Stabilisation of discrete physical Vlasov-Poisson equations based on new mathematical results for AW Hermite functions

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Symmetric/asymmetric functions

For simplicity consider the transport equation in velocity

$$\partial_t f(t,v) + e\partial_v f(t,v) = 0, \qquad e \in \mathbb{R}, \quad T > 0. \tag{1}$$

Let $(H_m)_{m\in\mathbb{N}}$ be the family of Hermite polynomials which are orthogonal with respect to the Gaussian weight e^{-v^2}

$$H_0(v) = 1, \quad H_1(v) = 2v, \quad H_2(v) = 4v^2 - 2, \quad \ldots$$

- The symmetric Hermite functions

$$\phi_m(v) = e^{-v^2/2T}\, T^{-\frac{1}{2}} (2^m m! \sqrt{\pi})^{-\frac{1}{2}} H_m(v/\sqrt{T}) = e^{\frac{v^2}{2T}} \psi_m(v) = e^{\frac{-v^2}{2T}} \psi^m(v)$$

forms a complete orthonormal family (Hilbertian family) of $L^2(\mathbb{R})$ and

$$\phi'_m(v) = -\sqrt{\frac{m+1}{2T}}\phi_{m+1}(v) + \sqrt{\frac{m}{2T}}\phi_{m-1}(v).$$

- The **asymmetric weighted (AW) bases** are $(\psi_m)_{m\in\mathbb{N}}$ and $(\psi^m)_{m\in\mathbb{N}}$

$$\begin{cases} \psi_m(v) = & e^{-v^2/T}\, T^{-\frac{1}{2}} (2^m m! \sqrt{\pi})^{-\frac{1}{2}} H_m(v/\sqrt{T}), \\ \psi^m(v) = & T^{-\frac{1}{2}} (2^m m! \sqrt{\pi})^{-\frac{1}{2}} H_m(v/\sqrt{T}) \quad = e^{\frac{v^2}{T}} \psi_m(v/\sqrt{T}). \end{cases}$$

One has for all m

$$(\psi_m)'(v) = -\sqrt{\frac{2(m+1)}{T}}\ \psi_{m+1}(v) \text{ and } (\psi^m)'(v) = \sqrt{\frac{2m}{T}}\ \psi^{m-1}(v).$$

Discretization

Start from the expansion $f(v) = \sum_{m\geq 0} u_m \psi_m(v)$ where the moments are $u_m(t)$.

By definition one has $e^{\frac{v^2}{2T}} f(v) = \sum_{m\geq 0} u_m \phi_m(v)$

$$e^{\frac{v^2}{2T}} f \in L^2(\mathbb{R}) \Longleftrightarrow \left\| e^{\frac{v^2}{2T}} f \right\|^2_{L^2(\mathbb{R})} = \sum_{m\geq 0} |u_m|^2 < \infty.$$

$$\text{One has} \begin{cases} \partial_t f(t,v) = \sum_{m\geq 0} u_m'(t)\psi_m(v), \\ \partial_v f(t,v) = -\sum_{m\geq 0} u_m(t)\sqrt{\frac{2(m+1)}{T}}\psi_{m+1}(v). \end{cases}$$

Taking the moments against $1, v, v^2, \ldots,$ on gets a triangular system

$$\begin{cases} u_0'(t) = 0, \\ u_1'(t) - e\sqrt{\frac{2}{T}} u_0(t) = 0, \\ \ldots, \\ u_m'(t) - e\sqrt{\frac{2m}{T}} u_{m-1}(t) = 0 \text{ for all } m \geq 1, \end{cases}$$

Lemma (Remarkable properties)

- the density $\int f(t,v)v$ is constant in time,
- the variation of u_m depends only on u_{m-1}.

Set the infinite triangular matrix $D = (d_{mn})_{m,n\geq 0}$ with $d_{mn} = -e\sqrt{\frac{2m}{T}}\delta_{m-1,n}$.
and the infinite vector of moments $U(t) = u_m(t))_{m\geq 0}. \in \mathbb{R}^{\mathbb{N}}$

$$\partial_t U + DU = 0.$$

Remark

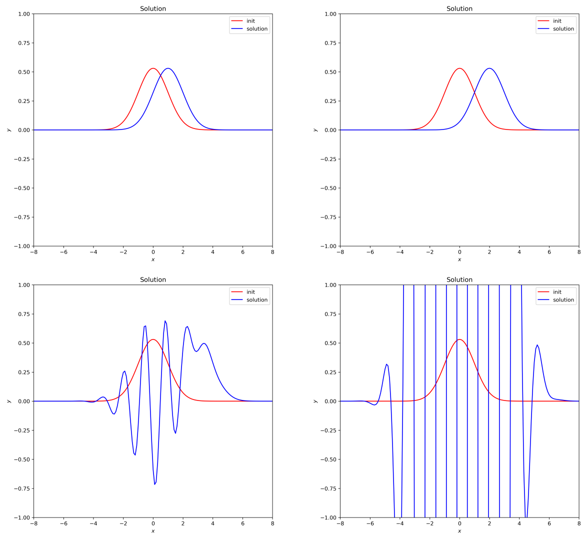
The infinite system of moments is formally equivalent to the transport equation.

Discretize $U_N(t) = (u_m(t))_{0\leq m\leq N} \in \mathbb{R}^{N+1}$ with $D^N = (d_{mn})_{0\leq m,n\leq N}$:

$$\partial_t U_N + D_N U_N = 0.$$

Our discretization is systematically performed with a Crank-Nicholson technique

$$\frac{U_N^{n+1} - U_N^n}{\Delta t} + D^N \frac{U_N^n + U_N^{n+1}}{2} = 0, \quad n \geq 0, \tag{2}$$



20 moments and $\Delta t = 0.1$. Until time $t \approx t_2$, the solution is correct. Then a numerical instability starts to be visible for $t \approx t_3$, and blows up exponentially for $t \geq t_4$.

More refs

General:

- Moments and modeling : kinetic equations $\int f(v)v^i dv$ with $i = 1, 2, 3$, Muller-Rugierri (extended thermodynamics 1993), Levermore 1996, ...

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- Grandgirard/.../Zarzoso: A 5D **gyrokinetic** full-f global semi-lagrangian code for flux-driven ion turb. sim., 2016.

- Adkins-Schekochihin, 2017: A solvable model of Vlasov-kinetic plasma turbulence in Fourier-Hermite phase space,

- Pham/Helluy/Crestetto 2012, Delzanno 2015, Manzini/.../Markidis 2016

- Charles+Dai+D.+Hirstoaga, Discrete moments models for Vlasov equations with non constant strong magnetic limit, CRAS 2023.

Stability:

- Schumer-Holloway, Vlasov simulations using velocity-scaled Hermite representations, 1998.

- D. Funaro, G. Manzini. Stability and conservation properties of Hermite- based approximations of the Vlasov-Poisson system, 2021.

- M. Bessemoulin-Chatard, F. Filbet, Stability of conservative discontinuous Galerkin/Hermite spectral methods for the Vlasov-Poisson system, 2022.

Gramm matrix A

- One has

$$\int_{\mathbb{R}} f(t,v)^2 dv = \int_{\mathbb{R}} \left(\sum_{m\geq 0} u_m(t)\psi_m(v) \right)^2 dv = \langle AU, U \rangle$$

where the infinite symmetric Gramm matrix is $A = (a_{mn})_{m,n\geq 0}$

$$a_{mn} = a_{nm} = \int \psi_m(v)\,\psi_n(v)\,dv.$$

- **In theory** a calculation of these coefficients is possible direct expansion is enough, but it is useless because of numerical stability as seen below

$$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.$$

The point is that there is **no answer in special-functions literature**
NIST, Abramovitz-Stegun, . . .

- **Imagine (JL 71'):** there is a way to calculate these numbers

$$\int_{\mathbb{R}} \left(e^{-v^2/2T} H_m(v) \right) \left(e^{-v^2/2T} H_n(v) \right) e^{-v^2/T} dv.$$

Property of infinite matrices

Lemma

$AD = -D^t A$.

Loosy proof: one has

$$0 = \frac{d}{dt}\int f^2 dv = \frac{d}{dt}\langle AU, U\rangle = 2\left\langle A\frac{d}{dt}U, U\right\rangle = 2\langle ADU, U\rangle \text{ for "all" } U.$$

So AD is skew symmetric.

Truncated matrices

Decompose

$$D = \begin{pmatrix} D^N_{11} & D^N_{12} \\ D^N_{21} & D^N_{22} \end{pmatrix}$$

where the blocks are

$$\begin{cases} D^N_{11} = D^N & \in \mathbb{R}^{(N+1)\times(N+1)}, \quad D^N_{12} = 0 & \in \mathbb{R}^{(N+1)\times\mathbb{N}}, \\ D^N_{21} & \in \mathbb{R}^{\mathbb{N}\times(N+1)}, \quad D^N_{22} & \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}. \end{cases}$$

Similarly decompose the infinite Gramm matrix

$$A = \begin{pmatrix} A^N_{11} & A^N_{12} \\ A^N_{21} & A^N_{22} \end{pmatrix}$$

where the blocks are

$$\begin{cases} A^N_{11} & \in \mathbb{R}^{(N+1)\times(N+1)}, \quad A^N_{12} & \in \mathbb{R}^{(N+1)\times\mathbb{N}}, \\ A^N_{21} & \in \mathbb{R}^{\mathbb{N}\times(N+1)}, \quad A^N_{22} & \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}. \end{cases}$$

Lemma

For all $N \geq 1$, one has $A_{11}^N D_{11}^N + (D_{11}^N)^T A_{11}^N \neq 0$.

Proof.

The equality $AD + D^T A = 0$ reduces to

$$\begin{cases} A_{11}^N D_{11}^N + A_{12}^N D_{21}^N + (D_{11}^N)^T A_{11}^N + (D_{21}^N)^T A_{21}^N = 0, \\ A_{12}^N D_{22}^N + (D_{22}^N)^T A_{21}^N = 0, \\ A_{21}^N D_{11}^N + A_{22}^N D_{21}^N + (D_{11}^N)^T A_{21}^N + (D_{21}^N)^T A_{21}^N = 0, \\ A_{22}^N D_{22}^N + (D_{22}^N)^T A_{21}^N = 0. \end{cases}$$

One obtains $A_{11}^N D_{11}^N + (D_{11}^N)^T A_{11}^N = -A_{12}^N D_{21}^N - (D_{21}^N)^T A_{21}^N \in \mathbb{R}^{(N+1)\times(N+1)}$.
Since D is an infinite triangular matrix with only one non zero diagonal just below the main diagonal, then

$$D_{21}^N = \left(\begin{array}{c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & \dots & \dots & \begin{matrix} -e\sqrt{\frac{2(N+1)}{T}} \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ \dots \end{matrix} \end{array} \right) \in \mathbb{R}^{\mathbb{N}\times(N+1)}.$$

The multiplication by A_{12}^N yields a matrix which is zero except its last column. □

How to fix it ?

$\Longrightarrow$ Try to recover skew-symmetry with respect to the Gramm matrix.

No other solution than to find a mathematical way to calculate

$$a_{mn} = a_{nm} = \int \psi_m(v)\, \psi_n(v)\, dv$$

or

$$\int_{\mathbb{R}} \left(e^{-v^2/2T} H_m(v)\right) \left(e^{-v^2/2T} H_n(v)\right) e^{-v^2/T} dv.$$

Theorem

If the sum of the indices is odd $m + n \in 2\mathbb{N} + 1$, then $a_{mn} = 0$. Otherwise

$$a_{m-l,m+l} = (-1)^l T^{-\frac{1}{2}} 2^{-2m-\frac{1}{2}} \frac{(2m)!}{m!\sqrt{(m-l)!(m+l)!}}.$$

Theorem (Stable calculation)

i) Diagonal coefficients of the Gramm-Schmidt matrix

$$\begin{cases} a_{00} = (2T)^{-\frac{1}{2}}, \\ a_{m+1,m+1} = \frac{2m+1}{2m+2} a_{mm}, \quad m \geq 0. \end{cases} \tag{3}$$

ii) Lower extra-diagonal coefficients of the Gramm-Schmidt matrix

$$a_{m-l-1,m+l+1} = \sqrt{\tfrac{m-l}{m+l+1}} a_{m-l,m+l}, \quad l = 0, \dots, m-1, \tag{4}$$

iii) Upper diagonal coefficients

$$a_{m+l,m-l} = a_{m-l,m+l} \text{ for } 1 \leq l \leq m. \tag{5}$$

A modified matrix

One has formally

$$D = D - \frac{1}{2}A^{-1}(AD + D^t A) = \frac{1}{2}D - \frac{1}{2}A^{-1}D^t A.$$

A first modified matrix with $\varepsilon > 0$ is

$$\begin{aligned} D_{11}^{N,mod} &= D_{11}^N - \tfrac{1}{2}(A_{11}^N + \varepsilon I^N)^{-1}\left((A_{11}^N + \varepsilon I^N)D_{11}^N + (D_{11}^N)^t(A_{11}^N + \varepsilon I^N)\right) \\ &= \tfrac{1}{2}D_{11}^N - \tfrac{1}{2}(A_{11}^N + \varepsilon I^N)^{-1}(D_{11}^N)^t(A_{11}^N + \varepsilon I^N). \end{aligned} \quad (6)$$

Lemma

The solution of

$$\partial_t U^N + D_{11}^{N,mod} U^N = 0$$

preserves the weighted quadratic norm with penalization

$$\frac{d}{dt}\left(\left\langle U^N, A_{11}^N U^N \right\rangle + \varepsilon \|U^N\|^2\right) = 0.$$

Lemma

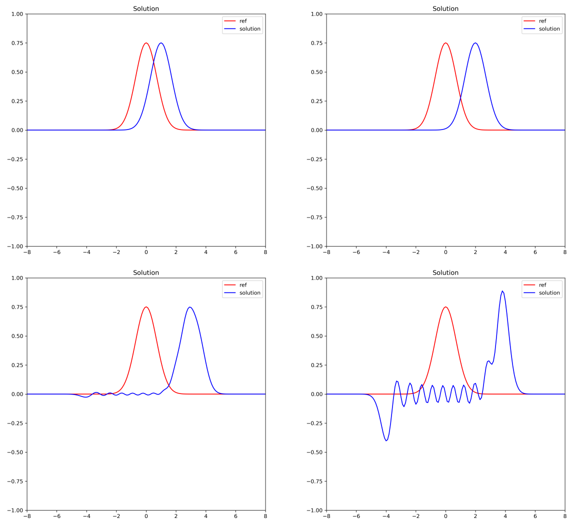
A second modified matrix is

$$D_{11}^{N,mod} = D_{11}^N + (A_{11}^N)^{-1}A_{12}^N D_{21}^N \Longleftrightarrow A_{11}^N D_{11}^{N,mod} = A_{11}^N D_{11}^N + A_{12}^N D_{21}^N$$

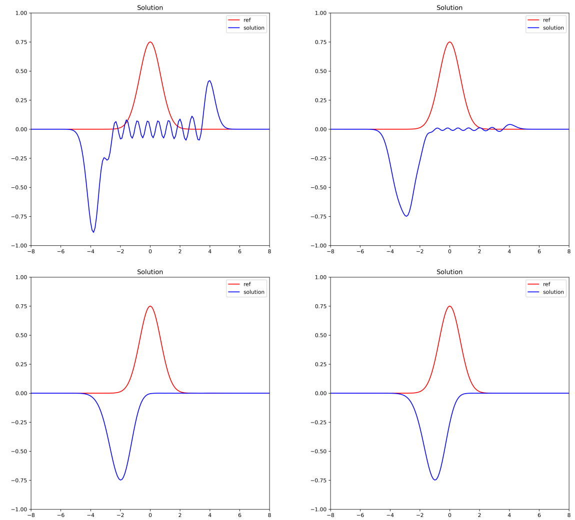
which is skew-symmetric and extremely simple to calculate exactly.

Results (with second method)

Results of the advection test computed the scheme (2) at time $t_1 = 1$ to $t_8 = 4$ (20 moments and $\Delta t = 0.1$).



A numerical recurrence phenomenon [Mehrenberger-Navoret-Pham] (Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh, Commun. in Comput. Phys., 2020.) with a change of sign is visible: $t_5 = 5$ to $t_8 = 8$

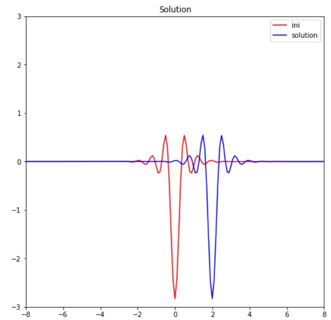


An accuracy test

Start from the formula in the sense of distribution

$$\delta(v) = \sum_{m \geq 0} \varphi_m(0)\varphi_m(v).$$

One has as well $\delta(v) = \delta(v)e^{-v^2/2T} = \sum_{m\geq 0} \varphi_m(0)\psi_m(v)$.
We consider the initial data $U^N = -(\varphi_m(0))_{0\leq m\leq N}$.



The reference solution is a Dirac mass which moves at velocity $e = 1$.

Diocotron

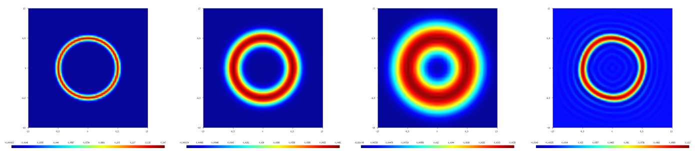
$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \right) \cdot \nabla_v f = 0, \\ \nabla \cdot \mathbf{E} = \int f dv - \rho_e. \end{cases}$$

$$f_0(\mathbf{x}, \mathbf{v}) = \begin{cases} \frac{n_0}{(\sqrt{2\pi})^3} (1 + \eta \cos(k\theta)) \exp^{-4(r-6.5)^2} \exp^{-|\mathbf{v}|^2/2}, & r^- \leq r \leq r^+, \\ 0, & \text{otherwise}, \end{cases}$$

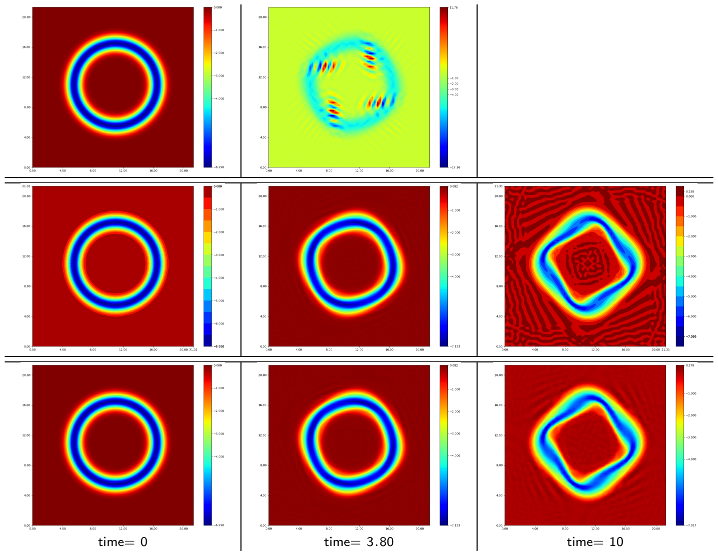
Non-homogenous magnetic field

$\mathbf{B}_0(\mathbf{x}) = \omega_c(\mathbf{x}) \frac{1}{\sqrt{1 + \alpha^2 x_3^2 + \alpha^2 x_2^2}} (1, \alpha x_2, -\alpha x_3)^\top$ and $\varepsilon > 0$ (=1 or 10^{-2}).

First seconds of a typical scenario:



Diocotron (our method)



Unstabilized method (first row), the stabilized method (second row), and the numerically conservative method (third row): 64^2 mesh grid. (Dai-D.-Hirtoaga in preparation)

Two-stream instability

$$f_0(x,v) = \frac{2}{7}\left(1 + \cos kx + \alpha(\cos 2kx + \cos 3kx)/1.2\right)(1 + v^2)\frac{1}{\sqrt{2\pi}}e^{-v^2/2}$$

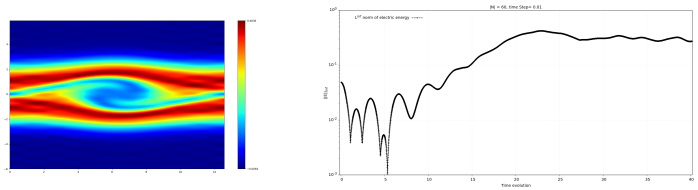


Figure: Left: density at time $t = 20$. Right: electric field versus time.

The simulation is nonlinearly stable.