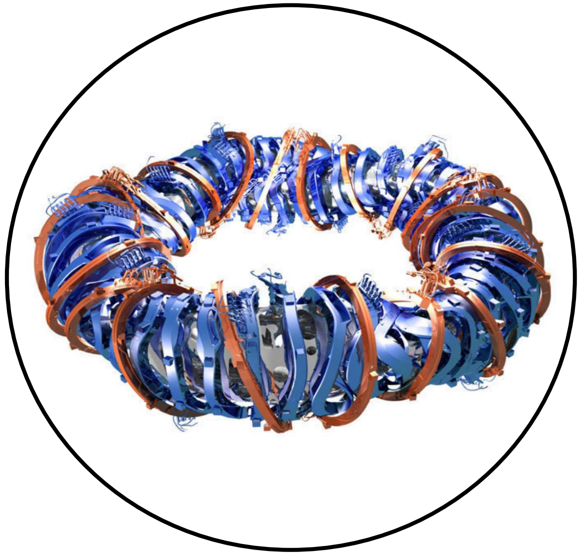
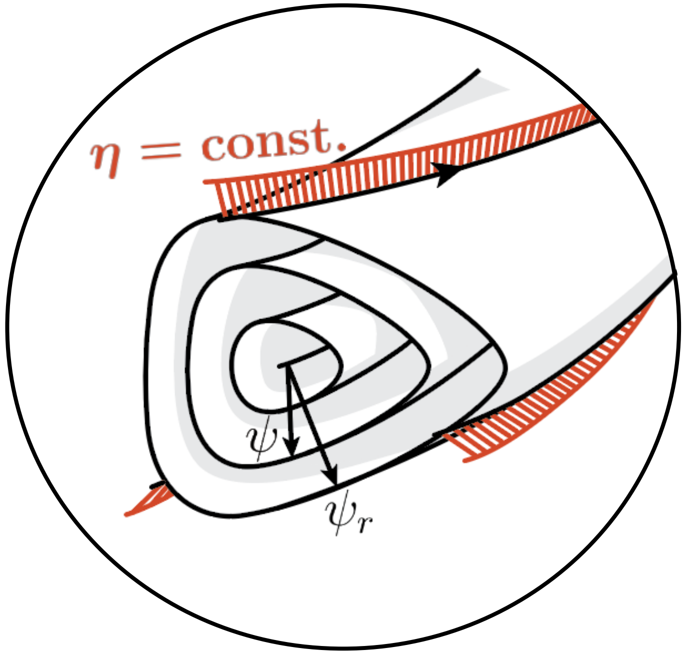
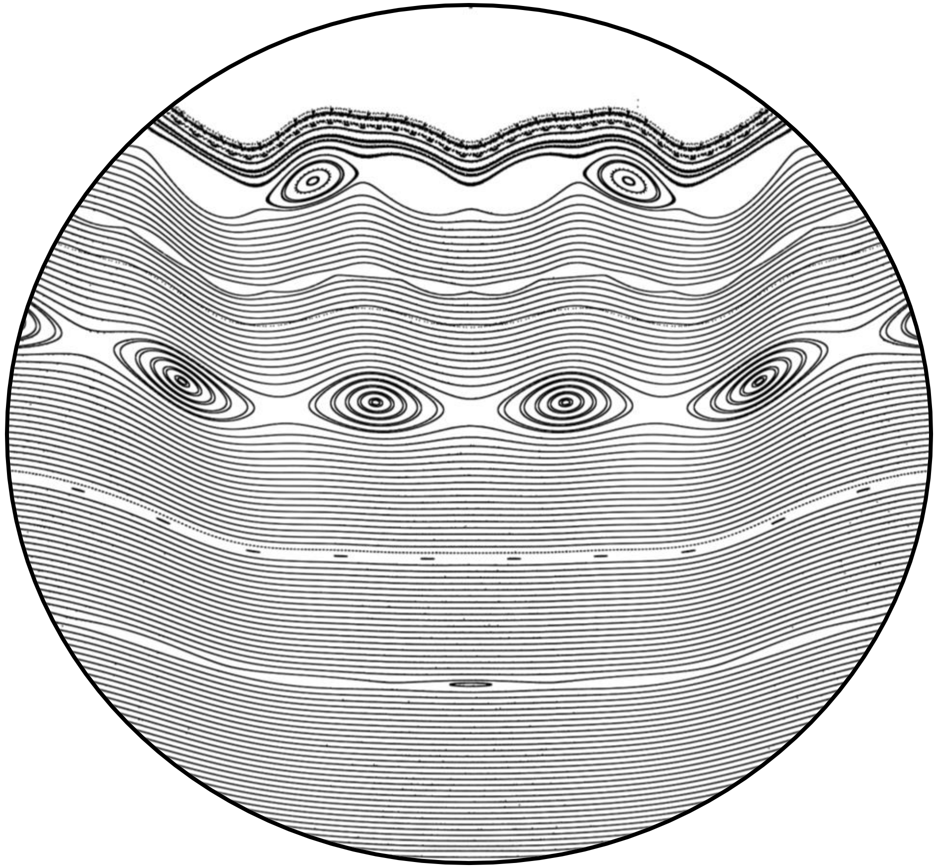
Orbits of energetic particles near rational flux surfaces in stellarators

Thomas E. Foster | **Princeton**

with Felix I. Parra, Roscoe B. White, José Luis Velasco





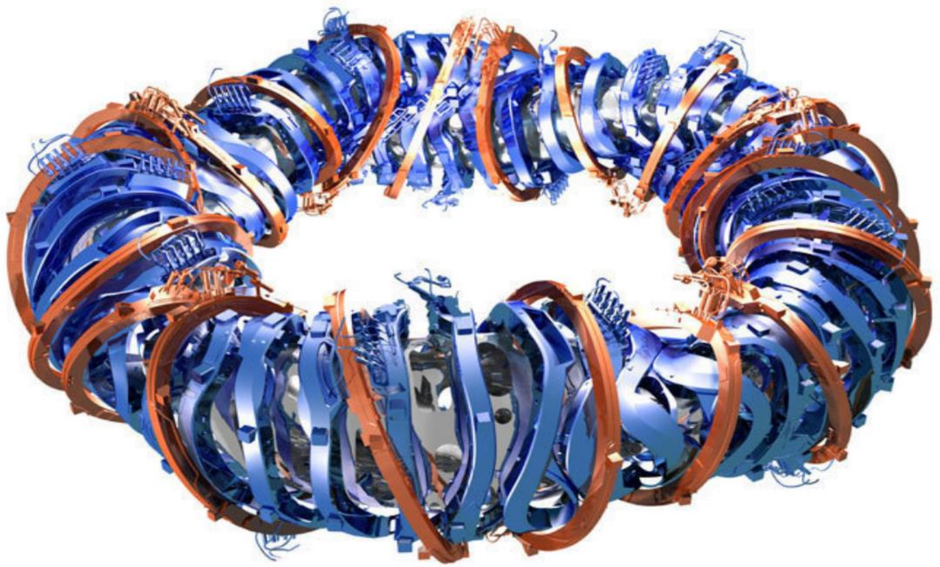


Motivation

- DT fusion produces $3.5\,\mathrm{MeV}$ alpha particles; speed is $\sim 10^7\,\mathrm{ms}^{-1}$.
- Collisional slowing-down time is $\sim 0.1\,\mathrm{s}$.

} Study collisionless orbits.

- In reactor, $\rho_\star = \rho/L \sim 0.03$. } Use guiding-centre equations.



M. Nagel *et al* 2017
IOP Conf. Ser.: Mater. Sci. Eng.

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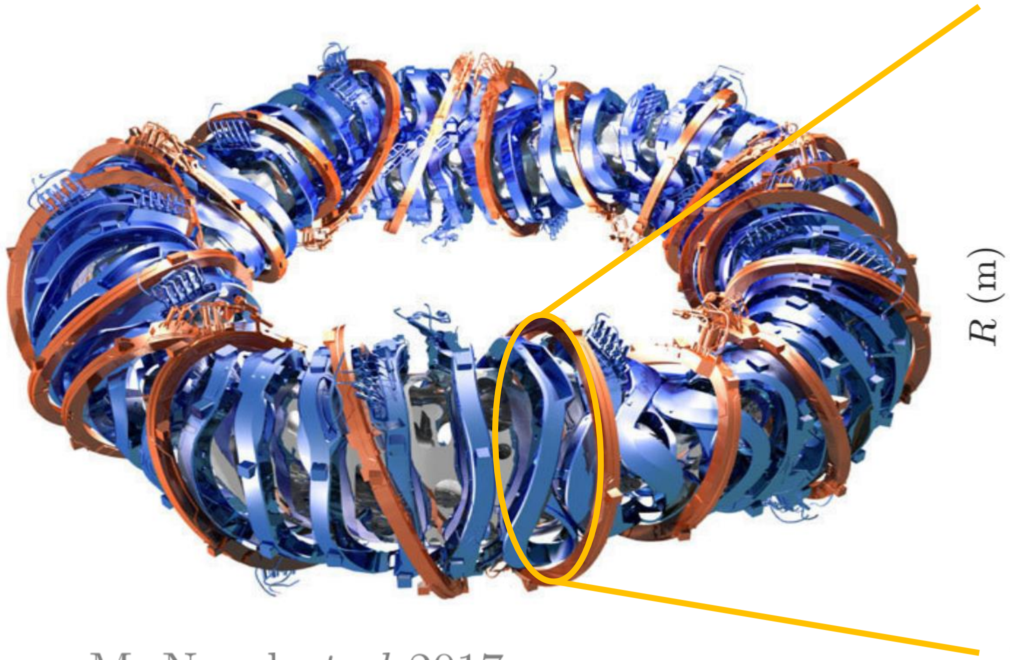

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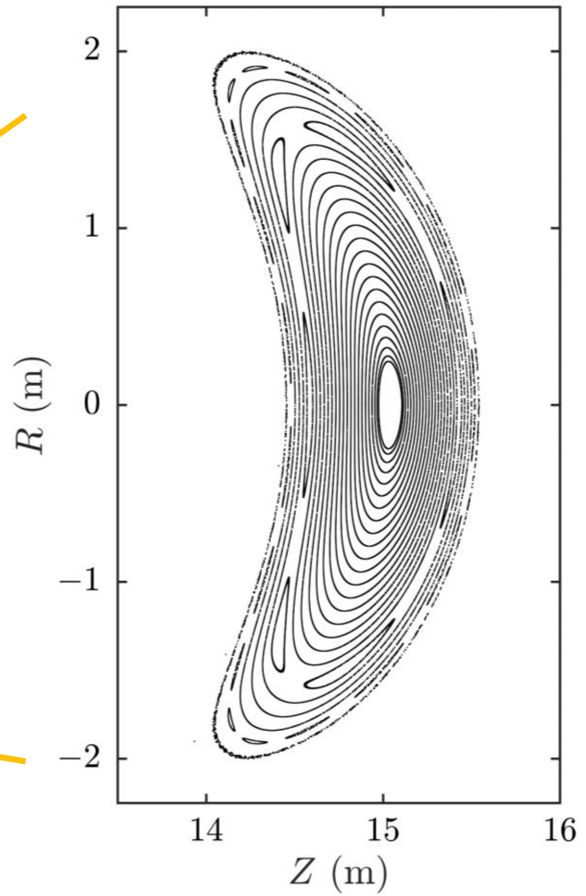

White *et al* 2022, White 2022:

- Magnetic field has nested toroidal flux surfaces
- But Poincaré plot shows islands in particle orbits
- These ‘drift islands’ grow with energy

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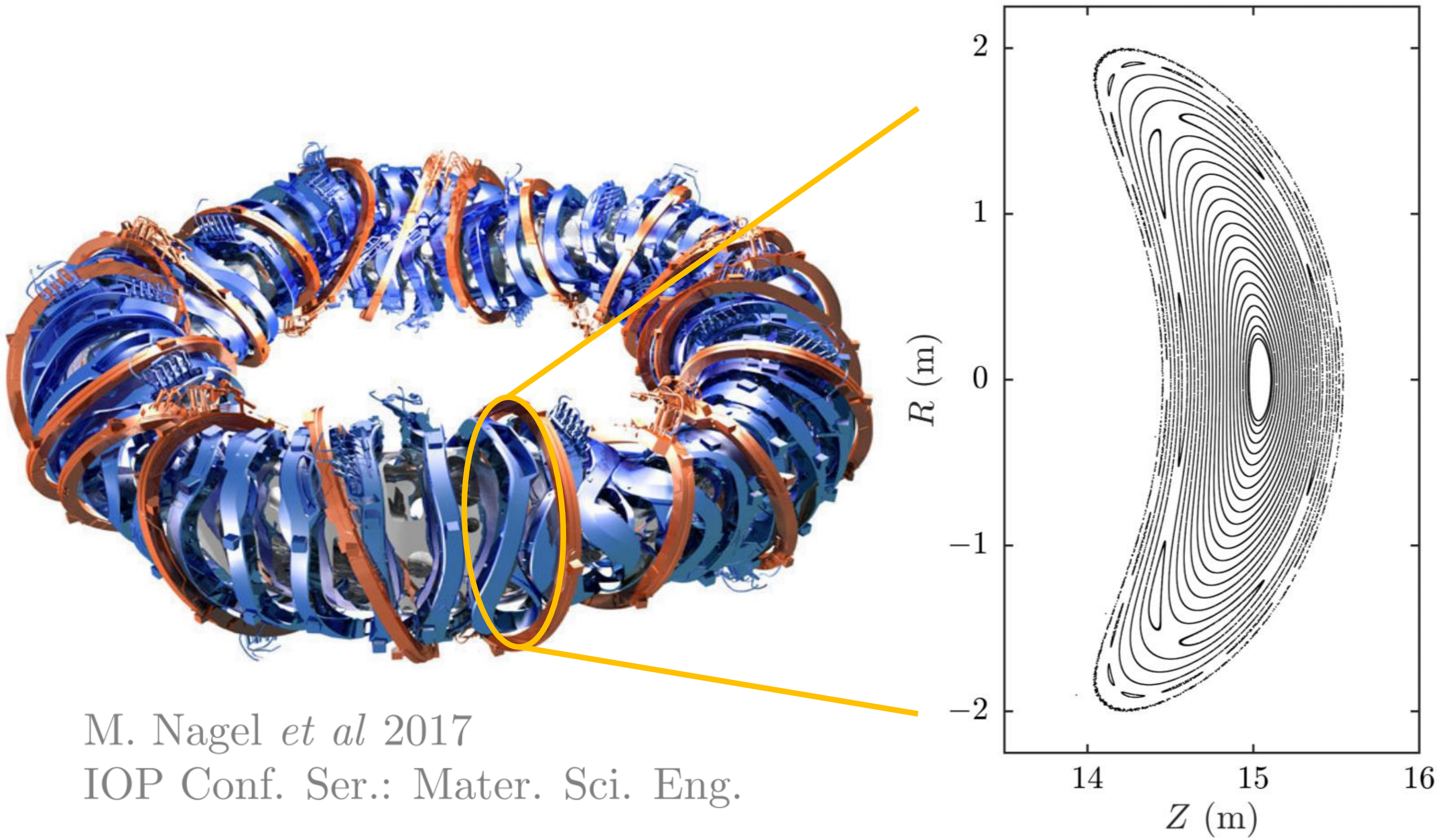

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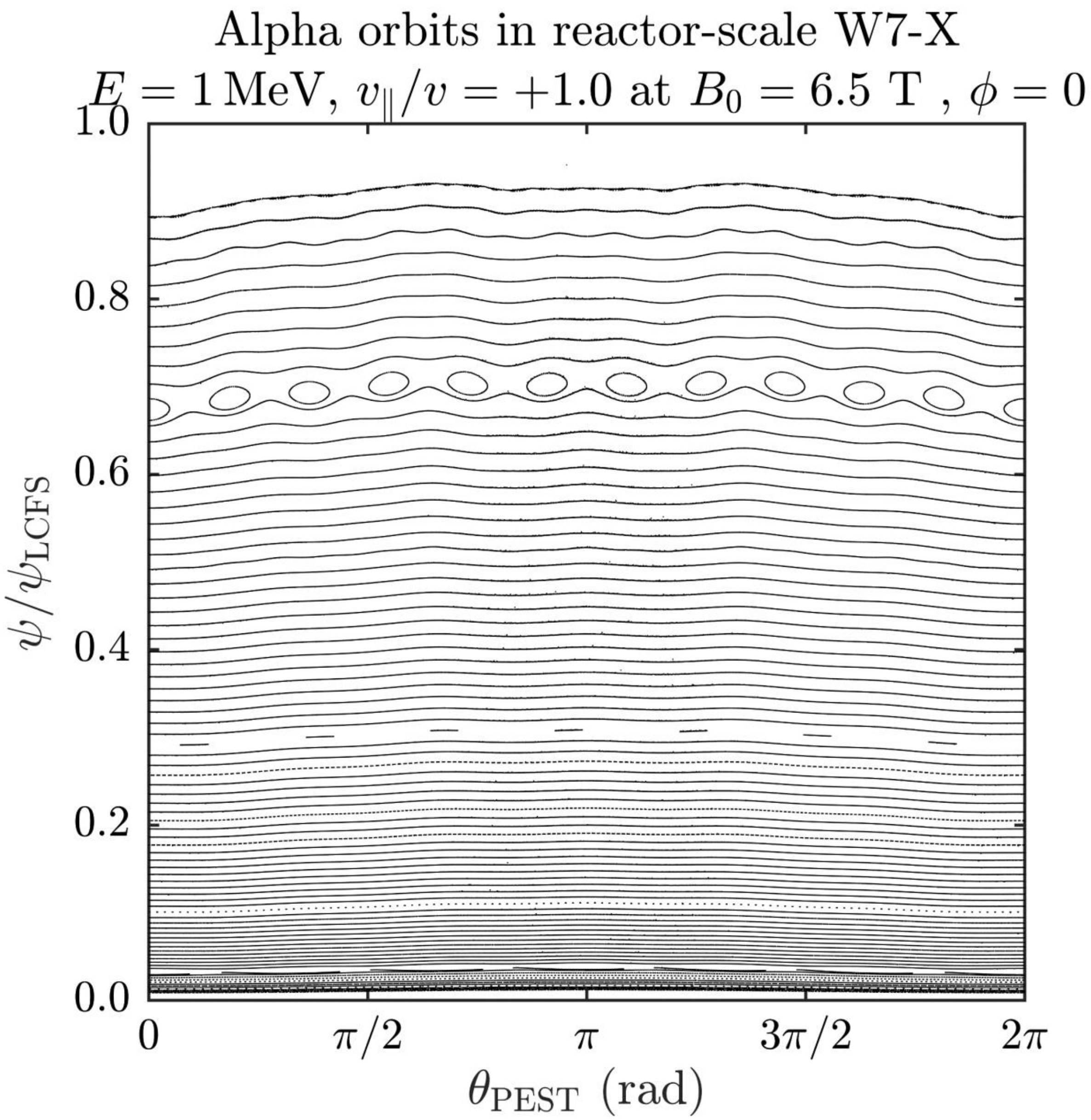
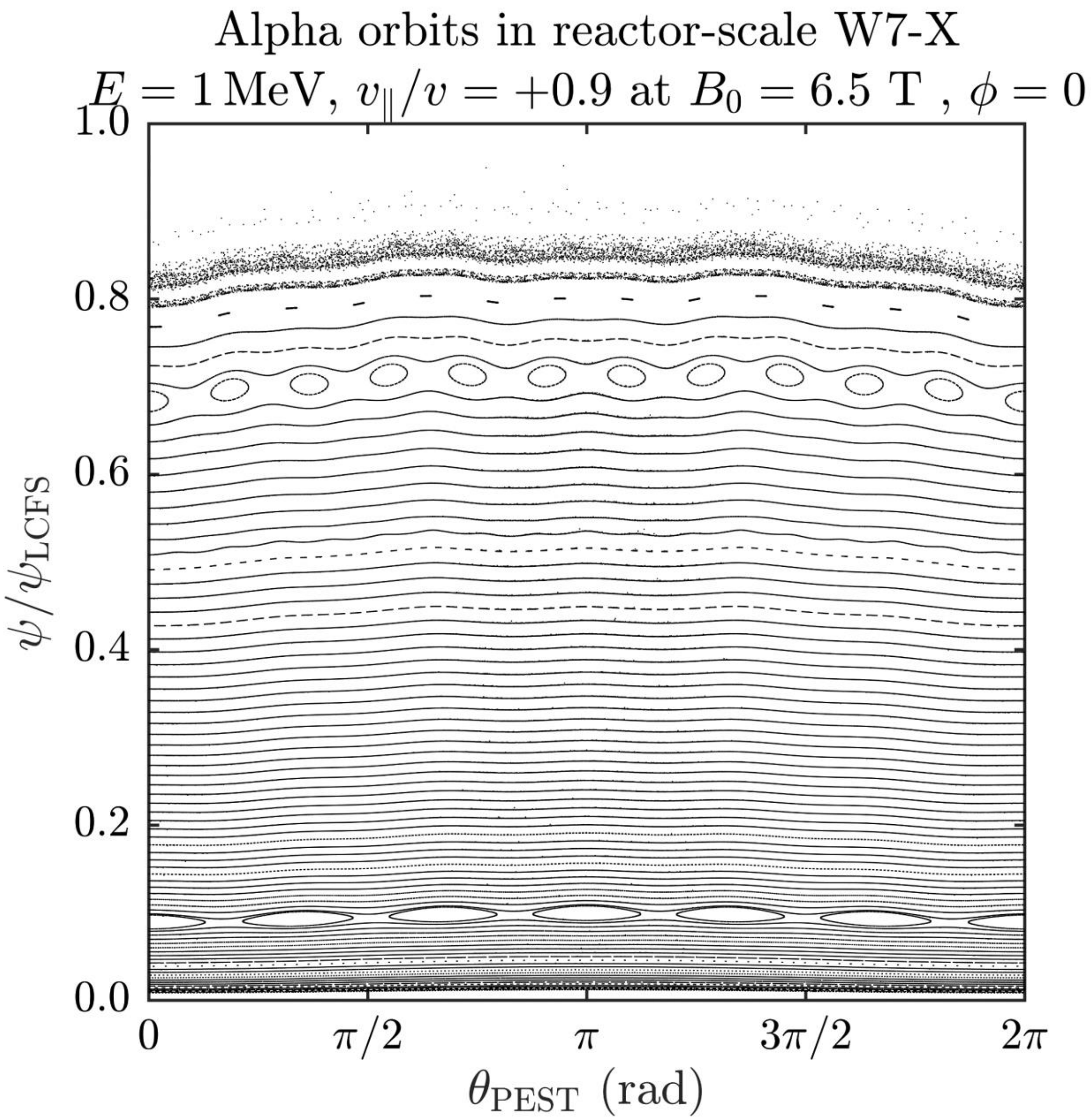
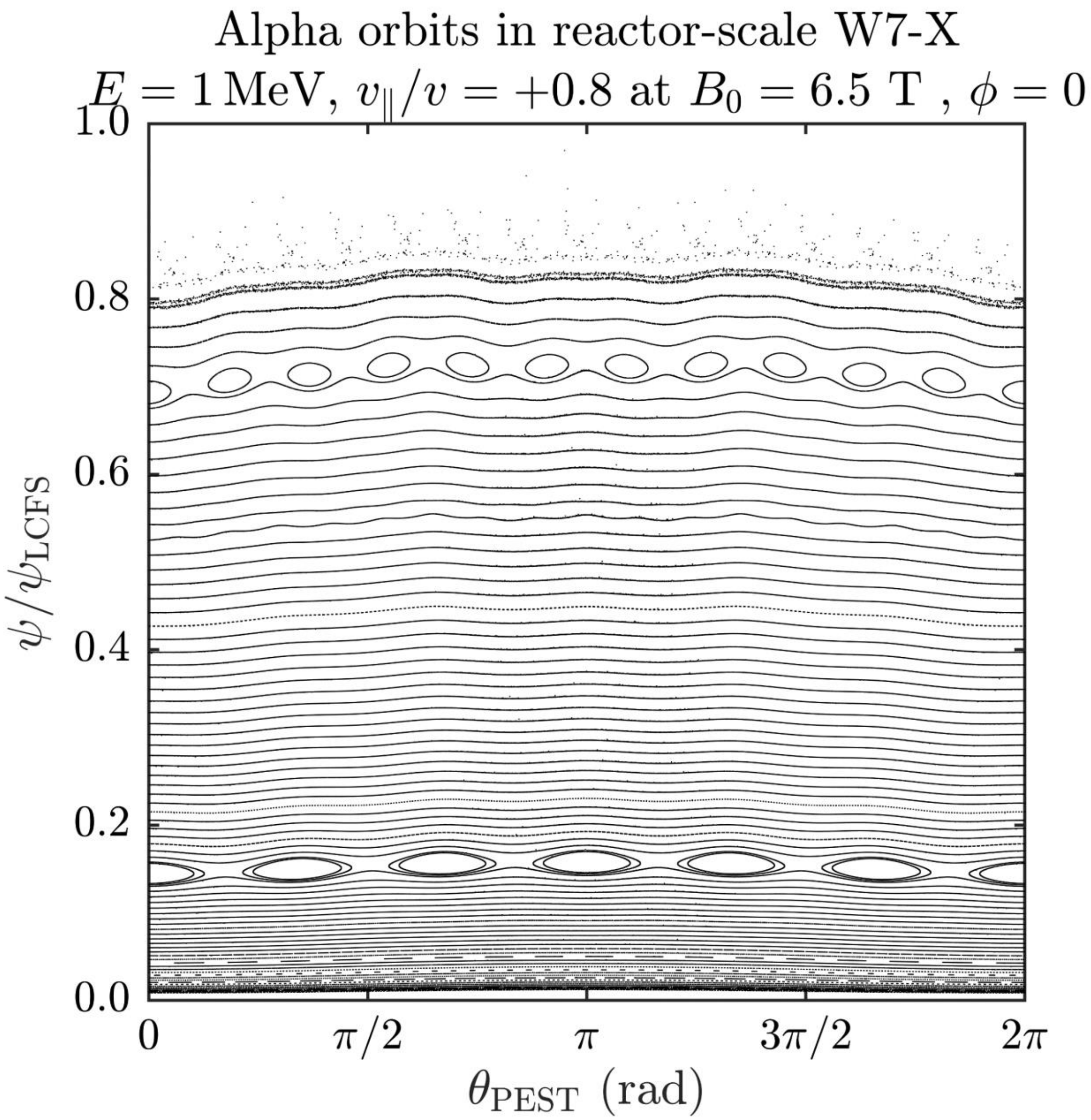
Aims:

1. Understand drift islands
2. Calculate shapes of orbits in Poincaré plot
3. Which rational surfaces matter?

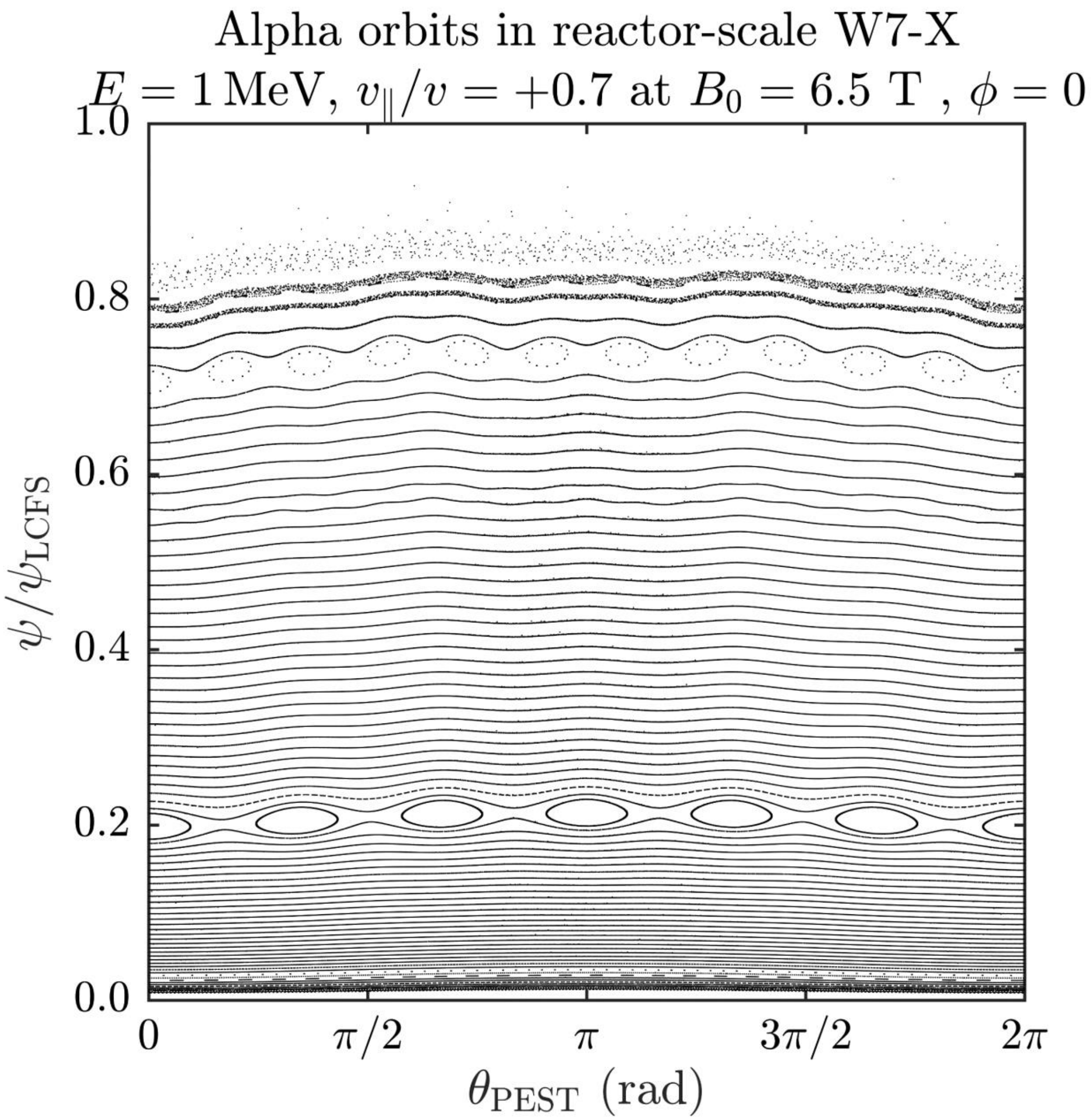

Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_\parallel/v = +1.0$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)



Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_{\parallel}/v = +0.9$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)



Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_\parallel/v = +0.8$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)


Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_\parallel/v = +0.7$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
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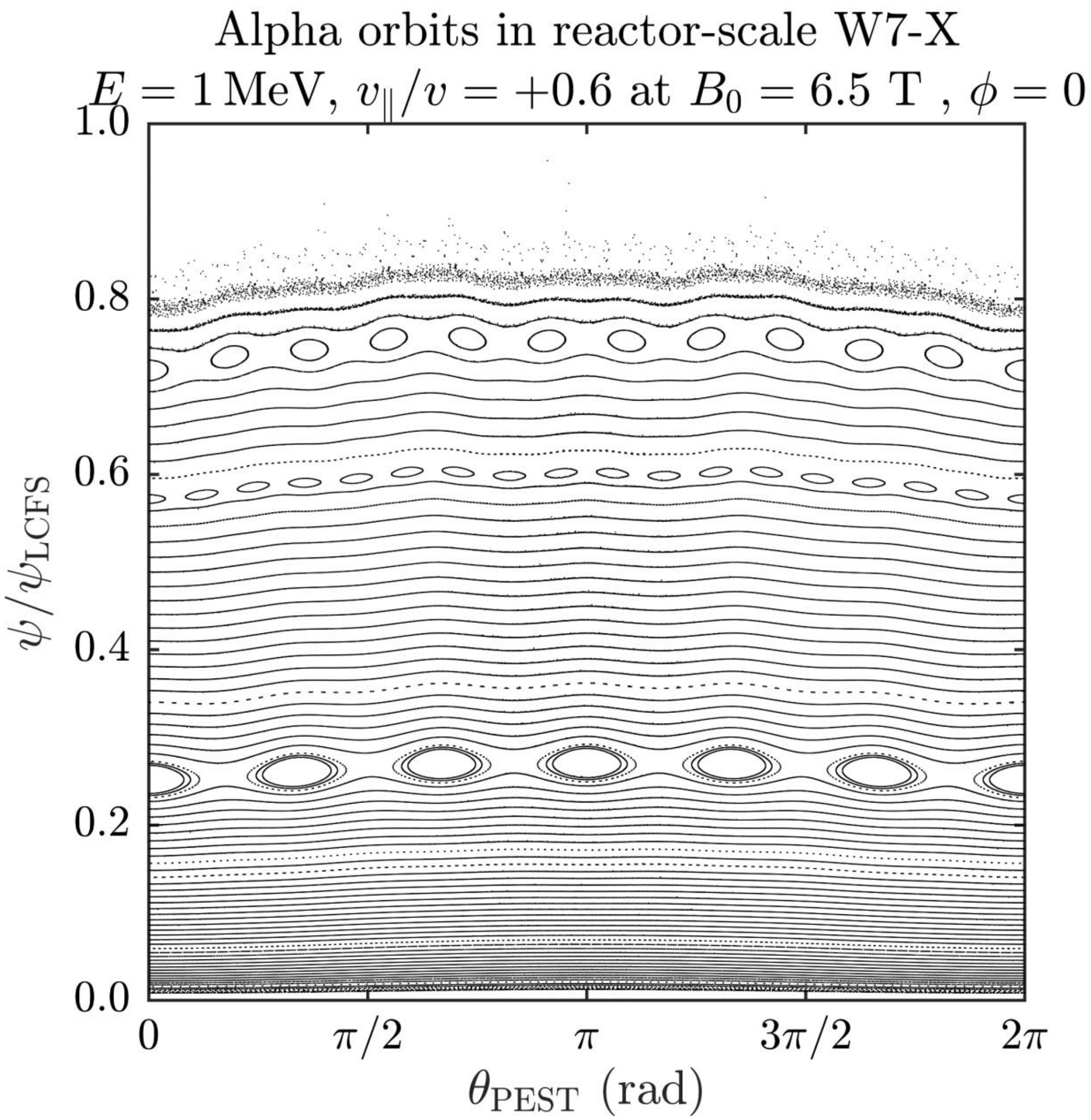
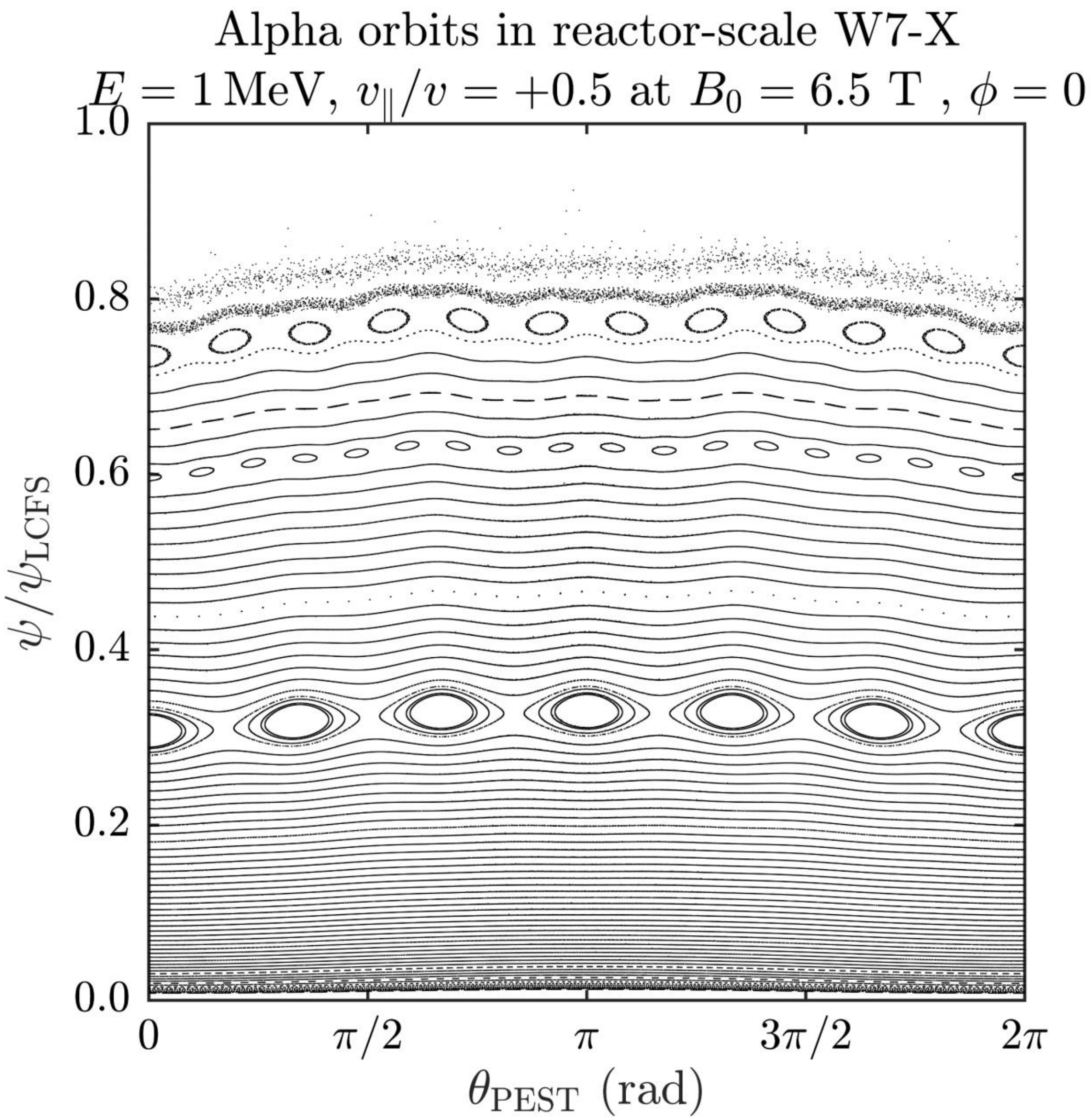

Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_\parallel/v = +0.6$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)



Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_\parallel/v = +0.5$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
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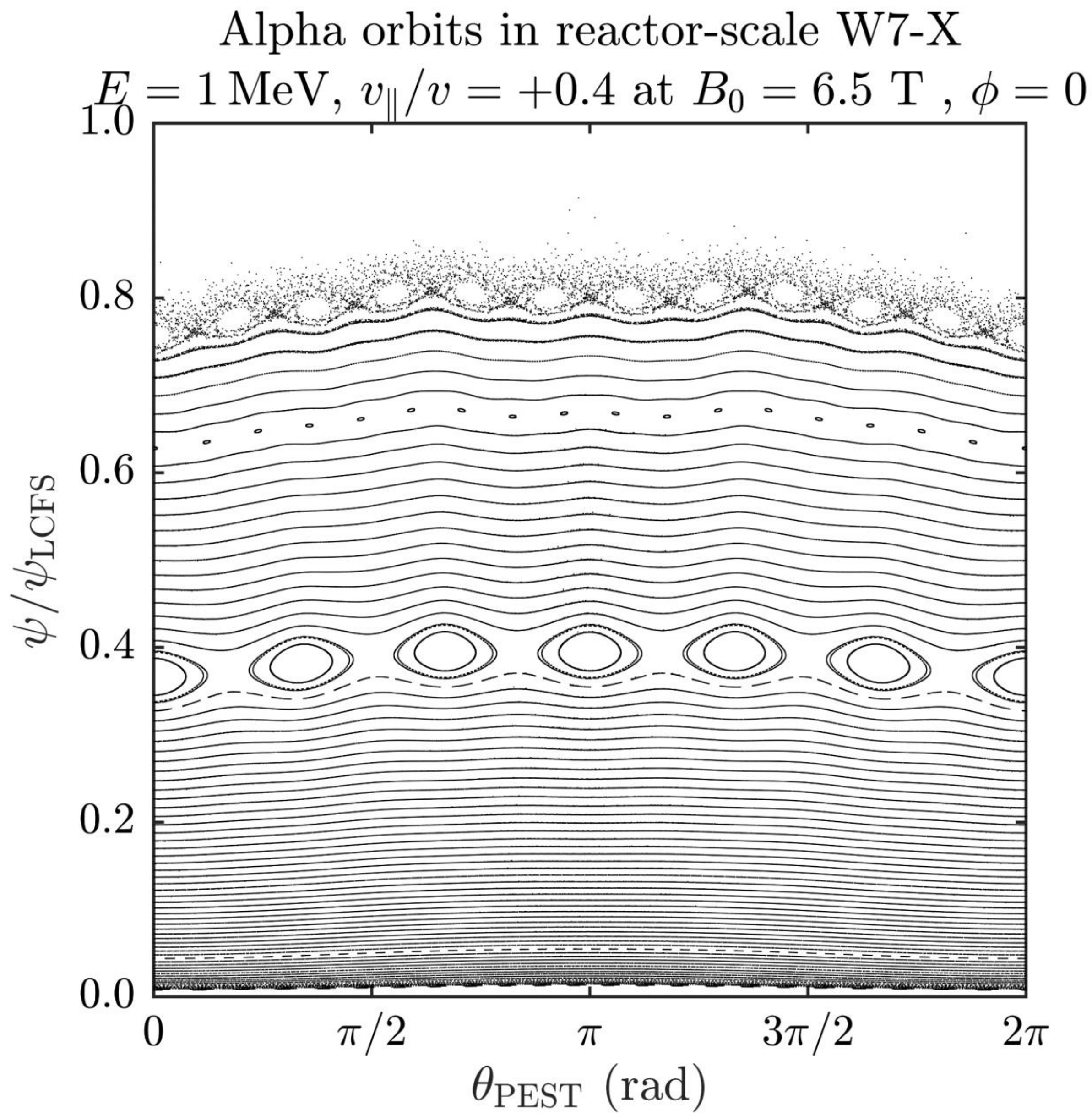
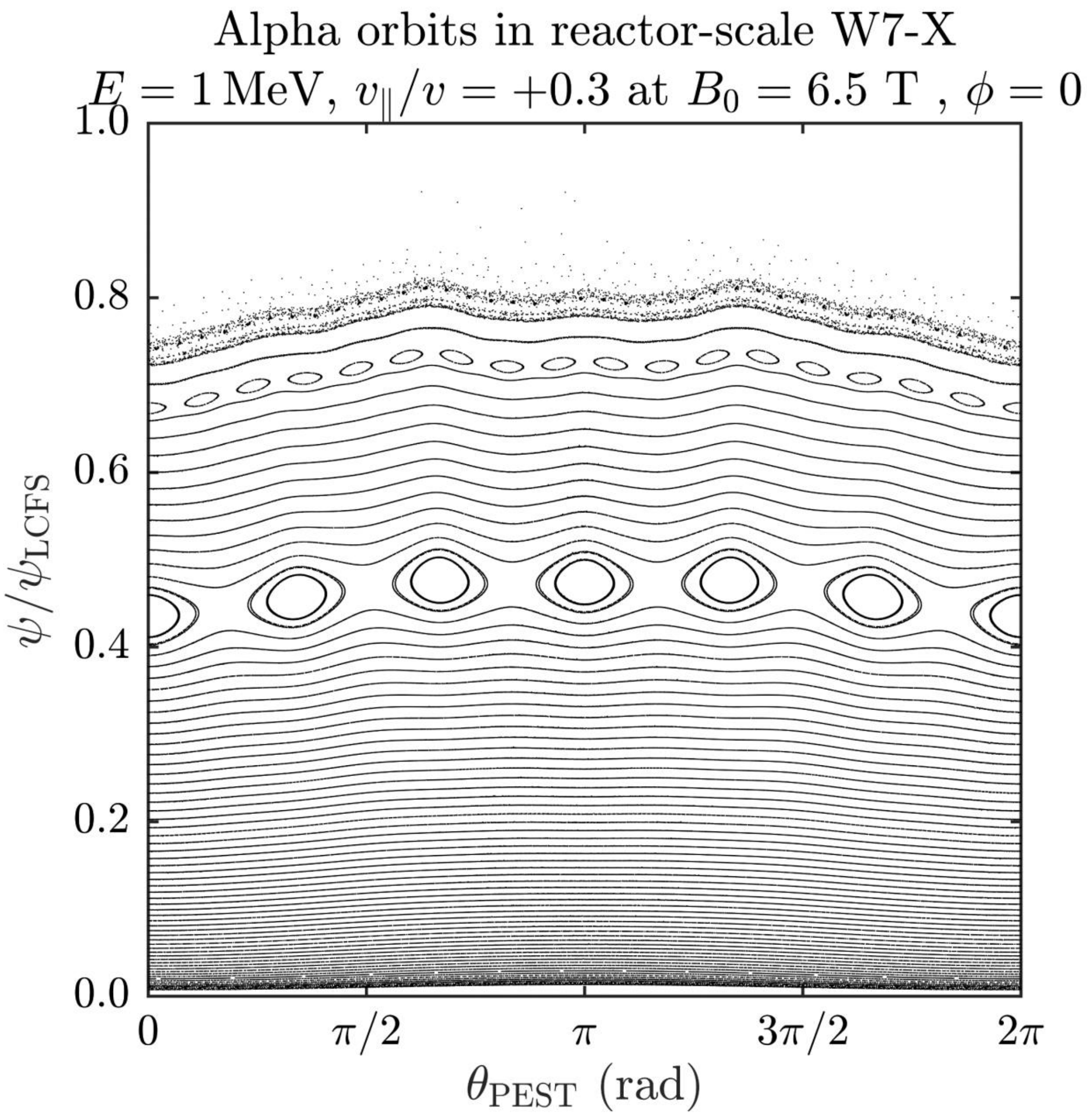

Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_{\|}/v = +0.4$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)


Alpha orbits in reactor-scale W7-X
$E = 1\,\mathrm{MeV}$, $v_{\parallel}/v = +0.3$ at $B_0 = 6.5$ T , $\phi = 0$
1.0
0.8
0.6
0.4
0.2
0.0
$\psi/\psi_{\mathrm{LCFS}}$
0
$\pi/2$
π
$3\pi/2$
2π
θ_{PEST} (rad)

Why drift islands?

- Guiding-centre equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{v_\parallel}{B}\left(\boldsymbol{B} + \boldsymbol{\nabla} \times \left(\frac{v_\parallel \boldsymbol{B}}{\Omega}\right)\right) \qquad \text{where} \quad v_\parallel = \pm\sqrt{2(\mathcal{E} - \mu B)}\,.$$

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- $\boldsymbol{B}^\star$ is a small perturbation to $\boldsymbol{B}$. In a general stellarator, this small perturbation creates islands: 'drift islands'.

Why drift islands?

Drift-island width:

$$w \sim \sqrt{\frac{\rho_\star}{s}}\, L$$

- Guiding-centre equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{v_\parallel}{B}\underbrace{\left(\boldsymbol{B} + \overbrace{\boldsymbol{\nabla}\times\left(\frac{v_\parallel \boldsymbol{B}}{\Omega}\right)}^{\text{Drifts: } \sim \rho_\star B}\right)}_{:= \boldsymbol{B}^\star \text{ (Morozov \& Solov'ev 66)}} \quad \text{where} \quad v_\parallel = \pm\sqrt{2(\mathcal{E} - \mu B)}\,.$$

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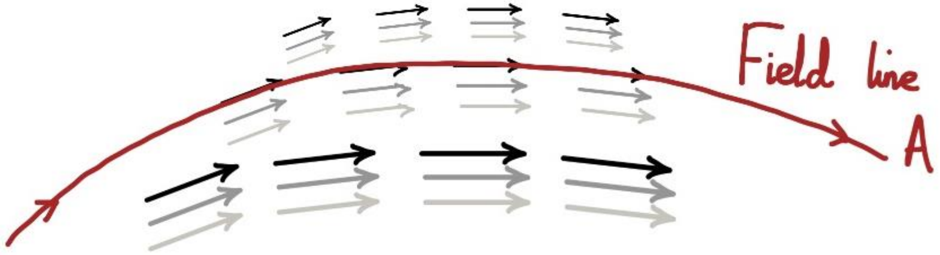
Field-line Lagrangian



- Variational principle for field-line equations:

$$S[\boldsymbol{x}] = \int_{\lambda_1}^{\lambda_2} \boldsymbol{A}(\boldsymbol{x}) \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_1^2 \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}$$

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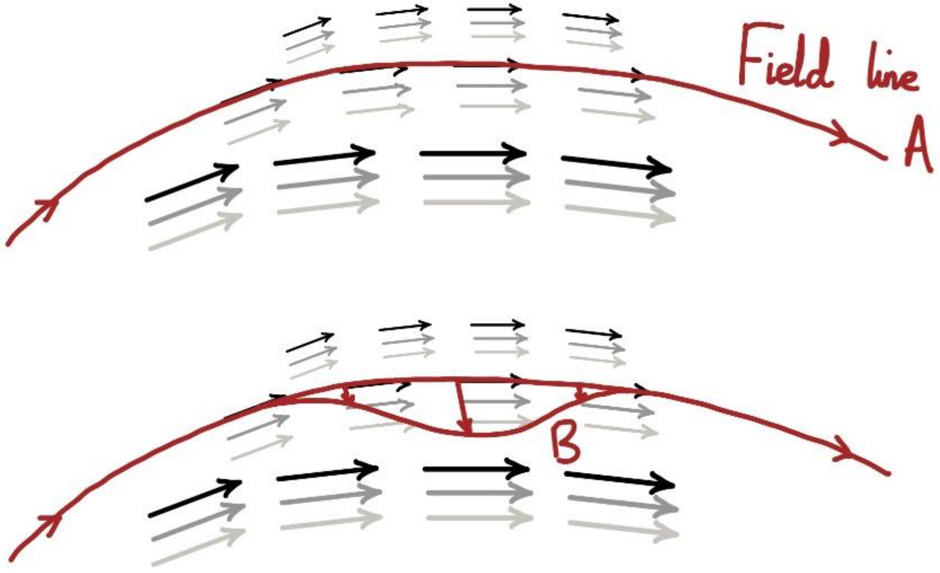

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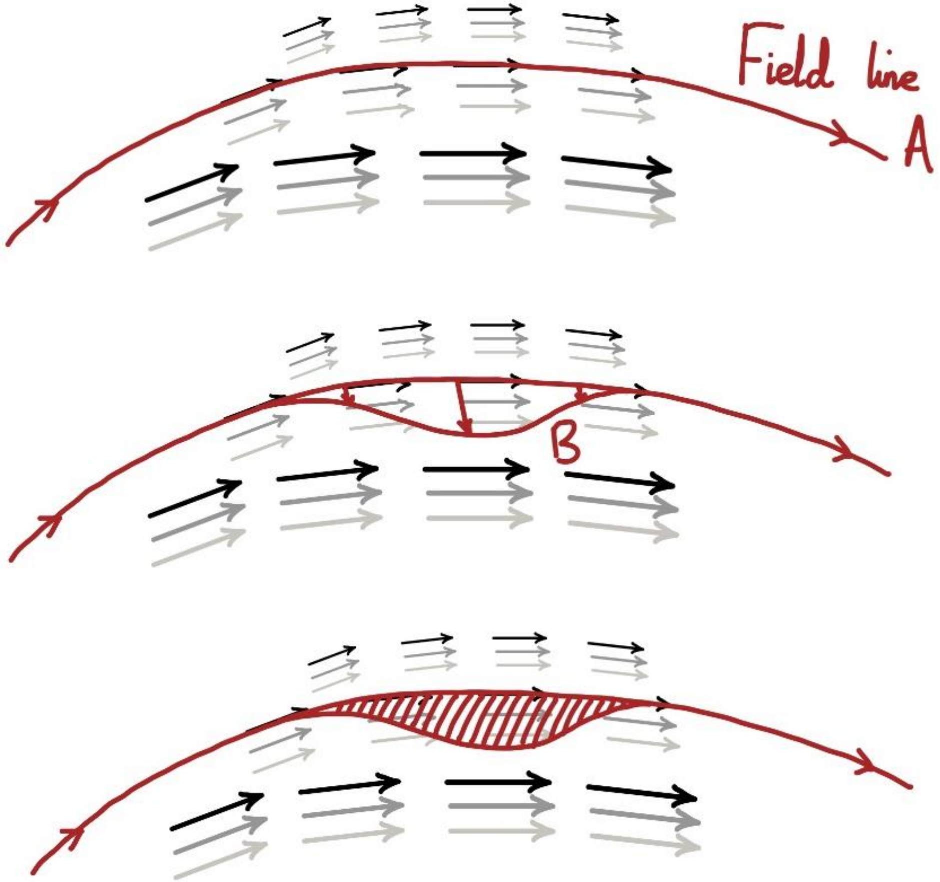

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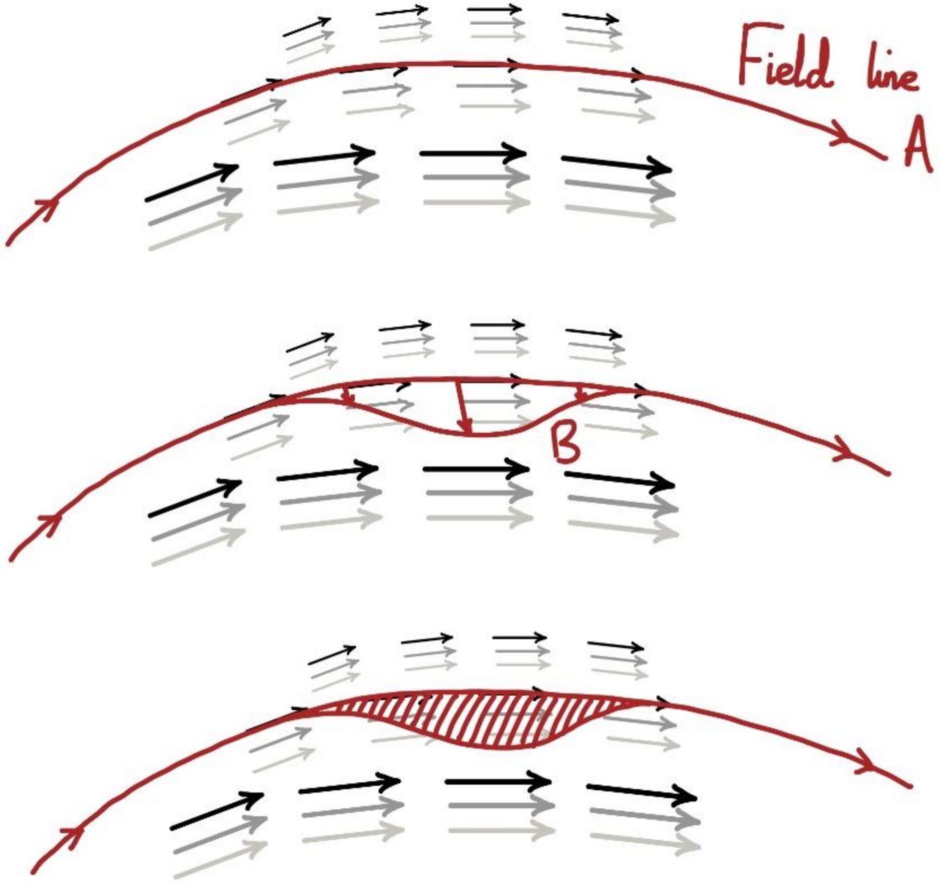
We use toroidal angle ζ to parameterise position along field line, so:

$$\mathcal{L} = \boldsymbol{A} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta}$$

- Field lines of $\boldsymbol{B}^\star$: $\boldsymbol{A}^\star = \boldsymbol{A} + \frac{v_\parallel \boldsymbol{B}}{\Omega}$



Unperturbed Lagrangian

$$\mathcal{L} = \boldsymbol{A} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta}$$

Unperturbed Lagrangian

$$\boldsymbol{A} = \psi \boldsymbol{\nabla} \theta - \chi(\psi) \boldsymbol{\nabla} \zeta$$

$$\downarrow$$

$$\mathcal{L} = \boldsymbol{A} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta}$$

Unperturbed Lagrangian

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$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \partial_\psi \boldsymbol{x}\,\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} + \partial_\theta \boldsymbol{x}\,\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \partial_\zeta \boldsymbol{x}$$

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$$\mathcal{L} = \boldsymbol{A}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \psi\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} - \chi(\psi)$$

Unperturbed Lagrangian

$$\boldsymbol{A} = \psi\,\boldsymbol{\nabla}\theta - \chi(\psi)\,\boldsymbol{\nabla}\zeta$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \partial_\psi \boldsymbol{x}\,\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} + \partial_\theta \boldsymbol{x}\,\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \partial_\zeta \boldsymbol{x}$$

$$\mathcal{L} = \boldsymbol{A}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \psi\,\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} - \chi(\psi)$$

Equations of motion trivial:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \iota(\psi)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = 0$$

Perturbed Lagrangian

$$\boldsymbol{A}^\star = \boldsymbol{A} + \frac{v_\parallel \boldsymbol{B}}{\Omega}$$

$$\downarrow$$

$$\mathcal{L} = \boldsymbol{A}^\star \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \left(\psi + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta \boldsymbol{x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \left(\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi \boldsymbol{x}\right)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \left(\chi(\psi) - \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta \boldsymbol{x}\right)$$

Perturbed Lagrangian

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$$\downarrow$$

$$\mathcal{L} = \boldsymbol{A}^\star \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\zeta} = \left(\psi + \frac{v_\parallel}{\Omega}\boldsymbol{B} \cdot \partial_\theta \boldsymbol{x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \left(\frac{v_\parallel}{\Omega}\boldsymbol{B} \cdot \partial_\psi \boldsymbol{x}\right)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \left(\chi(\psi) - \frac{v_\parallel}{\Omega}\boldsymbol{B} \cdot \partial_\zeta \boldsymbol{x}\right)$$

Equations of motion suck:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \iota(\psi) - \frac{mc}{Ze}\left[\partial_\psi(v_\parallel \hat{\boldsymbol{b}} \cdot \partial_\zeta \boldsymbol{x}) - \partial_\zeta(v_\parallel \hat{\boldsymbol{b}} \cdot \partial_\psi \boldsymbol{x})\right] + O(\rho_\star^2)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = \frac{mc}{Ze}\left[\partial_\theta(v_\parallel \hat{\boldsymbol{b}} \cdot \partial_\zeta \boldsymbol{x}) - \partial_\zeta(v_\parallel \hat{\boldsymbol{b}} \cdot \partial_\theta \boldsymbol{x})\right] + O\left(\rho_\star^2\right)$$

Near-identity change of coordinates

$$\mathcal{L} = \left(\psi + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta \boldsymbol{x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \left(\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi \boldsymbol{x}\right)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \left(\chi(\psi) - \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta \boldsymbol{x}\right)$$

- Why do the equations of motion suck? Because the Lagrangian depends on both angles θ and ζ.
- Our mission: find new coordinates $(\bar{\psi}, \bar{\theta}, \zeta)$ in which Lagranian does not depend on $\bar{\theta}$ or ζ.

Near-identity change of coordinates

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- Why do the equations of motion suck? Because the Lagrangian depends on both angles θ and ζ.
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$$\psi = \bar{\psi} + \psi^{(1)}(\bar{\psi}, \bar{\theta}, \zeta)$$

$$\theta = \bar{\theta} + \theta^{(1)}(\bar{\psi}, \bar{\theta}, \zeta)$$

inspired by

$$\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{\rho}(\boldsymbol{x}, \mathcal{E}, \mu, \varphi)$$

Near-identity change of coordinates

$$\mathcal{L} = \left(\psi + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \left(\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi\boldsymbol{x}\right)\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \left(\chi(\psi) - \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta\boldsymbol{x}\right)$$

$$\psi = \bar{\psi} + \psi^{(1)}(\bar{\psi},\bar{\theta},\zeta)$$

$$\theta = \bar{\theta} + \theta^{(1)}(\bar{\psi},\bar{\theta},\zeta)$$

$$\Downarrow$$

$$\mathcal{L} = \left(\bar{\psi} + \psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x} - \partial_\theta R^{(1)}\right)\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} + \left(-\theta^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi\boldsymbol{x} - \partial_\psi R^{(1)}\right)\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\zeta}$$
$$+ \left(-\chi(\bar{\psi}) - \iota(\bar{\psi})\psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta\boldsymbol{x} - \partial_\zeta R^{(1)}\right)$$

Also add total derivative: $+\frac{\mathrm{d}R^{(1)}}{\mathrm{d}\zeta} = \partial_\psi R^{(1)}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} + \partial_\theta R^{(1)}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \partial_\zeta R^{(1)}$

Near-identity change of coordinates

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$$+ \left(-\chi(\bar{\psi}) - \iota(\bar{\psi})\,\psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta \boldsymbol{x} - \partial_\zeta R^{(1)}\right)$$

Near-identity change of coordinates

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$$+ \left(-\chi(\bar{\psi}) - \iota(\bar{\psi})\psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta\boldsymbol{x} - \partial_\zeta R^{(1)}\right)$$

$$\psi^{(1)} = -\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x} + \partial_\theta R^{(1)}$$

$$\theta^{(1)} = +\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi\boldsymbol{x} - \partial_\psi R^{(1)}$$

$$\mathcal{L} = \bar{\psi}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} - \chi(\bar{\psi}) + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot(\partial_\zeta\boldsymbol{x} + \iota\partial_\theta\boldsymbol{x}) - \left(\partial_\zeta R^{(1)} + \iota\partial_\theta R^{(1)}\right)$$

Near-identity change of coordinates

$$\mathcal{L} = \left(\bar{\psi} + \psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x} - \partial_\theta R^{(1)}\right)\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} + \left(-\theta^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi\boldsymbol{x} - \partial_\psi R^{(1)}\right)\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\zeta}$$
$$+ \left(-\chi(\bar{\psi}) - \iota(\bar{\psi})\psi^{(1)} + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\zeta\boldsymbol{x} - \partial_\zeta R^{(1)}\right)$$

$$\downarrow$$

$$\mathcal{L} = \bar{\psi}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} - \chi(\bar{\psi}) + \underbrace{\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot(\partial_\zeta\boldsymbol{x} + \iota\partial_\theta\boldsymbol{x})}_{=V} - \left(\partial_\zeta R^{(1)} + \iota\partial_\theta R^{(1)}\right)$$

$$\partial_\zeta R^{(1)} + \iota\partial_\theta R^{(1)} = V - \langle V\rangle$$

$$\langle f\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f\,\mathrm{d}\theta\,\mathrm{d}\zeta$$

Final result

$$\psi^{(1)} = -\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x} + \partial_\theta R^{(1)}$$

$$\theta^{(1)} = +\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi\boldsymbol{x} - \partial_\psi R^{(1)}$$

$$\partial_\zeta R^{(1)} + \iota\,\partial_\theta R^{(1)} = V - \langle V\rangle$$

$$\mathcal{L} = \bar{\psi}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} - \chi(\bar{\psi}) + \left\langle\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot(\partial_\zeta\boldsymbol{x} + \iota\,\partial_\theta\boldsymbol{x})\right\rangle$$

Conserved quantity: $\bar{\psi}$

$$\psi = \bar{\psi} + \psi^{(1)}(\bar{\psi},\bar{\theta},\zeta) \quad\longrightarrow\quad \bar{\psi} \simeq \psi - \psi^{(1)}(\psi,\theta,\zeta) = \psi + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta\boldsymbol{x} - \partial_\theta R^{(1)}$$

Problem :(

$$\bar{\psi} = \psi + \frac{v_\parallel}{\Omega}\boldsymbol{B} \cdot \partial_\theta \boldsymbol{x} - \partial_\theta R^{(1)}$$

- If all passing particles conserved $\bar{\psi}$, then they would all have orbit widths of size $w \sim \rho_\star L$.
- Must fail around rational surfaces!

Problem :(

$$\bar{\psi} = \psi + \frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\theta \boldsymbol{x} - \partial_\theta R^{(1)}$$

- If all passing particles conserved $\bar{\psi}$, then they would all have orbit widths of size $w \sim \rho_\star L$.
- Must fail around rational surfaces!
- Return to:

$$\partial_\zeta R^{(1)} + \iota\partial_\theta R^{(1)} = V - \langle V\rangle$$

$$\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla} R^{(1)} = v_\parallel - \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\zeta\left\langle\frac{v_\parallel}{\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\zeta}\right\rangle := U$$

$$R^{(1)} = \int^l U\,\mathrm{d}l'$$

Magnetic differential equation

$$\partial_\zeta R^{(1)} + \iota \partial_\theta R^{(1)} = V - \langle V \rangle$$

- Better idea is to Fourier expand:

$$R^{(1)} = \sum_{(p,q)\in\mathbb{Z}^2} R^{(1)}_{pq}\, e^{i(p\theta - q\zeta)} \qquad V = \sum_{(p,q)\in\mathbb{Z}^2} V_{pq}\, e^{i(p\theta - q\zeta)}$$

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$$R^{(1)} = \sum_{(p,q)\in\, \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{iV_{pq}}{q - \iota p}\, e^{i(p\theta - q\zeta)}$$

When does this converge?

$$R^{(1)} = \sum_{(p,q)\in \mathbb{Z}^2\setminus\{(0,0)\}} \frac{iV_{pq}}{q - \iota p} e^{i(p\theta - q\zeta)}$$

- Clearly diverges when ι is rational: resonances. Does it converge anywhere?
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- Two ways to make sum converge:
  1. Make numerators vanish whenever denominators do: omnigeneous stellarators
  2. Make ι 'sufficiently irrational' so that numerators decay faster with $|p|$ and $|q|$ than denominators

  e.g. sum guaranteed to converge if V is analytic and ι is *Diophantine*: there exist C, $\sigma > 0$ such that, for all p, q,

$$\left|\iota - \frac{p}{q}\right| > \frac{C}{q^{2+\sigma}}$$

Still problems :(

$$R^{(1)} = \sum_{(p,q)\in \mathbb{Z}^2\setminus\{(0,0)\}} \frac{iV_{pq}}{q - \iota p}\, e^{i(p\theta - q\zeta)}$$

- This sum diverges for arbitrarily small deviation from omnigeneity – no stellarator is perfectly omnigeneous!

- Even though the sum converges for sufficiently irrational ι, we need it to be differentiable because we need its ψ-derivative in

$$\theta^{(1)} = +\frac{v_\parallel}{\Omega}\boldsymbol{B}\cdot\partial_\psi \boldsymbol{x} - \partial_\psi R^{(1)}\,.$$

- Must be a better way!

Dealing with resonances

- Two key ideas. First, we don't need to eliminate all angle-dependence from the Lagrangian: only to $O(\rho_\star^2)$. Second, give up on describing all of phase space at once.

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$$\mathcal{L} = \bar{\psi}\,\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} - \chi(\bar{\psi}) + V_{<n} + \overbrace{V_{\geq n}}^{\sim\rho_\star^2} - \Big(\partial_\zeta R^{(1)} + \iota\,\partial_\theta R^{(1)}\Big)$$

$$V_{<n} = \sum_{(p,q)\in\mathbb{Z}^2_{<n}} V_{pq}\, e^{i(p\theta - q\zeta)} \qquad \mathbb{Z}^2_{<n} = \{(p,q)\in\mathbb{Z}^2 : |p| + |q| < n\}$$

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Non-resonant case

- There are finitely many rational surfaces that resonate with modes of $V_{<n}$. Assume we are working in a region that doesn't contain any of them.

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Physical interpretation

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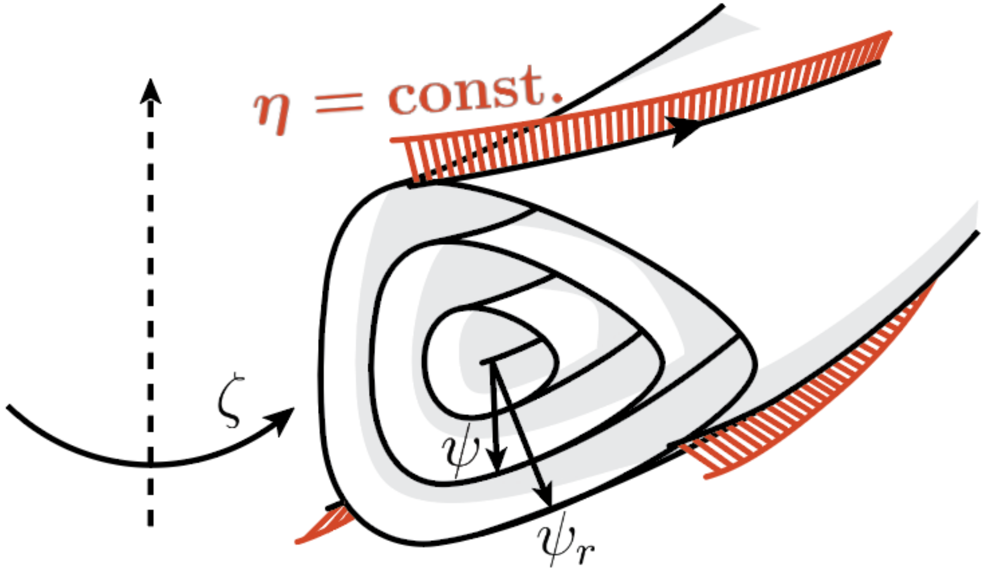

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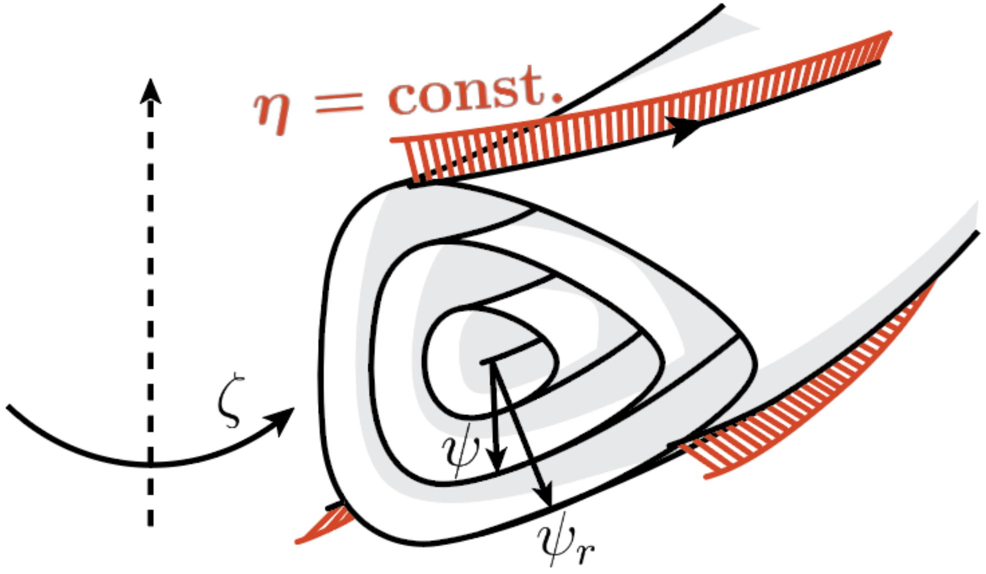
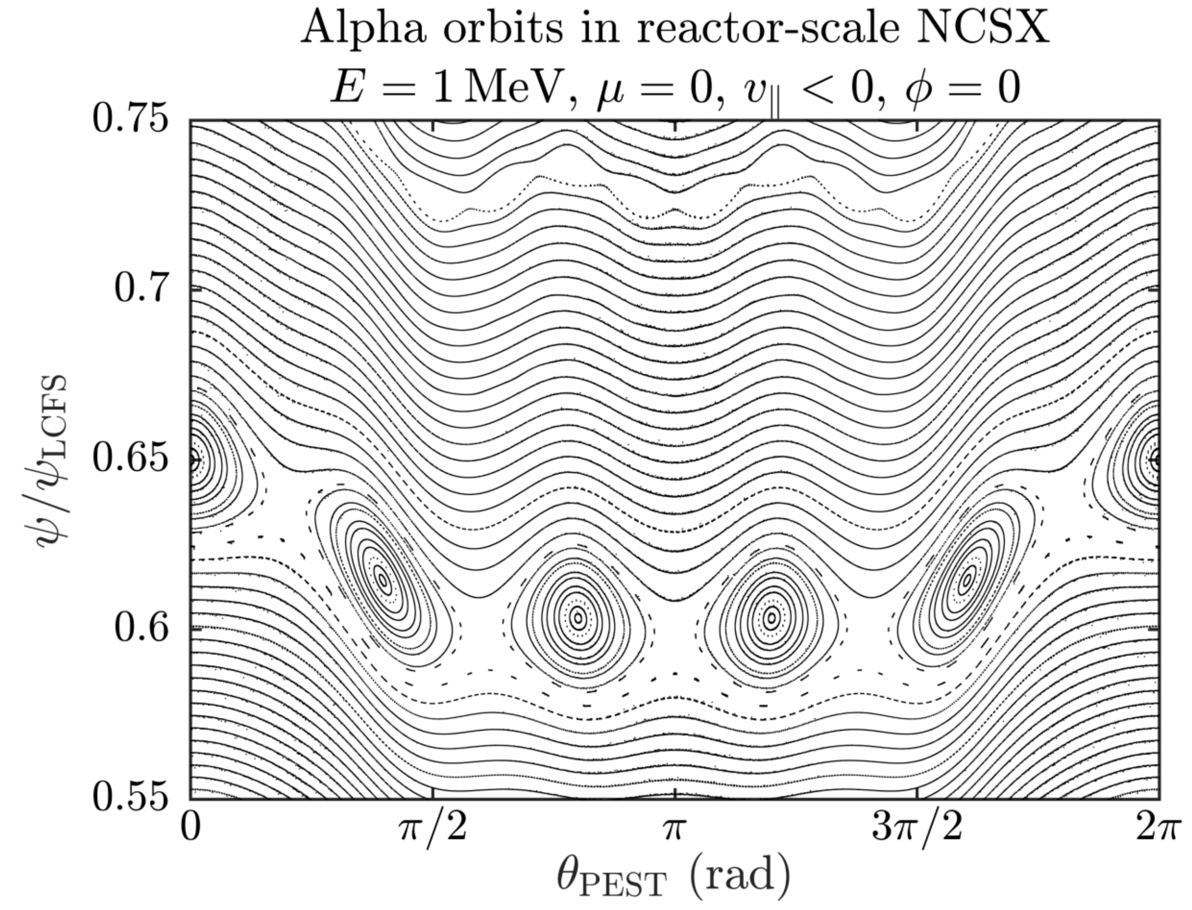
$$\propto \oint\left(v_\parallel\hat{\boldsymbol{b}} + \frac{Ze}{mc}\boldsymbol{A}\right)\cdot\mathrm{d}\boldsymbol{x}\,.$$



Comparison with simulation



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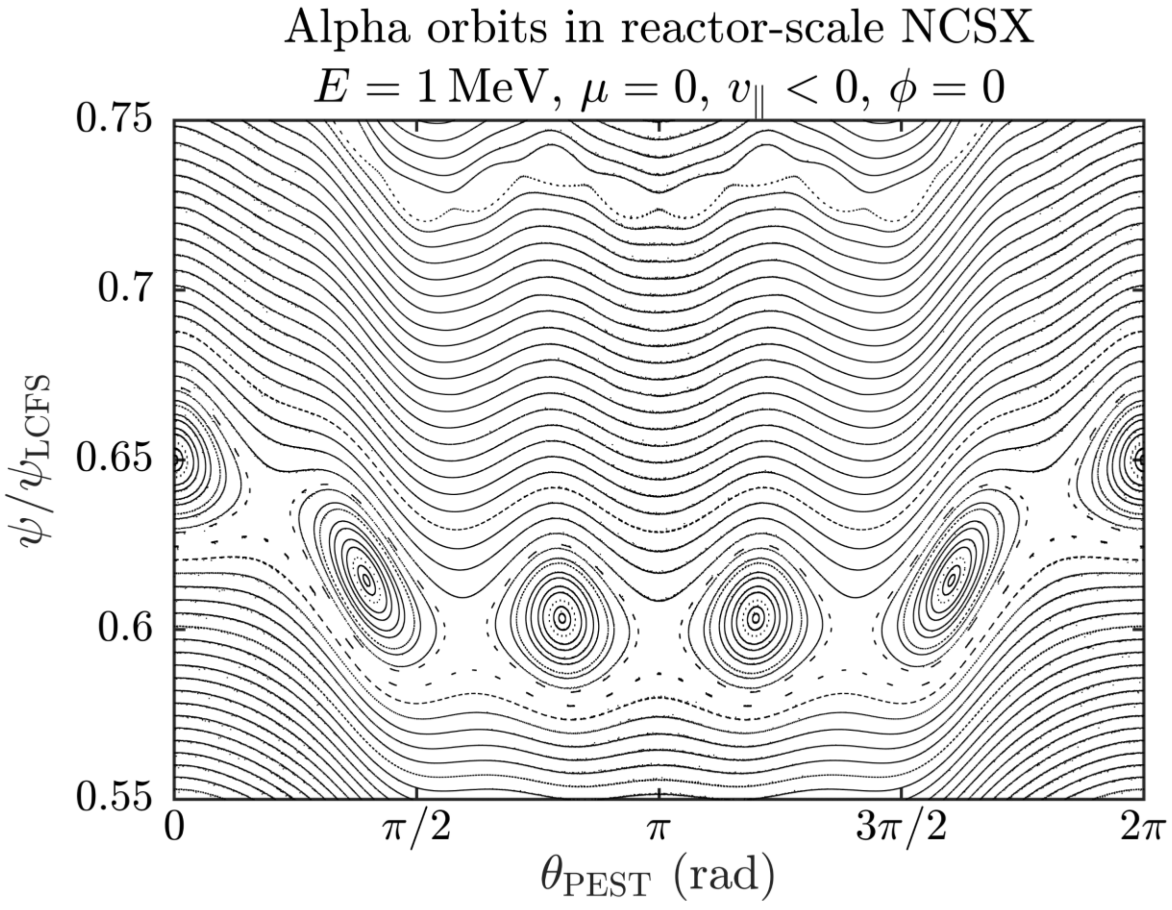

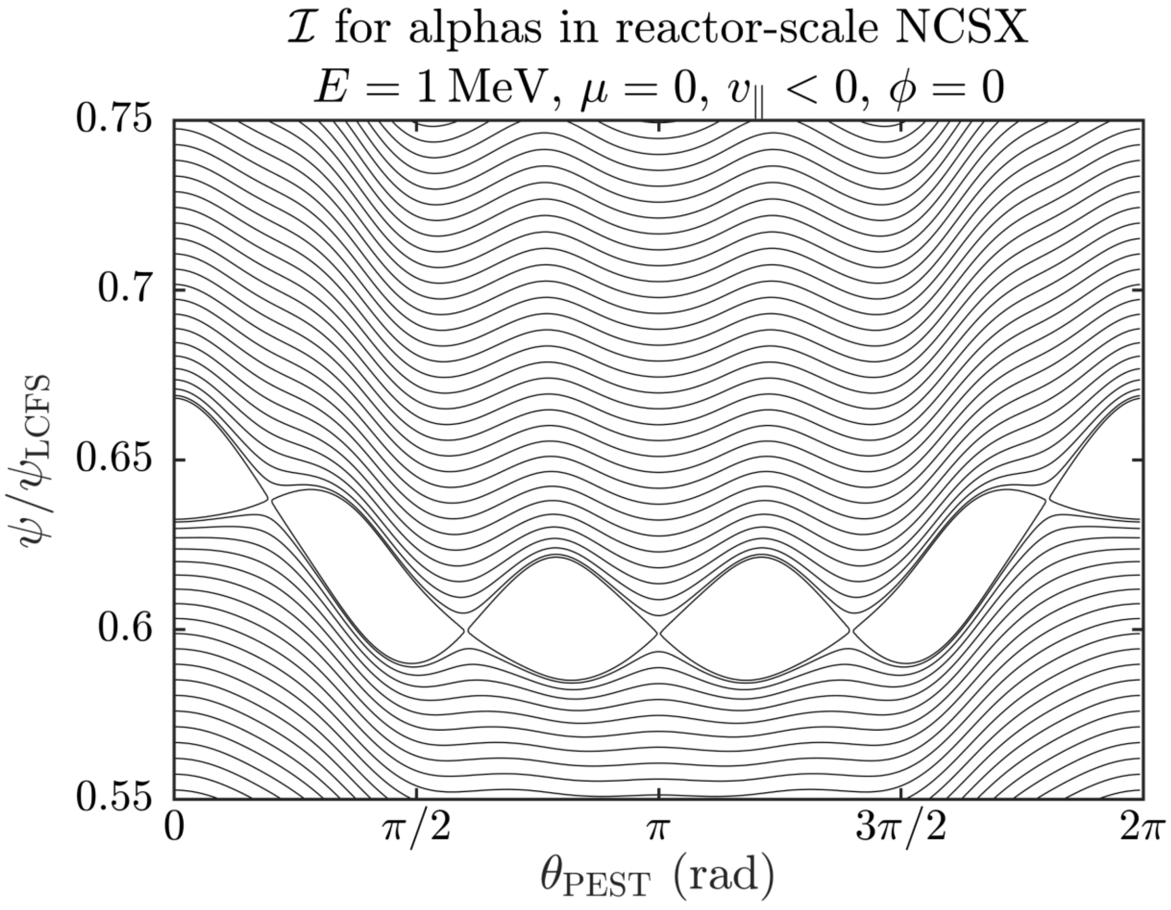

Aside: Going to higher order

$$\mathcal{L} = (\psi + O(\rho_\star))\ \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + (O(\rho_\star))\,\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} - \chi(\psi) + O(\rho_\star)$$

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$$\Rightarrow \quad \mathcal{L} = \left(\bar{\psi} + O(\rho_\star^2)\right)\ \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\zeta} + \left(O(\rho_\star^2)\right)\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\zeta} - \bar{\chi}(\bar{\psi}) + O(\rho_\star^2)$$

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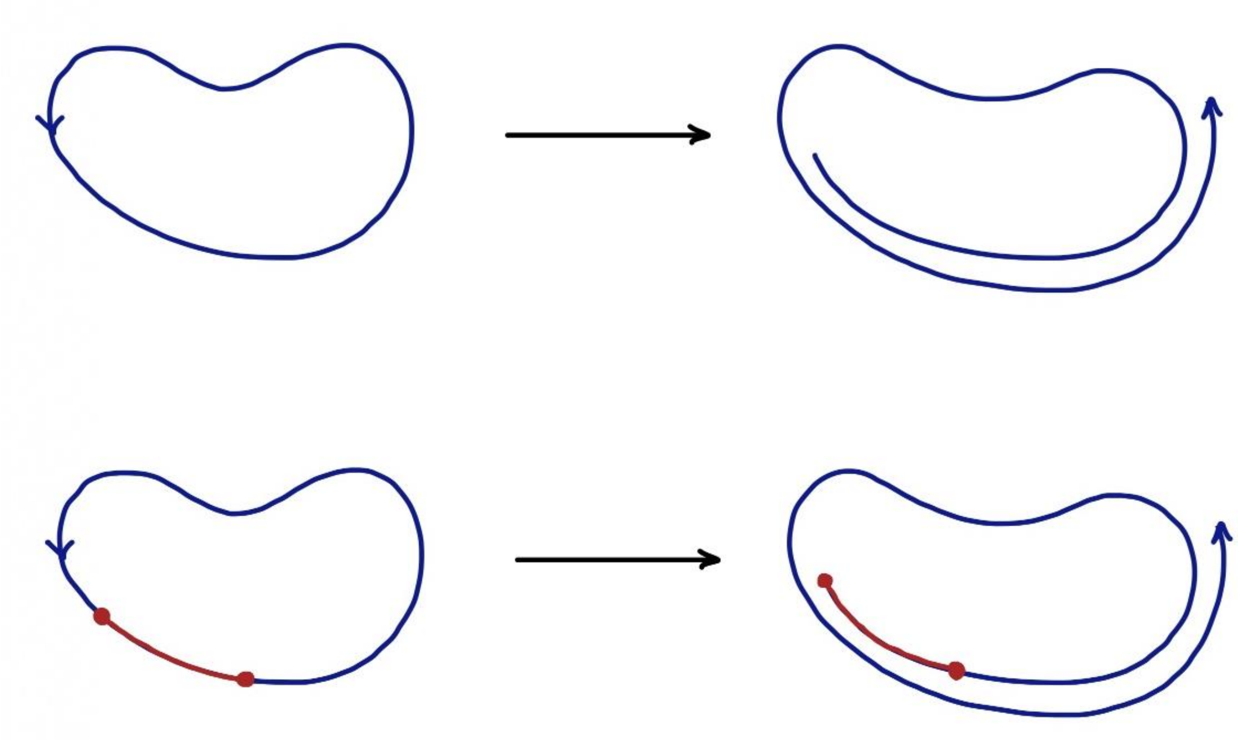
Aside: Going to higher order

$$(\boldsymbol{J}, \boldsymbol{\theta}) \longrightarrow (\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})$$



Summary

- Guiding-centre motion follows field lines of $\boldsymbol{B}^{\star}$.
- Drift islands exist around rational surfaces, width $w \sim \sqrt{\rho_{\star}/s}$.
- Island shape from adiabiatic invariant:

$$\mathcal{I} \propto \oint \left(v_{\parallel} \hat{\boldsymbol{b}} + \frac{Ze}{mc} \boldsymbol{A} \right) \cdot \mathrm{d}\boldsymbol{x}$$

- Whether a surface is 'resonant' (and therefore needs to be treated differently when calculating orbits) depends on the numerical value of ι, the size of resonant Fourier harmonics of the field, the values of $\mathcal{E}$ and μ, and how high order we go. Not as simple as rational or irrational!