# Beyond Quasilinear Theory: What Happens When Quasilinear Theory Fails in a Stellarator

(Catto 2024 + earlier effort by Catto & Tolman in 2021 JPP paper)


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**Quasilinear theory has many meanings**

# Quasilinear (QL) theory here

*Requires resonant particles & confined well particles

*QL theory requires a velocity dependent resonance

*Need collisions in steady state to define delta function

(not interested in bump on tail, modern QL theory :-)

*Collisions simplest and relevant for magnetic fusion

*Can be velocity space (RF) or drift kinetic

*Drift kinetic for "optimized" stellarator transport

# Stellarator transport

*Quasilinear = resonant plateau = superbanana plateau

*Collisions resolve a singularity: resonant plateau (RP)

*RP involves narrow collisional layers resonance

*RP behavior occurs when $\nu$ cancels out: "plateau"

*Drift reversal resonance occurs for trapped alphas

*Local f flattening suggests breakdown of QL theory

# Trapped alphas in a nearly quasiymmetric stellarator

Bounce averaged drift kinetic equation

$$\bar{\omega}_\alpha \frac{\partial \tilde{f}}{\partial \varphi} - \bar{V} sin\varphi \, \frac{\partial(\bar{f} + \tilde{f})}{\partial r} = \bar{\nu}\epsilon \frac{\partial^2 \tilde{f}}{\partial \lambda^2}$$

with $\bar{f}$ = unperturbed (slowing down tail solution)

$\tilde{f}$ = perturbed distribution ($\bar{f} \gg \tilde{f}$)

$\bar{\omega}_\alpha$ = bounce average trapped drift in a flux surface

$\bar{V}$ = bounce average radial drift due to QS departure $\delta$

$\varphi$ = QS breaking angular variation

$\lambda = 2\mu B_0/v^2$ = pitch angle variable (adiabatic invariant)

$\bar{\nu} = v_\lambda^3/v^3\tau_s$ pitch angle collision freq. of alphas by ions

# Trapped alpha resonance

* $\bar{\omega}_\alpha$ reverses direction at some pitch angle⟹resonance
* Drift reversal leads to collisional transport
* SBP = RP just like QL neglects the nonlinear term
* Islands may form at resonances
* Islands introduce a small radial scale = island width
* Island width ~ collisional boundary layer width when

$$\partial \tilde{f}/\partial r \sim \partial \bar{f}/\partial r$$

* QL = RP = SBP treatment fails
* What happens when collisions still matter?

# Form of resonance

* At large aspect ratio $\bar{\omega}_\alpha$ reverses at $2E(\kappa_0) = K(\kappa_0)$

$$\kappa^2 = [1 - (1 - \epsilon)\lambda]/2\epsilon\lambda$$

* Trapped boundary depends on inverse aspect ratio $\epsilon$

$$1/(1 + \epsilon) < \lambda < 1/(1 - \epsilon)$$

* Expanding by writing $\bar{\omega}_\alpha = -2(\kappa^2 - \kappa_0^2)\bar{\omega}'_\alpha$ then

$$\bar{\omega}_\alpha = \{\lambda - [1 - (2\kappa_0^2 - 1)\epsilon]\}\bar{\omega}'_\alpha/\epsilon$$

* Resonance depends on $\lambda$ and $\epsilon$ with $\kappa_0^2 = 0.83$ and

$$\bar{\omega}'_\alpha \sim \bar{\omega}_\alpha \sim qv^2/\Omega_0 R_0^2 \epsilon$$

* Trapped fraction $\sim\sqrt{\epsilon}$ , $\bar{V} \sim \bar{\omega}_\alpha R_0 \delta$ & $\epsilon \gg \delta$ = non-QS

# RP = QL limit

* Full nonlinear eq.

$$[\lambda - (1 + \epsilon - 2\kappa_0^2 \epsilon)] \frac{\bar{\omega}_\alpha'}{\epsilon} \frac{\partial \tilde{f}}{\partial \varphi} - \frac{\bar{V}}{R_0} sin\varphi \frac{\partial(\bar{f} + \tilde{f})}{\partial \epsilon} = \bar{\nu} \epsilon \frac{\partial^2 \tilde{f}}{\partial \lambda^2}$$

* RP or QL limit just an inhomogeneous Airy eq.

$$[\lambda - \lambda_0(\epsilon)] \frac{\bar{\omega}_\alpha'}{\epsilon} \frac{\partial \tilde{f}}{\partial \varphi} - \frac{\bar{V}}{R_0} sin\varphi \frac{\partial \bar{f}}{\partial \epsilon} = \bar{\nu} \epsilon \frac{\partial^2 \tilde{f}}{\partial \lambda^2}$$

* Drift ~ collisions ⟹ RP boundary width $\Delta\lambda \sim (\epsilon^2 \bar{\nu} / \bar{\omega}_\alpha)^{1/3}$

$$\bar{\nu}_{eff} \sim \bar{\nu}\epsilon / (\Delta\lambda)^2 \sim \bar{\nu}\epsilon (\bar{\omega}_\alpha / \epsilon^2 \bar{\nu})^{2/3}$$

* RP diffusivity independent of collisions: $v_0$ = birth speed

$$D_{rp} \sim (\Delta\lambda / \epsilon^{1/2}) (\bar{V} / \bar{\nu}_{eff})^2 \bar{\nu}_{eff} \sim \epsilon^{1/2} \bar{V}^2 / \bar{\omega}_\alpha \sim q v_0^2 \delta^2 / \Omega_0 \epsilon^{1/2}$$

# RP = QL limit failure

* Island width: nonlinear term ~ drift, $\bar{V}/R_0 \Delta\epsilon \sim \bar{\omega}'_\alpha \Delta\epsilon/\epsilon$

$$\Delta\epsilon \sim (\bar{V}\epsilon/\bar{\omega}_\alpha R_0)^{1/2} \sim (\epsilon\delta)^{1/2} \ll \epsilon^{1/2}$$

* QL = RP neglects islands: $\Delta\lambda \gg \Delta\epsilon$
* Collisional boundary layer larger than island width if

$$(\epsilon^2\bar{\nu}/\bar{\omega}_\alpha)^{1/3} \gg (\epsilon\delta)^{1/2}$$

* QL assumes $\partial\tilde{f}/\partial\epsilon \sim \tilde{f}/\Delta\epsilon \ll \partial\bar{f}/\partial\epsilon \sim \bar{f}/\epsilon$ or

$$\tilde{f}/\bar{f} \ll \Delta\epsilon/\epsilon \sim (\delta/\epsilon)^{1/2} \ll \Delta\lambda/\epsilon \sim (\bar{\nu}/\bar{\omega}_\alpha\epsilon)^{1/3}$$

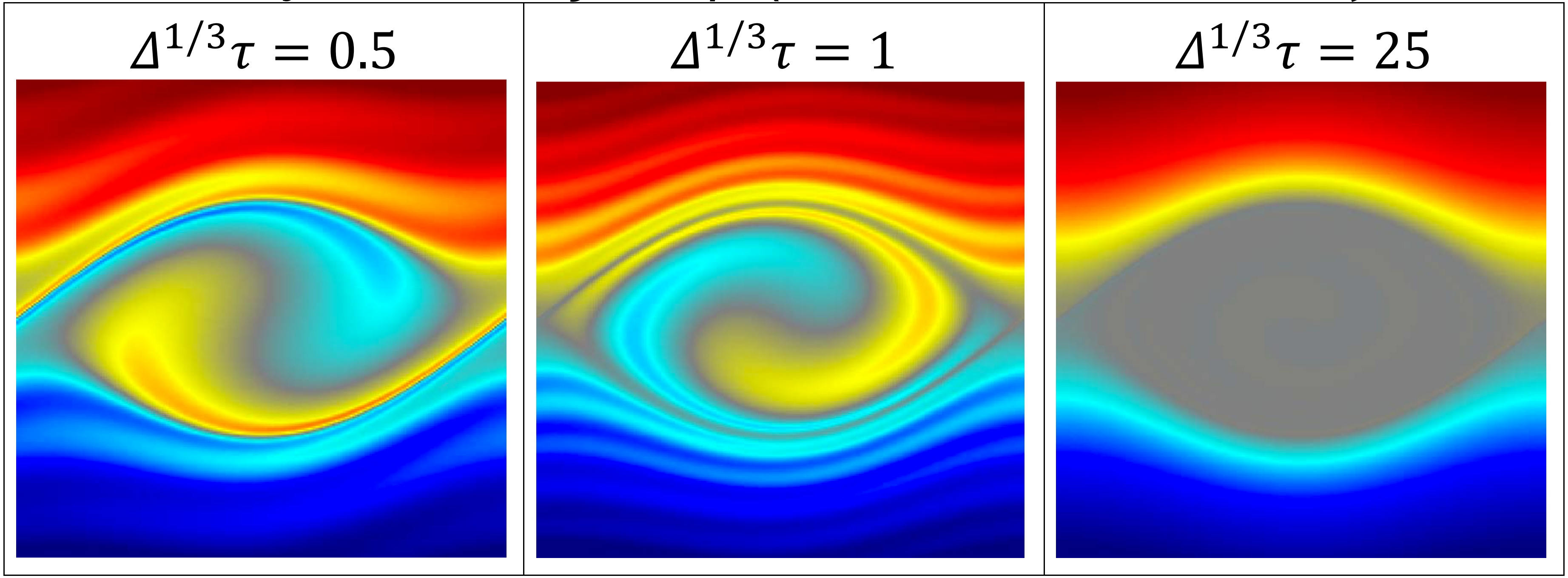
* $\tilde{f}/\bar{f}$ small is not enough!
* What happens if $\Delta\lambda \sim \Delta\epsilon$ or $\Delta\lambda \ll \Delta\epsilon$?

# Nonlinear effects & collisional phase mixing

* Hamilton,Tolman,Arzamasskiy, Duarte (AJ 2023) solve

$$\partial g/\partial\tau + j\partial g/\partial\varphi - sin\varphi\, \partial g/\partial j = \Delta\partial^2 g/\partial j^2$$

to find steady state for $j$ vs $\varphi$ (shown for $\Delta = 0.001$)



# Collisionless vs collisional contrasted

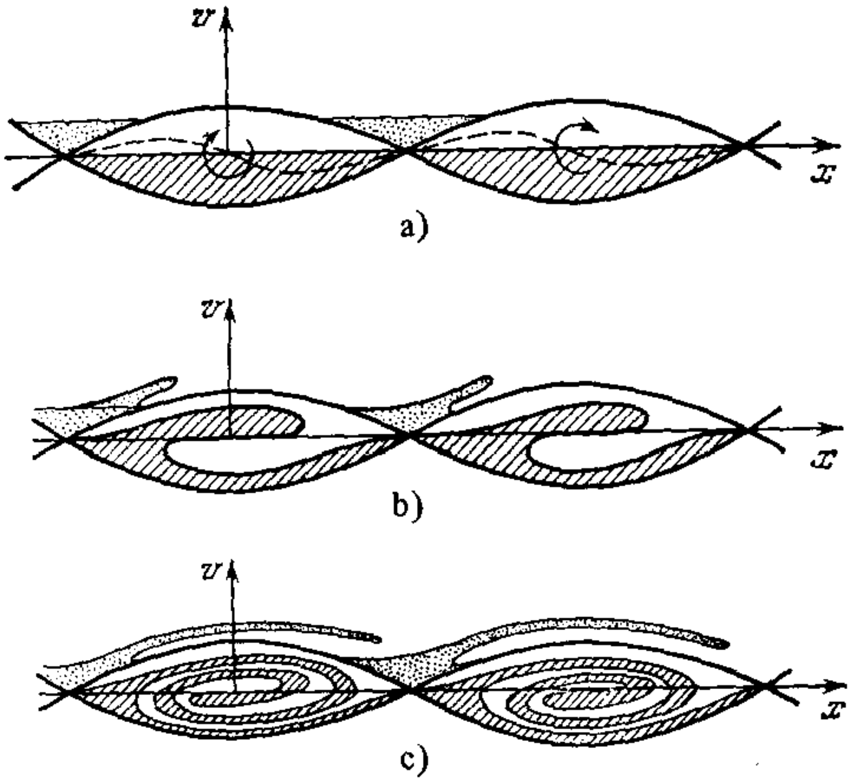
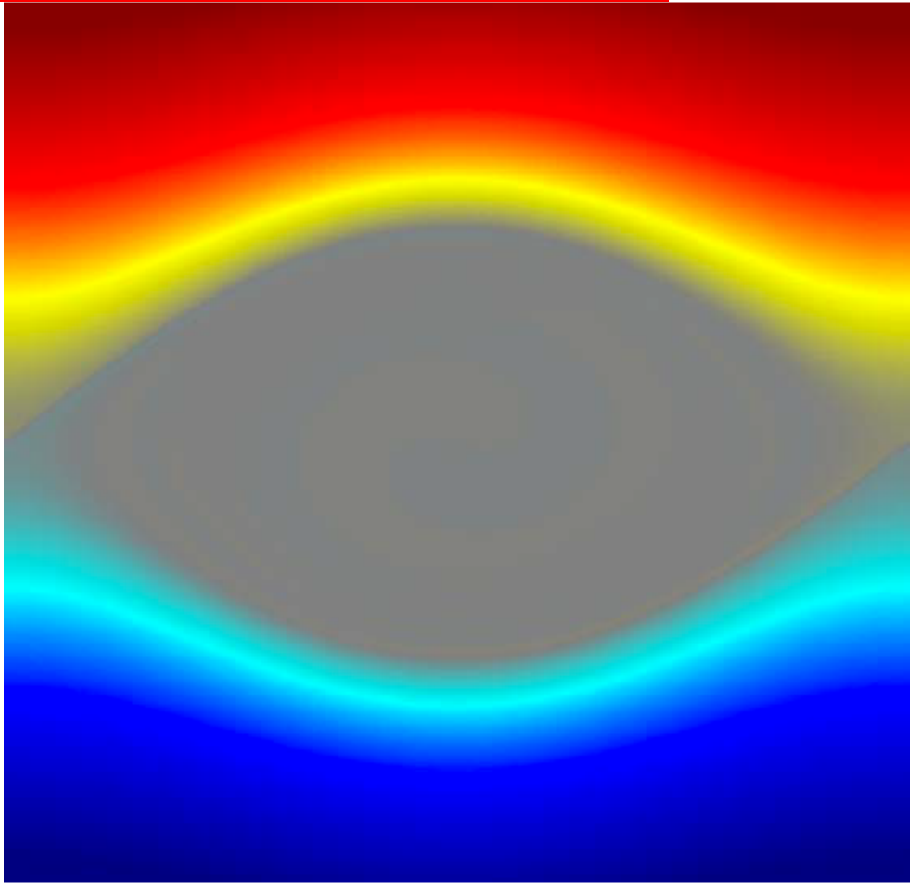


FIG. 3. Motion of resonant particles in a wave.



**Kadomstev 1968 Sov. Phys. Usp. vs Hamilton *et al.* 2023 AJ**

# Nonlinear effects for $\Delta \ll 1$

* Can't solve $\Delta \sim 1$, but can solve $\mathit{\Delta} \ll 1$
* Normalizing the trapped nonlinear drift kinetic equation

$$(x-\Lambda)\frac{\partial \tilde{f}}{\partial \varphi} - sin\varphi\left(\bar{f}' + \frac{\partial \tilde{f}}{\partial x}\right) = \mathit{\Delta}\frac{\partial^2 \tilde{f}}{\partial \mathit{\Lambda}^2}$$

with $x \propto \epsilon$ , $\mathit{\Lambda} \propto (1-\lambda)$ and $\bar{f}' = \partial \bar{f}/\partial x = constant$

* To obtain form nearly same eq. as Hamilton *et al.* let

$$\tilde{f} = g - (x-\Lambda)\bar{f}'$$

$$(x-\Lambda)\frac{\partial g}{\partial \varphi} - sin\varphi\,\frac{\partial g}{\partial x} = \Delta\frac{\partial^2 g}{\partial \Lambda^2}$$

# Reduced Hamiltonian

* Making believe I'm a real physicist, introduce

$$h = (x - \Lambda)^2/2 - cos\varphi$$

* Also, let $\sigma = \pm 1$

$$j(x, \Lambda) = x - \Lambda = \sigma\sqrt{2(h + cos\varphi)} = j(h, \varphi)$$

* Hamiltonian properties give Hamilton *et al.* form

$$j\frac{\partial g}{\partial \varphi} - sin\varphi\,\frac{\partial g}{\partial j} = \Delta\frac{\partial^2 g}{\partial j^2}$$

* Unlike Hamilton *et al*., only interested in steady state
* Collisional boundary layer about separatrix at h = 1

# Change Variables

* Changing variables from $x, \Lambda$ to $h, \varphi$

$$\left.\frac{\partial g}{\partial \varphi}\right|_h = \Delta \left.\frac{\partial}{\partial h}\right|_\varphi \left(j \left.\frac{\partial g}{\partial h}\right|_\varphi\right)$$

* For $\Delta \ll 1$, lowest order motion is collisionless with

$$g = g_1(h) + g_2(h, \varphi, \Lambda) + \ldots$$

* No need to solve next order, but needed for solubility

$$\left.\frac{\partial g_2}{\partial \varphi}\right|_h = \Delta \left.\frac{\partial}{\partial h}\right|_\varphi \left(j \left.\frac{\partial g_1}{\partial h}\right|_\varphi\right)$$

# Solubility

* Terminology: bound or librating for island motion and unbound or circulating outside island (alphas trapped)
* In Hamilton *et al.* no x dependence in $j$
* Here, every flux surface has island centered at $\Lambda = x$
* Islands are fully kinetic, need $\Lambda$ & $x$
* Integrate over a full bound period or full circulation

$$\left.\frac{\partial}{\partial \mathrm{h}}\right|_{\varphi} [(\oint_h d\varphi\, j) \left.\frac{\partial g_1}{\partial \mathrm{h}}\right|_{\varphi})] = 0$$

* Bound $g_1 = 0$, unbound $g_1 = \pi\sigma\bar{f}' \int_k^1 \frac{dt}{t^2 E(t)}$ & $k = \sqrt{\frac{2}{h+1}}$

# Full Solution

* Using $\sigma = \pm 1$ for circulating, and $\sigma = 0$ for bound

$$\tilde{f} = g_1 - j\bar{f}' = \bar{f}'[\sigma\pi \int_k^1 \frac{dt}{t^2 E(t)} - j]$$

* Using $\tilde{f}$ can form the alpha energy flux

$$Q = \frac{M_\alpha}{2} \langle \int d^3v \, f v^2 \vec{v}_d \cdot \nabla r \rangle$$

* Find $\delta^{1/2}/\tau_s$ dependence for energy diffusivity

$$D_\nu \sim \delta^{1/2} (\epsilon \delta R_0^2)(v_\lambda^3 / v_c v_0^2 \tau_s \delta) \sim \delta^{1/2} \epsilon R_0^2 v_\lambda^3 / v_0^2 v_c \tau_s$$

with $v_\lambda \sim v_c$ critical speed & $\tau_s$ slowing down time

# Estimating kinetic island energy diffusivity

* Step size is the island width: $R_0(\epsilon\delta)^{1/2}$

* Effective fraction being transport: $(\epsilon\delta)^{1/2}/\epsilon^{1/2} = \delta^{1/2}$

* Effective collision frequency: $\bar{\nu}\epsilon/[(\epsilon\delta)^{1/2}]^2 = \bar{\nu}/\delta$

* $\bar{\nu} = v_\lambda^3/v_0^3\tau_s$ modified by $v_0/v_c$ due to speed $\int dv$

$$D_v \sim \delta^{1/2}(\epsilon\delta R_0^2)(v_\lambda^3/v_c v_0^2 \tau_s \delta)$$

# QL vs kinetic island

* Kinetic island ~ RP = QL at $\Delta \sim 1$

$$\frac{D_\nu}{D_{rp}} \sim \left(\frac{\epsilon}{\delta}\right)^{3/2} \frac{\Omega_0 R_o^2 v_\lambda^3}{q v_0^4 v_c \tau_s} \sim \Delta \ll 1$$

which is roughly

$$(\epsilon\delta)^{1/2} \gg (\epsilon^2 \bar{\nu}/\bar{\omega}_\alpha)^{1/3}$$

* QL = RP of restricted validity: Expect similar procedure modifies transport of passing alphas, resonant bulk ions, TAE & NTM modes in stellarators and tokamaks; RF heating & current drive for tokamaks & stellarators