Lagrangian regularity of the electron magnetohydrodynamic (e-MHD) flow on a bounded domain

Nicolas Besse

Observatoire de la Côte d'Azur
Nicolas.Besse@oca.eu

Contents

1. **The incompressible e-MHD equations in Eulerian form**
2. Main result: Lagrangian regularity of e-MHD
3. Elements of a constructive proof
4. Numerical application: 3D-axisymmetric wall-bounded and potentially singular incompressible Euler flows
5. Epilogue

The incompressible e-MHD equations in Eulerian form

- Ω is a bounded domain of $\mathbb{R}^3$. $x \in \Omega$.
- $u = u(t, x)$ is the velocity field.
- $b = b(t, x)$ is the magnetic field.
- $e = e(t, x)$ is the electric field.
- The incompressible e-MHD equations:

$$
\begin{aligned}
\partial_t u + u \cdot \nabla u + e + u \times b = 0, &\quad x \in \Omega, \quad t \in]0, T[,\\
-\nabla \times b = u, &\quad x \in \Omega, \quad t \in]0, T[,\\
\partial_t b + \nabla \times e = 0, &\quad x \in \Omega, \quad t \in]0, T[,\\
\nabla \cdot b = 0, \quad \rho = 1, &\quad x \in \Omega, \quad t \in]0, T[,\\
u \cdot \nu = 0, \quad b \cdot \nu = 0, \quad e \times \nu = 0, &\quad x \in \partial\Omega, \; t \in]0, T[,\\
(u, b, e)_{|_{t=0}} = (u_0, b_0, e_0), &\quad x \in \Omega.
\end{aligned}
$$

Reformulation of the incompressible e-MHD equations in Eulerian form

- $a = a(t,x)$ is the vector potential such that $b = \nabla \times a$ and $\nabla \cdot a = 0$.
- $p = p(t,x) := u - a$ is the canonical momentum.
- $\omega = \omega(t,x) := \nabla \times u$ is the standard fluid vorticity.
- $\omega_* := \nabla \times p = \omega - b$ is the generalized vorticity.
- The incompressible e-MHD equations:

$$\begin{aligned} \partial_t \omega_* = \nabla \times (u \times \omega_*) \equiv -\pounds_u \omega_* \quad & x \in \Omega, \quad t \in]0,T[, \\ u = -\nabla \times b \quad & x \in \Omega, \quad t \in]0,T[, \\ e = -(\partial_t + u \cdot \nabla)u - u \times b \quad & x \in \Omega, \quad t \in]0,T[, \\ (u,b,e)_{|_{t=0}} = (u_0, b_0, e_0), \quad & x \in \Omega. \end{aligned}$$

$$(\star) \begin{cases} -(1-\Delta)b = \omega_* \text{ on } \Omega, \\ \nabla \cdot b = 0 \text{ on } \overline{\Omega}, \\ \nu \cdot \nabla \times b = 0, \text{ and } b \cdot \nu = 0 \text{ on } \partial\Omega, \end{cases}$$

- Elliptic BVP $(\star)$ in b can be replaced by $b = \nabla \times a$ and the Elliptic BVP in a :

$$\begin{cases} \Delta a = u \text{ on } \Omega, \\ \nabla \cdot a = 0 \text{ on } \overline{\Omega}, \\ a \times \nu = 0 \text{ on } \partial\Omega. \end{cases}$$

Theorem (Well-posedness of e-MHD equations on a bounded domain)

Let Ω be a bounded and simply-connected domain of $\mathbb{R}^3$ with $\mathscr{C}^\infty$ boundary $\partial\Omega$.
Let $s > 3/2 + 1$. Let $u_0 \in H^s(\Omega)$ (initial fluid vorticity $\omega_0 = \nabla \times u_0 \in H^{s-1}(\Omega)$) such that $\nabla \cdot u_0 = 0$ on Ω and $u_0 \cdot \nu = 0$ on $\partial\Omega$.
Let the initial fields (a_0, b_0) be the unique solutions of the following boundary value problems,

$$\begin{cases} \Delta b_0 = \nabla \times u_0 \ \ \text{on } \Omega, \\ \nabla \cdot b_0 = 0 \ \ \text{on } \overline{\Omega}, \\ \nu \cdot \nabla \times b_0 = 0, \ \ \text{and } \ b_0 \cdot \nu = 0 \ \ \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta a_0 = u_0 \ \ \text{on } \Omega, \\ \nabla \cdot a_0 = 0 \ \ \text{on } \overline{\Omega}, \\ a_0 \times \nu = 0 \ \ \text{on } \partial\Omega. \end{cases}$$

Consequently $a_0 \in H^{s+2}(\Omega)$ (initial magnetic field $b_0 = \nabla \times a_0 \in H^{s+1}(\Omega)$) and the initial generalized vorticity $\omega_{*0} = \omega_0 - b_0 \in H^{s-1}(\Omega)$. Let $e_0 \in H^{s-1}(\Omega)$ with $e_0 \times \nu = 0$.

Then there exist a time $T > 0$ and a unique solution to the e-MHD equations, such that

$$\begin{aligned} & u \in \mathscr{C}(0,T;H^s(\Omega)) \cap W^{1,\infty}(0,T;H^{s-1}(\Omega)) \cap \mathscr{C}^1([0,T) \times \Omega), \\ & a \in \mathscr{C}(0,T;H^{s+2}(\Omega)) \cap W^{1,\infty}(0,T;H^{s+1}(\Omega)) \cap \mathscr{C}^1(0,T;\mathscr{C}^{2,\gamma}(\Omega)), \ \ 0 < \gamma < 1, \\ & b \in \mathscr{C}(0,T;H^{s+1}(\Omega)) \cap W^{1,\infty}(0,T;H^{s}(\Omega)) \cap \mathscr{C}^1(0,T;\mathscr{C}^{1,\gamma}(\Omega)), \ \ 0 < \gamma < 1, \\ & e \in L^\infty(0,T;H^{s-1}(\Omega)), \\ & \omega_* \in \mathscr{C}(0,T;H^{s-1}(\Omega)). \end{aligned}$$

Lagrangian fields, and the generalized Cauchy Invariants equation (CIE)

- Let $\alpha \mapsto X_t(\alpha) = X(t,\alpha)$ be the Lagrangian flow-map tracking at time t the position of a particle starting at $\alpha \in \overline{\Omega}$.
- The Lagrangian flow-map X satisfies the following ordinary differential equation,

$$\partial_t X(t,\alpha) = u(t, X(t,\alpha)), \quad X(0,\alpha) = \alpha \in \overline{\Omega}.$$

- $A = A(t,\alpha) := a(t, X(t,\alpha))$ is the Lagrangian magnetic vector potential.
- $B = B(t,\alpha) := b(t, X(t,\alpha))$ is the Lagrangian magnetic field.
- $E = E(t,\alpha) := e(t, X(t,\alpha))$ is the Lagrangian electric field.
- $U = U(t,\alpha) := u(t, X(t,\alpha))$ is the Lagrangian velocity field.
- $P = P(t,\alpha) := p(t, X(t,\alpha))$ is the Lagrangian canonical momentum.
- A Lagrangian formulation of Lie-advection of the generalized vorticity ω_* , i.e.

$$\partial_t \omega_* + \pounds_u \omega_* = 0,$$

is given by

the Cauchy or vorticity-transport equation: $\omega_*(t, X(t,\alpha)) = \nabla_\alpha^T X(t,\alpha)\, \omega_{*0}(\alpha)$,

or by

the Cauchy Invariants Equation (CIE): $\displaystyle\sum_{i=1}^{3} \nabla_\alpha P^i(t,\alpha) \times \nabla_\alpha X^i(t,\alpha) = \omega_{*0}(\alpha).$

Digression on the Cauchy Invariants Equation (CIE)

- CIE is a generalization of the Cauchy invariants equation in $\mathbb{R}^3$ obtained by Cauchy (1825):

$$\sum_i \nabla \dot{X}^i \times \nabla X^i = \omega_0.$$

In Cauchy's paper, where cross product and vector do not exist, CIE is just a step of calculus to prove the Cauchy formula.

- CIE can be proved by a variationnal principle and the rellabeling symmetry (cf. N.Besse-U.Frisch JFM17, CMP17).

- CIE can be generalized to any Lie-advected exact k-form on manifolds of any dimension (cf. NB-UF JFM17, NB CMP20).

  Here, the generalized vorticity 2-form $\omega_*^{(2)}(t,x) = dp^{(1)}(t,x)$ satisfies the (CIE):

$$dP_i^{(1)}(t,\alpha) \wedge dX^i(t,\alpha) = \omega_0^{(2)}(\alpha)$$

- CIE can be generalized to other (magneto-)hydrodynamics models which may be incompressible or compressible, in fact to any system which is described by Lie transport (cf. NB-UF JFM17, NB JMAA22).

- CIE can be generalized to dissipative systems where Lie-advection is supplemented by the sum of squares of Lie derivative. CIE still holds but in average or in a statistical sense (reminiscent to Feynman–Kac formula)
  (cf. NB NA23)

- The Hodge dual of CIE is the Cauchy formula.

Main result

Theorem (Lagrangian regularity of the e-MHD flow on a bounded domain [NB JMAA22])

- *Assume that the hypotheses of well-posedness Theorem hold.*
- *Assume that the boundary* $\partial\Omega$ *is analytic (in space).*
- *Then, there exists a time* $T = C(\Omega, \|u_0\|_{H^s(\Omega)}, \|a_0\|_{H^s(\Omega)})$ *such that the Lagrangian fields* (X, A, B, E) *satisfy*

$$X,\ A \in \mathcal{A}(]0, T[; H^s(\Omega)), \quad \text{and} \quad B, E \in \mathcal{A}(]0, T[; H^{s-1}(\Omega)).$$

where the functional space,

$$\mathcal{A}(]0, T[; H^s(\Omega))$$

is the space of functions which are analytic in time with values in $H^s(\Omega)$ *(Sobolev spaces) in the physical space.*

Corollary

- $U, P \in \mathcal{A}(]0, T[; H^s(\Omega))$, *since* $U = \dot{X}$ *and* $P = U - A$.
- *This result can be extended to a large class of ultradifferentiable regularity, which contains among others the Gevrey regularity class.*

Lagrangian formulation of e-MHD on a bounded domain (1/2)

A Lagrangian formulation in the variables $(t,\alpha) \in [0,T[\times\overline{\Omega}$ of the incompressible e-MHD equations on the bounded domain Ω is given by

$$\begin{aligned}
\nabla \dot{X}^k \times \nabla X^k + b_0 &= \nabla A^k \times \nabla X^k + \omega_0, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
\det\left(\frac{\partial X}{\partial \alpha}\right) &= 1, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
\nabla \cdot (G\nabla A) &= \dot{X}, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
\nabla \cdot (\mathcal{A}\,A) = \mathcal{A} : \nabla A &= 0, &\quad \alpha \in \overline{\Omega}, &\quad t \in [0,T[,\\
B^i = (\nabla X^k \times \nabla X^i) \cdot \nabla A^k&, \ \ \forall i, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
E &= -\ddot{X} - \dot{X} \times B, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
U = \dot{X}, \quad P = U - A &= \dot{X} - A, &\quad \alpha \in \Omega, &\quad t \in [0,T[,\\
\dot{X}\cdot\nu(X) = 0, \quad A \times \nu(X) &= 0, &\quad \alpha \in \partial\Omega, &\quad t \in [0,T[.
\end{aligned}$$

with

$$G := G(X) = \mathcal{A}\mathcal{A}^T,$$

where the inverse Jacobian matrix $\mathcal{A}$ is given by

$$(\mathcal{A})_{ij} := (\mathcal{A})_{ij}(X) = \left(\frac{\partial X}{\partial \alpha}\right)^{-1}_{ij} = \left(\frac{\partial \alpha}{\partial X}\right)_{ij} = \frac{1}{2}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\frac{\partial X^{j_1}}{\partial \alpha^{i_1}}\frac{\partial X^{j_2}}{\partial \alpha^{i_2}}.$$

Lagrangian formulation of e-MHD on a bounded domain (2/3)

We introduce the following decomposition for the Lagrangian flow-map X and the Lagrangian magnetic vector potential A :

$$X(t,\alpha) = \alpha + \xi(t,\alpha), \quad \text{and} \quad A(t,\alpha) = a_0(\alpha) + \Psi(t,\alpha),$$

with $\xi(0,\alpha) = 0$, and $\Psi(0,\alpha) = 0$.

In terms of the new unknowns (ξ, Ψ), the Lagrangian formulation of the e-MHD equations on a bounded domain becomes

$$\begin{aligned}
\nabla \times \dot{\xi} &= \omega_0 + \nabla \times \Psi + \nabla(a_0^k + \Psi^k - \dot{\xi}^k) \times \nabla \xi^k, && \alpha \in \Omega, \quad t \in [0,T[,\\
\nabla \cdot \xi + \frac{1}{2}(\partial_i \xi^i \partial_j \xi^j - \partial_i \xi^j \partial_j \xi^i) + \frac{1}{6}\varepsilon_{i_1 i_2 i_3}\varepsilon_{j_1 j_2 j_3}\partial_{i_1}\xi^{j_1}\partial_{i_2}\xi^{j_2}\partial_{i_3}\xi^{j_3} &= 0, && \alpha \in \Omega, \quad t \in [0,T[,\\
\Delta(a_0 + \Psi) + \nabla \cdot \left(\mathfrak{g}\, \nabla(a_0 + \nabla \Psi)\right) &= \dot{\xi}, && \alpha \in \Omega, \quad t \in [0,T[,\\
(1 + \nabla \cdot \xi)\nabla \cdot \Psi - (\partial_j a_0^i + \partial_j \Psi^i)\Big(\partial_j \xi^i - \frac{1}{2}\varepsilon_{i i_1 i_2}\varepsilon_{j j_1 j_2}\partial_{i_1}\xi^{j_1}\partial_{i_2}\xi^{j_2}\Big) &= 0, && \alpha \in \overline{\Omega}, \quad t \in [0,T[,\\
B^i = (\nabla \times a_0)^i + (\nabla \times \Psi)^i + (\nabla(a_0^k + \Psi^k) \times \nabla \xi^k)^i & \\
+ (\nabla \xi^i \times \nabla(a_0^k + \Psi^k))^k + (\nabla \xi^k \times \nabla \xi^i) \cdot \nabla(a_0^k + \Psi^k), & \quad \forall i, && \alpha \in \Omega, \quad t \in [0,T[,\\
E = -\ddot{\xi} - \dot{\xi} \times B&, && \alpha \in \Omega, \quad t \in [0,T[,\\
U = \dot{\xi}, \quad P = \dot{\xi} - a_0 - \Psi&, && \alpha \in \Omega, \quad t \in [0,T[,\\
\dot{\xi} \cdot \nu(\alpha + \xi) = 0, \quad a_0 \times \nu(\alpha + \xi) + \Psi \times \nu(\alpha + \xi) &= 0, && \alpha \in \partial\Omega, \quad t \in [0,T[,
\end{aligned}$$

where the matrix $\mathfrak{g}$ is given by

Lagrangian formulation of e-MHD on a bounded domain (3/3)

$$
\begin{aligned}
\mathfrak{g}_{ij} \quad = \quad & \delta_{ij}\nabla\cdot\xi \;+\; (1+\nabla\cdot\xi)\,(\delta_{ij}\nabla\cdot\xi - \partial_i\xi^j - \partial_j\xi^i) \;+\; \partial_k\xi^i\partial_k\xi^j \\
& + \frac{1}{2}(1+\nabla\cdot\xi)\,(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}\partial_{l_2}\xi^{k_2} \\
& - \frac{1}{2}\varepsilon_{kk_1k_2}(\varepsilon_{il_1l_2}\partial_k\xi^j + \varepsilon_{jl_1l_2}\partial_k\xi^i)\partial_{l_1}\xi^{k_1}\partial_{l_2}\xi^{k_2} \\
& + \frac{1}{4}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{k_1}\partial_{i_2}\xi^{k_2}(\partial_{j_1}\xi^{k_1}\partial_{j_2}\xi^{k_2} - \partial_{j_1}\xi^{k_2}\partial_{j_2}\xi^{k_1}).
\end{aligned}
$$

Construction of the e-MHD flow as a formal time series: General Picture

- We introduce the following formal time-Taylor expansions of ξ and Ψ,

$$\xi(t,\alpha) = \sum_{\sigma>0} \xi_\sigma(\alpha) t^\sigma, \quad \text{and} \quad \Psi(t,\alpha) = \sum_{\sigma>0} \Psi_\sigma(\alpha) t^\sigma.$$

- Plugging these time-Taylor series in the previous Lagrangian formulation of the e-MHD we obtain a constructive scheme to determine recursively all the time-Taylor coefficients $\{\xi_\sigma\}_{\sigma>0}$ and $\{\Psi_\sigma\}_{\sigma>0}$.

- Schematically we obtain the following recursive procedure, for $\sigma > 1$,

$$\begin{aligned} \xi_\sigma &= \mathcal{F}_\xi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma}\big), \\ \Psi_{\sigma-1} &= \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'\le\sigma}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big), \end{aligned}$$

  where the functionals $\mathcal{F}_\xi[a_0](\cdot)$ and $\mathcal{F}_\Psi[a_0](\cdot)$, which depends on a_0, can be seen as some integro-differential or pseudo-differential operators of order zero.

- This recursive scheme is initialized with $\xi_1 = u_0$ and $a_0 = \mathcal{L}^{-1}\xi_1$, where $\mathcal{L}$ refers to the linear differential operator associated with a boundary value problem of elliptic type.

Construction of the e-MHD flow as a formal time series: Recursive Scheme (1/5)

1) **Initialization of the recursive algorithm**.

The time-Taylor coefficient ξ_1 is given by

$$\xi_1 = u_0.$$

The initial magnetic vector potential a_0 is solution of the following non-homogeneous elliptic boundary value problem,

$$\begin{cases} \Delta a_0 = \xi_1, & \alpha \in \Omega, \\ \nabla \cdot a_0 = 0, & \alpha \in \partial\Omega, \\ a_0 \times \nu = 0, & \alpha \in \partial\Omega. \end{cases}$$

2) **Determination of the time-Taylor coefficients ξ_σ for $\sigma > 1$**.

The Helmholtz–Hodge decomposition of the time-Taylor coefficient ξ_σ reads

$$\xi_\sigma = \nabla\varphi_\sigma + \nabla \times \Phi_\sigma, \quad \nabla \cdot \Phi_\sigma = 0, \quad \alpha \in \Omega,$$

where the Helmholtz–Hodge potentials φ_σ and Φ_σ are respectively a scalar and a three-dimensional vector.

The scalar potential φ_σ satisfies the following non-homogeneous elliptic boundary value problem,

$$\begin{cases} \Delta\varphi_\sigma = \nabla \cdot \xi_\sigma, & \alpha \in \Omega, \\ \partial_\nu\varphi_\sigma = \xi_\sigma \cdot \nu, & \alpha \in \partial\Omega, \end{cases}$$

Construction of the e-MHD flow as a formal time series: Recursive Scheme (2/5)

where

$$\begin{aligned}\nabla\cdot\xi_\sigma &= -\frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}}\left(\partial_i\xi^i_{\sigma_1}\partial_j\xi^j_{\sigma_2}-\partial_i\xi^j_{\sigma_1}\partial_j\xi^i_{\sigma_2}\right)\\ &\quad -\frac{1}{6}\varepsilon_{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma\\ \sigma_1,\,\sigma_2,\,\sigma_3>0}}\partial_{i_1}\xi^{j_1}_{\sigma_1}\partial_{i_2}\xi^{j_2}_{\sigma_2}\partial_{i_3}\xi^{j_3}_{\sigma_3},\\ \xi_\sigma\cdot\nu &= -\sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}}\frac{\sigma_1}{\sigma}\,\xi_{\sigma_1}\cdot\nu_{\sigma_2},\end{aligned}$$

with

$$\nu_\sigma(\alpha):=\sum_{1\leq|\beta|\leq\sigma}\partial^\beta\nu(\alpha)\sum_{i=1}^{\sigma}\sum_{P_i(\sigma,\beta)}\prod_{j=1}^{i}\frac{(\xi^1_{\ell_j})^{k^1_j}}{k^1_j!}\cdots\frac{(\xi^3_{\ell_j})^{k^3_j}}{k^3_j!}.$$

Using Faà-di-Bruno formula, the set $P_i(\sigma,\beta)$ is defined by

$$P_i(\sigma,\beta):=\Big\{(\ell_1,\ldots,\ell_i),\ (k_1,\ldots,k_i);\ \ 0<\ell_1<\ldots<\ell_i;\\ |k_j|>0,\ j\in[1,i];\ \sum_{j=1}^{i}k_j=\beta,\ \sum_{j=1}^{i}|k_j|\ell_j=\sigma\Big\}.$$

Construction of the e-MHD flow as a formal time series: Recursive Scheme (3/5)

The vector potential Φ_σ satisfies the following non-homogeneous elliptic boundary value problem:

$$\begin{cases} \Delta\Phi_\sigma = -\nabla \times \xi_\sigma, & \alpha \in \Omega, \\ \nabla \cdot \Phi_\sigma = 0, & \alpha \in \partial\Omega, \\ \Phi_\sigma \times \nu = 0, & \alpha \in \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \nabla \times \xi_\sigma \quad &= \quad \frac{1}{\sigma}\nabla \times \Psi_{\sigma-1} + \frac{1}{\sigma}\nabla a_0^k \times \nabla \xi_{\sigma-1}^k \\ &+ \sum_{\substack{\sigma_1+\sigma_2+1=\sigma \\ \sigma_1,\, \sigma_2>0}} \frac{1}{\sigma}\nabla\Psi_{\sigma_1}^k \times \nabla\xi_{\sigma_2}^k - \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\, \sigma_2>0}} \frac{\sigma_1}{\sigma}\nabla\xi_{\sigma_1}^k \times \nabla\xi_{\sigma_2}^k. \end{aligned}$$

3) **Determination of the time-Taylor coefficients Ψ_σ for $\sigma > 0$.**

The time-Taylor coefficient Ψ_σ satisfies the following non-homogeneous elliptic boundary value problem:

$$\begin{cases} \Delta\Psi_\sigma = f_\sigma, & \alpha \in \Omega, \\ \nabla \cdot \Psi_\sigma = h_\sigma, & \alpha \in \partial\Omega, \\ \Psi_\sigma \times \nu = g_\sigma, & \alpha \in \partial\Omega, \end{cases}$$

Construction of the e-MHD flow as a formal time series: Recursive Scheme (4/5)

where

$$
\begin{aligned}
f_\sigma \quad &:= \quad (\sigma+1)\xi_{\sigma+1} - \nabla\cdot(\mathfrak{g}_\sigma \nabla a_0) - \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} \nabla\cdot(\mathfrak{g}_{\sigma_1}\nabla\Psi_{\sigma_2}),\\
h_\sigma \quad &:= \quad \partial_j a_0^i \partial_j \xi_\sigma^i + \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} \left\{\partial_j \xi_{\sigma_1}^i \partial_j \Psi_{\sigma_2}^i - \nabla\cdot\xi_{\sigma_1}\,\nabla\cdot\Psi_{\sigma_2} - \frac{1}{2}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi_{\sigma_1}^{j_1}\partial_{i_2}\xi_{\sigma_2}^{j_2}\partial_j a_0^i\right\}\\
&\qquad -\frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma\\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi_{\sigma_1}^{j_1}\partial_{i_2}\xi_{\sigma_2}^{j_2}\partial_j\Psi_{\sigma_3}^i,\\
g_\sigma \quad &:= \quad -a_0\times\nu_\sigma - \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} \Psi_{\sigma_1}\times\nu_{\sigma_2},
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{g}_\sigma)_{ij} \quad := \quad & 2\delta_{ij}\nabla\cdot\xi_\sigma - \partial_i\xi^j_\sigma - \partial_j\xi^i_\sigma \; + \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} \Big\{ \nabla\cdot\xi_{\sigma_1}\,(\delta_{ij}\nabla\cdot\xi_{\sigma_2} - \partial_i\xi^j_{\sigma_2} - \partial_j\xi^i_{\sigma_2}) \\
& + \partial_k\xi^i_{\sigma_1}\,\partial_k\xi^j_{\sigma_2} \; + \; \frac{1}{2}(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}_{\sigma_1}\,\partial_{l_2}\xi^{k_2}_{\sigma_2} \Big\} \\
& + \frac{1}{2} \sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma\\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \Big\{ \nabla\cdot\xi_{\sigma_3}\,(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}_{\sigma_1}\,\partial_{l_2}\xi^{k_2}_{\sigma_2} \\
& - \varepsilon_{kk_1k_2}(\varepsilon_{il_1l_2}\partial_k\xi^j_{\sigma_3} + \varepsilon_{jl_1l_2}\partial_k\xi^i_{\sigma_3})\partial_{l_1}\xi^{k_1}_{\sigma_1}\,\partial_{l_2}\xi^{k_2}_{\sigma_2} \Big\} \\
& + \frac{1}{4} \sum_{\substack{\sigma_1+\sigma_2+\sigma_3+\sigma_4=\sigma\\ \sigma_1,\,\sigma_2,\,\sigma_3,\,\sigma_4>0}} \varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{k_1}_{\sigma_1}\,\partial_{i_2}\xi^{k_2}_{\sigma_2}(\partial_{j_1}\xi^{k_1}_{\sigma_3}\,\partial_{j_2}\xi^{k_2}_{\sigma_4} - \partial_{j_1}\xi^{k_2}_{\sigma_3}\,\partial_{j_2}\xi^{k_1}_{\sigma_4}).
\end{aligned}
$$

A nonlinear recursive scheme

- Fixing $\sigma > 1$, we assume that we know all the following time-Taylor coefficients $\{\xi_{\sigma'}\}_{\sigma'<\sigma}$ and $\{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}$. From these known time-Taylor coefficients, the aim is to obtain the next unknown time-Taylor coefficients ξ_σ and $\Psi_{\sigma-1}$, called the current time-Taylor coefficients at the rank σ .
- Introducing the notation $\mathcal{X} := \xi_\sigma$ and $\mathcal{Y} := \Psi_{\sigma-1}$ for the current time-Taylor coefficients, the scheme rewrites as

$$\begin{aligned} \mathcal{X} &= \mathcal{F}_\xi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}, \mathcal{Y}\big), \\ \mathcal{Y} &= \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}, \mathcal{X}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big), \end{aligned}$$

  Then, the boundary value problems are coupled together and thus constitute a closed nonlinear system in terms of the current time-Taylor coefficients $(\xi_\sigma, \Psi_{\sigma-1})$ or $(\mathcal{X}, \mathcal{Y})$.
- This situation is very different from the incompressible Euler equations, which corresponds in the scheme to set $a_0 = 0$, and $\{\Psi_\sigma = 0\}_{\sigma>0}$, i.e. $\mathcal{F}_\Psi \equiv 0$ and

$$\mathcal{X} = \xi_\sigma = \mathcal{F}_\xi[0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}, \{\Psi_{\sigma'} = 0\}_{\sigma'<\sigma}\big) = \mathcal{CZ}\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}\big),$$

  where $\mathcal{CZ}(\cdot)$ stands for a Calderón–Zygmund integro-differential operator of order zero.

  In this case we clearly observe that for any $\sigma > 1$, the current time-Taylor coefficient $\mathcal{X} = \xi_\sigma$ is obtained only from coefficients $\{\xi_{\sigma'}\}_{\sigma'<\sigma}$ by solving linear boundary value problems in terms of the current time-Taylor coefficient $\mathcal{X} = \xi_\sigma$.

Convergence analysis (1/3)

- We aim at proving that $\xi,\ \Psi \in \mathcal{A}(]-T,T[;H^s(\Omega))$.
- It will be the case if and only if there exists a real positive number ρ such that

$$\forall \psi \in \{\xi, \Psi\}, \quad \text{the sets} \quad \left\{ \frac{\|\partial_t^\sigma \psi\|_{H^s}}{\rho^\sigma \sigma!}, \quad \sigma \in \mathbb{N}, \quad t \in]-T,T[\right\} \quad \text{are bounded.}$$

- This will be the case if the generatrice function $t \mapsto \zeta(t)$, defined by

$$\zeta(t) = \sum_{\sigma > 0} \left(\|\xi_\sigma\|_{H^s(\Omega)} + \|\Psi_\sigma\|_{H^s(\Omega)} \right) \varrho^{-\sigma} t^\sigma, \quad \text{is uniformly bounded on }]-T,T[.$$

- Using Hodge decomposition, and elliptic estimates for the Neumann & Dirichlet BVPs we obtain for $\sigma > 0$,

$$\|\Psi_\sigma\|_{H^s(\Omega)} \leq C_3 \left(\|f_\sigma\|_{H^{s-2}(\Omega)} + \|g_\sigma\|_{H^{s-1/2}(\partial\Omega)} + \|h_\sigma\|_{H^{s-3/2}(\partial\Omega)} \right),$$

and

$$\begin{aligned} \|\xi_\sigma\|_{H^s(\Omega)} &\leq \|\nabla \cdot \varphi_\sigma\|_{H^s(\Omega)} + \|\nabla \times \Phi_\sigma\|_{H^s(\Omega)} \leq \|\varphi_\sigma\|_{H^{s+1}(\Omega)} + \|\Phi_\sigma\|_{H^{s+1}(\Omega)} \\ &\leq C_{12} \left(\|\nabla \times \xi_\sigma\|_{H^{s-1}(\Omega)} + \|\nabla \cdot \xi_\sigma\|_{H^{s-1}(\Omega)} + \|\xi_\sigma \cdot \nu\|_{H^{s-1/2}(\partial\Omega)} \right). \end{aligned}$$

Therefore we obtain,

$$\text{(I)} \quad \zeta(t) \leq C_{123} \sum_{\sigma > 0} \Big(\|\nabla \cdot \xi_\sigma\|_{H^{s-1}(\Omega)} + \|\nabla \times \xi_\sigma\|_{H^{s-1}(\Omega)} + \|\xi_\sigma \cdot \nu\|_{H^{s-1/2}(\partial\Omega)}$$
$$+ \|f_\sigma\|_{H^{s-2}(\Omega)} + \|g_\sigma\|_{H^{s-1/2}(\partial\Omega)} + \|h_\sigma\|_{H^{s-3/2}(\partial\Omega)} \Big) \varrho^{-\sigma} t^\sigma.$$

Convergence analysis (2/3)

Since $\partial\Omega$ is analytic, there exists a positive constant C_ν, such that, for $0 \le s < \infty$, and $|\beta| \ge 0$, $\|\partial^\beta \nu\|_{H^s(\partial\Omega)} \le C_\nu R_\nu^{-|\beta|} |\beta|!$. Using this estimate we get

Proposition

Let $s > 3/2 + 1$. Then there exist positive constants $C_d, C_r, C_n, C_f, C_g, C_h$, such that

$$
\begin{aligned}
\sum_{\sigma>0} \|\nabla \cdot \xi_\sigma\|_{H^{s-1}(\Omega)} \varrho^{-\sigma} t^\sigma &\le C_d \zeta^2(t)(1+\zeta(t)),\\
\sum_{\sigma>0} \|\nabla \times \xi_\sigma\|_{H^{s-1}(\Omega)} \varrho^{-\sigma} t^\sigma &\le \|u_0\|_{H^s} \varrho^{-1} t + C_r \zeta(t)\big(t + (1+t)\zeta(t)\big),\\
\sum_{\sigma>0} \|\xi_\sigma \cdot \nu\|_{H^{s-1/2}(\partial\Omega)} \varrho^{-\sigma} t^\sigma &\le C_n \zeta(t)\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1},\\
\sum_{\sigma>0} \|f_\sigma\|_{H^{s-2}(\Omega)} \varrho^{-\sigma} t^\sigma &\le \dot\zeta(t) + C_f \zeta(t)\big(1 + \zeta(t) + \zeta^2(t) + \zeta^3(t) + \zeta^4(t)\big),\\
\sum_{\sigma>0} \|g_\sigma\|_{H^{s-1/2}(\partial\Omega)} \varrho^{-\sigma} t^\sigma &\le C_g (1+\zeta(t))\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1},\\
\sum_{\sigma>0} \|h_\sigma\|_{H^{s-3/2}(\partial\Omega)} \varrho^{-\sigma} t^\sigma &\le C_h \zeta(t)\big(1 + \zeta(t) + \zeta^2(t)\big),
\end{aligned}
$$

with $K_\nu^{-1} = C_a C_\partial / R_\nu$.

Convergence analysis (3/3)

Combining (I) and estimates of the previous proposition we obtain the differential inequality:

$$\zeta(t) \leq C_{123}\Big\{\|u_0\|_{H^s}\varrho^{-1}t \;+\; \dot{\zeta}(t) \;+\; C_{drfh}(1+t)\zeta(t)\big(1+\zeta(t)\big) \\ +\; C_{dfh}\zeta^3(t) + C_f\zeta^4(t) + C_f\zeta^5(t) \;+\; C_{ng}\big(1+\zeta(t)\big)\big(1-K_\nu^{-1}\zeta(t)\big)^{-1}\Big\}, \tag{1}$$

where $C_{drfh} = C_d + C_r + C_f + C_h$, $C_{dfh} = C_d + C_f + C_h$, and $C_{ng} = C_n + C_g$. Setting

$$\begin{aligned} \lambda(t) &:= \|u_0\|_{H^s}\varrho^{-1}t, \\ Q(t) &:= \lambda(t) - C_{123}^{-1}\zeta(t) + C_{drfh}(1+t)\zeta(t)\big(1+\zeta(t)\big) + C_{dfh}\zeta^3(t) + C_f\zeta^4(t) + C_f\zeta^5(t), \\ Z(t) &:= Q(t) + C_{ng}\big(1+\zeta(t)\big)\big(1-K_\nu^{-1}\zeta(t)\big)^{-1}, \end{aligned}$$

inequality (1) can be recast as $-\dot{\zeta}(t) \leq Z(t)$, which gives, after time integration, the following final inequality

$$\zeta(t) + \int_0^t Z(\tau)d\tau \geq 0. \tag{2}$$

A sufficient condition for inequality (2) to hold is to have both

$$Q(t) \geq 0, \quad \text{and} \quad \zeta(t) \leq K_\nu. \tag{3}$$

Following standard argument, we can then show that there exists a time $T > 0$, with

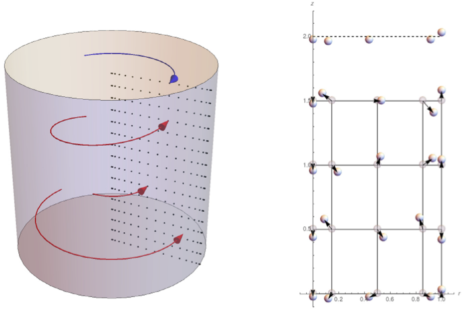
$$T = T(\|u_0\|_{H^s}, \|a_0\|_{H^s}, C_a, C_\partial, C_\nu, K_\nu, \varrho)$$

such that for all $t \in]0, T[$, the sufficient condition (3) is satisfied.

Numerical application: 3D-axisymmetric wall-bounded and potentially singular incompressible Euler flows: PhD T. Hertel.

- 3D incompressible Euler equations in a wall-bounded infinite cylinder (Euclidean 3D-axisymmetry geometry).
- Axisymmetry implies invariance by rotation in θ around the cylinder z-axis: $(r, \theta, z) \to (r, z)$.
- Boundary conditions: periodic in z and rigid impermeable boundary in r.
- Initial condition for potentially singular flow: Luo–Hou initial condition,

$$v_0^r = 0, \quad v_0^z = 0, \quad v_0^\theta(r, z) = 100r \exp(-30(1-r^2)^4) \sin(2\pi z/L), \quad L = 1/6.$$



(a) *Particles move onto the slice that contains the Chebyshev-Fourier grid.*

(b) *Particle displacement on the (r, z)-plane on a 5×4 grid. The upper row of particles have been added by periodicity.*

Fig The modeled slice of the cylindrical domain $D(1, 2)$ with the underlying Chebyshev-Fourier grid is shown. Only the radial and vertical movements of the particles are relevant. The top particle positions in (b) are obtained by periodicity of the flow from those that correspond to the first horizontal grid line.

The Cauchy–Lagrange algorithm

- It belongs to the class of forward semi-Lagrangian methods.
- We compute the time-Taylor coefficients of the Lagrangian map $a \mapsto X(t, a)$ at any chosen order by using the above recursion relations (with Helmholtz–Hodge decompositions) and the initial (or current) vorticity.

  This requires to solve some non-homogeneous elliptic Boundary Value Problems (BVPs) in space.
- To solve the non-homogeneous elliptic BVPs problems we use a Chebyshev–Fourier pseudo-spectral decomposition (polynomials) in space, which leads to spectral accuracy in space.
- We udpate of the Lagrangian vorticity (along the flow X) at the next time-step by using the previously computed Lagrangian map $a \mapsto X(t, a)$ and the Cauchy formula.

  This leads to the knowledge of the new vorticity (at the next time step) on a distribution of scattered spatial points.
- We resample the Lagrangian vorticity on the Eulerian (Chebyshev–Fourier) grid by using a high-order interpolation scheme (B-Splines) on a non-uniform grid (cascade interpolation scheme). The loop is closed.
- All details are in the paper Hertel-Besse-Frisch JCP22.

Conservation laws (helicity, Kelvin circulation, kinetic energy) and $\|\omega(t)\|_\infty$

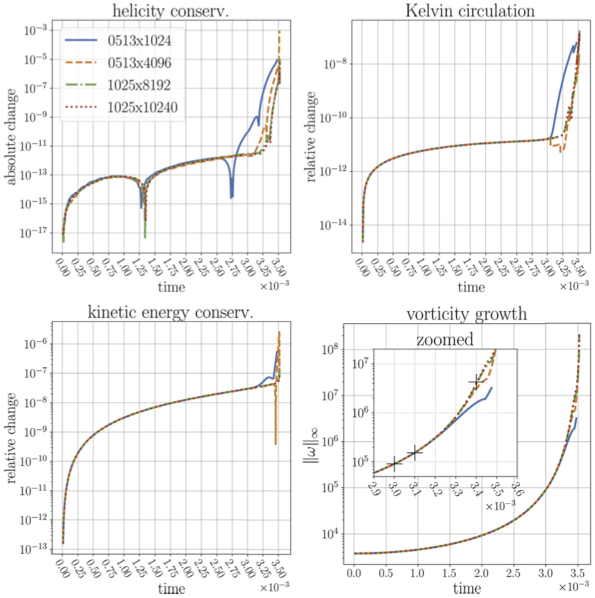
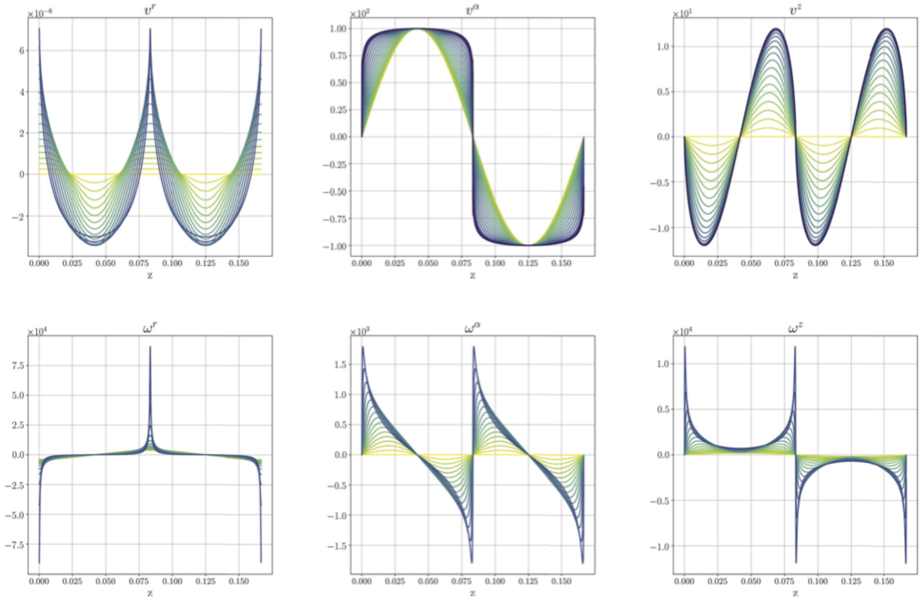
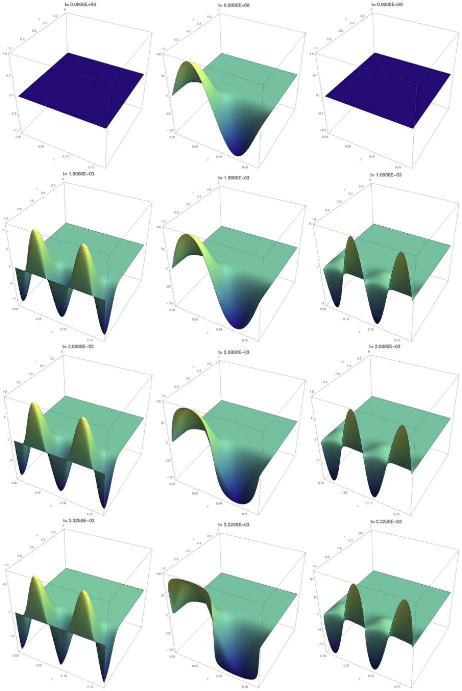


Fig. Higher resolution runs are shown. Here, the truncation order was set to $S=18$ to maintain the initial time-step 10^{-5} longer. The interpolation was done by the 5th order B-spline cascade; no dealiasing is used here. Sudden breaks in the conserved quantities arrive much later than in the previous plots, indicating a lasting stability with increased resolution.

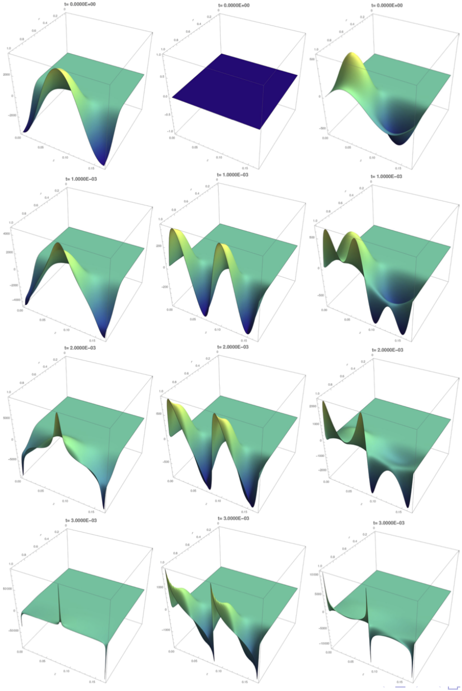
$(v^r, v^\theta, v^z)(t, r = 1)$ **and** $(\omega^r, \omega^\theta, \omega^z)(t, r = 1)$ **for** $t \in [0, 3.51 \times 10^{-3}]$



$(v^r, v^\theta, v^z)(t, r, z)$:



$(\omega^r, \omega^\theta, \omega^z)(t, r, z)$:



Why it works ? Answer: same kind of structure as the incompressible Euler equations

- As far as it concerns time regularity, there is a remarkable difference between the Eulerian and Lagrangian solutions to the incompressible Euler equations in the spatial non-too-smooth regime.

- For instance, in Eulerian coordinates, we know that there exists, locally in time, a unique regular solution $u \in L^\infty_t([0,T]; C^{1,\alpha}_x(\Omega))$ for initial data $u_0 \in C^{1,\alpha}_x(\Omega)$ with $0 < \alpha < 1$ ($C^{1,\alpha}_x$ are standard Hölder spaces). But the initial-data-to-solution map of Euler equations is not continuous in Hölder spaces as a map from $C^{1,\alpha}$ to $L^\infty_t C^{1,\alpha}_x$ (cf. Misiolek–Yoneda 2018). In other words the solution does not depend continuously on initial data in such Hölder spaces.

- By contrast in Lagrangian coordinates, where one is focusing on Lagrangian particles trajectories, an initial velocity field with limited smoothness (typically in Sobolev or Hölder classes) launches geodesic curves whose temporal smoothness (ultradifferentiable) widely exceeds the limited spatial smoothness.

- **Non-constructive proofs in $\mathbb{R}^n$:** J.-Y. Chemin 1992 (C^∞-regularity, paradifferential calculus), P. Serfati 1992, 1995 (analytic regularity, EDOs in Banach spaces), Gamblin 1994, Constantin-Vicol-Wu 2015 (SQG in $\mathbb{R}^2$), Hernandez 2017 (Serfati's method renewed + counterexamples).
- **Constructive proof in periodic domains of $\mathbb{R}^3$:** Zheligovsky-Frisch 2014 (Fourier series).
- **Non-constructive proofs on bounded domains of $\mathbb{R}^n$:** T. Kato 2000 (C^∞-regularity), Glass-Sueur-Takahashi 2012 (analytic regularity in $\Omega \subset \mathbb{R}^3$).
- **Constructive proof on bounded domains of $\mathbb{R}^3$:** Besse-Frisch 2017.
- **Constructive proof on manifolds of any dimension (w/o boundary):** Besse 2020

Other Hamiltonian systems, and in particular other MHD models ?

$\mathcal{O}_1$ The first obstruction, named $\mathcal{O}_1$, is the presence of several coupled fluids.

This concerns two-fluid models and a fortiori multi-fluid models, or models which arise as a derivation from a two-fluid or a multi-fluid theory.

Indeed, in a two-fluid transport model the two Lagrangian flow-maps (associated with the velocity field of each fluid) are coupled together through some equations for the electromagnetic fields, which in return determine the velocity fields. Because of this coupling, one Lagrangian flow-map experiences directly the roughness (with respect to Lagrangian variables) of the other Lagrangian flow-map.

Everything happens as if one Lagrangian flow-map comes across the other one and thus sees its relative roughness.

$\mathcal{O}_2$ The second obstruction, named $\mathcal{O}_2$, is the finite speed of propagation property which is not compatible with the Lagrangian analyticity property.

A system, in which waves propagate at a finite speed, can not sustain the Lagrangian analyticity property because, for this, some information must propagate at infinite speed.

For examples, in the incompressible Euler equations (resp. pressureless compressible one-fluid Euler–Poisson system) this is the pressure (resp. electric scalar potential) which propagates at infinite speed, while for the e-MHD this is the magnetic field or the magnetic vector potential.

By contrast it has been shown that the 2D barotropic (isentropic) compressible Euler equations, where the pressure propagates at a finite speed do not satisfy the Lagrangian analyticity property for its corresponding Lagrangian flow-map X.

THANK YOU FOR YOUR ATTENTION