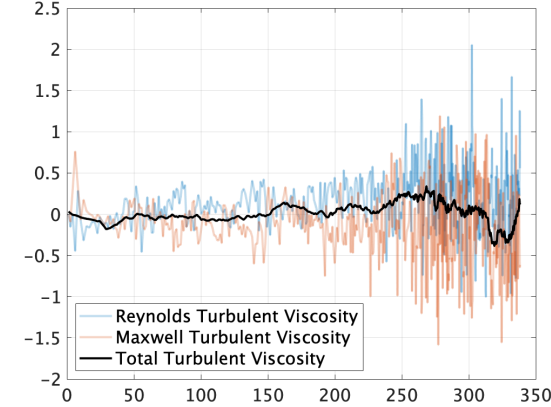
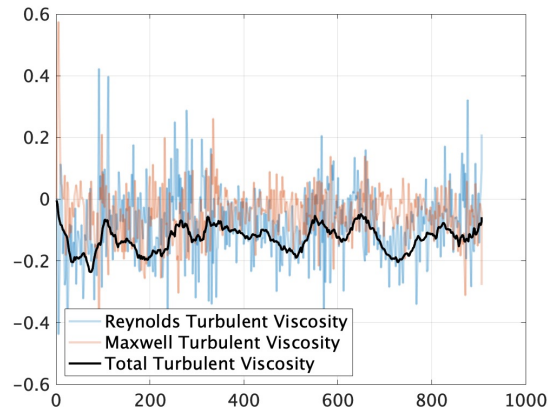
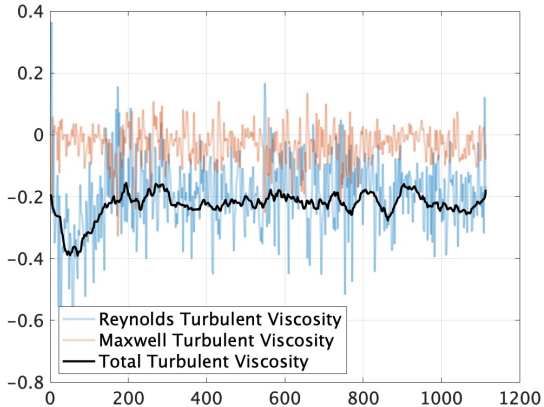
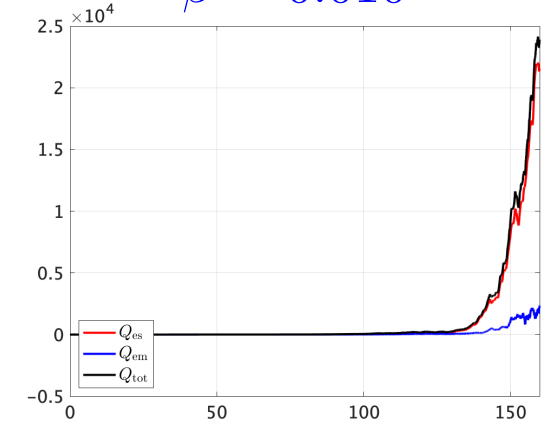
$\beta = 0.006$



$\beta = 0.008$

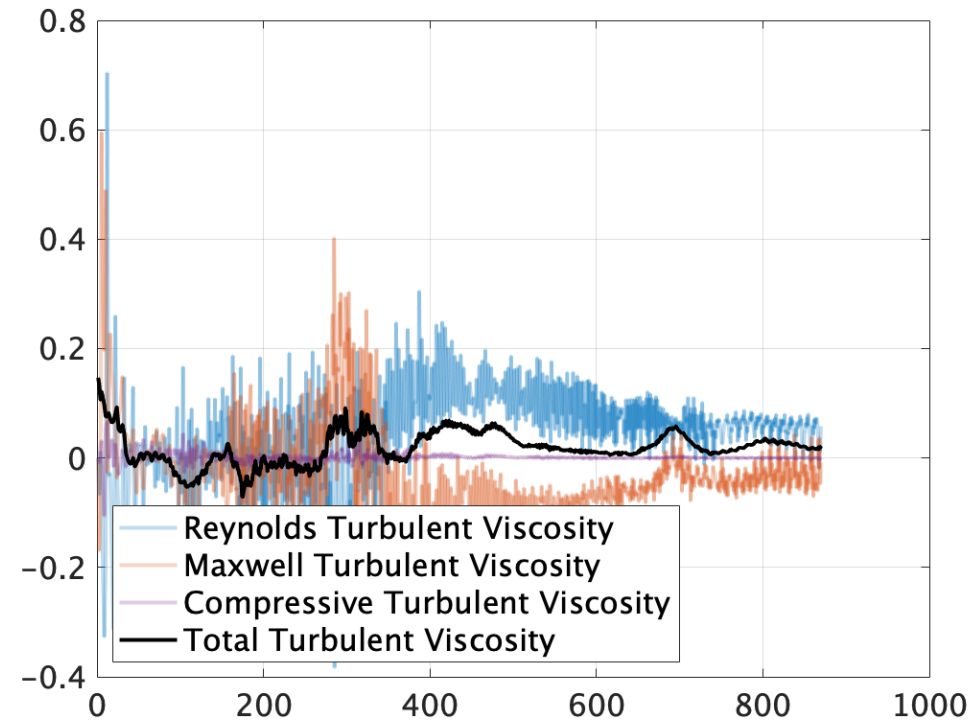
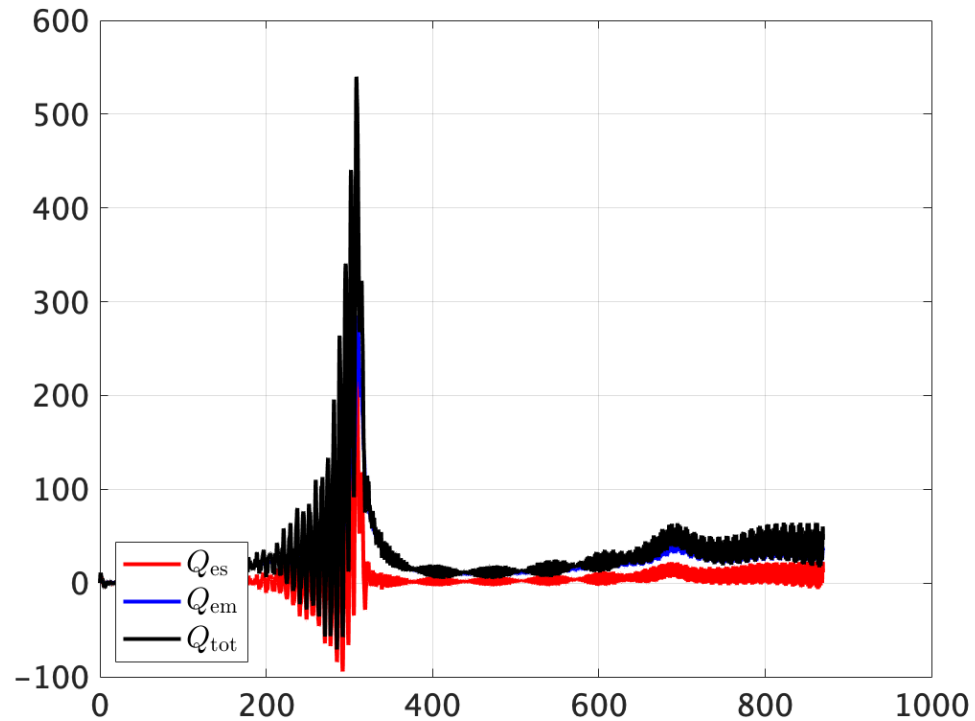


$\beta = 0.010$



Non-zonal transition – **STEP** simulations

- Q: Is the same picture borne out in STEP?
- A: possibly... which I concede is not really an answer.



Non-zonal transition – STEP simulations

- Why does STEP care?
- It seems like a broad class of EM simulations in δf gyrokinetics live or die by the sign of some dimensionless parameter:

$$D \equiv \frac{\Pi_{\delta A_{\parallel}}}{\Pi_{\delta\phi}}$$

$$D \sim \frac{1}{\beta_s} \left\langle (1 - \Gamma_{1s}) \left(\frac{\overline{\delta A_{\parallel}}}{\rho_s B} \right)^2 \right\rangle_{\psi} / \left\langle (1 - \Gamma_{0s}) \left(\frac{q_s \overline{\delta\phi}}{T_{0s}} \right)^2 \right\rangle_{\psi}$$

- Can we predict this parameter (or the victor of the competition of stresses) from the system inputs?
- Way of predicting where the no-go-zone is in parameter space.

Conclusions and outlook

Spherical Tokamak for Energy Production (**STEP**) is a UK programme, aiming to develop a compact **prototype reactor** that aims to deliver **net electric power** to the national grid.

Early designs of STEP plasmas were far away from gyrokinetic steady state (ultra-large-flux state).

- Available heating and fuelling rates are only consistent at shearing rates unlikely to be achieved in STEP.
- Global GK calculations support the existence of the large-flux-state.
- Observe well-defined transitions between a finite-amplitude saturated state dominated by strong zonal-flows, and a blow-up state that fails to saturate.
- The breakup of the low-transport regime is linked to a competition between the two different sources of poloidal momentum in the system; the Reynolds stress and the Maxwell stress.
- FUTURE: semi-analytical model for the transition threshold.

Complementary approach: M. Giacomin towards flux-driven simulations.