Qualitative behaviours of the solution of the Vlasov equation \ Penrose stability criterian

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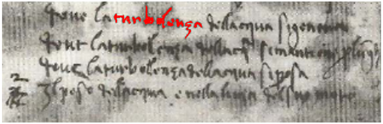
Two notions, strong and weak turbulence , in particular how they appear in the hierarchy of equations which range from particles dynamic to macroscopic problems via kinetic equations and above that to the issue of closure.
Fluid turbulence concerns the case of **family , weak limit ..** of unstable solutions such that in some region for the Reynolds stress tensor one has for weak limit (say with $\nu \to 0$).

$$\mathcal{R} = \overline{u_\nu \otimes u_\nu} \neq \overline{u_\nu} \otimes \overline{u_\nu}$$

Uriel Frisch noticed the following observation made by Leonardo da Vinci



Where the 'turbolenza' of the water is generated.

Where the 'turbolenza' of the water persists for a long distance.

2. Where the 'turbolenza' of the water settles.

What is important for the present discussion is the existence of turbulent region and in modern time, mostly for aeronautic engineering (Prandlt, Von Karman, Taylor, Kolmogorov , Taylor..) In such region space-time-locally the fluid can be considered as isolated (periodic) homogenous isotropic and stationary.
One introduce closures there are many (for instance $\epsilon\ k$ models) in agreement with phenomenological criteria as Kolmogorov K 41, for the energy spectrum,

$$E(|k|) \simeq \nu \Big\langle (\nabla \wedge u_\nu)^2 \Big\rangle^{\frac{2}{3}} |k|^{-\frac{5}{3}}$$

Weak turbulence as been developed for Plasma Physic starting from Macroscopic MhD equation with solution compared as waves R. E. Peierls. Zur kinetischen Theorie der Warmeleitung in Kristallen. Annalen Physik 3 (1929) and now the object of engineering Physic . For instance one deduces from the MHD equation for QUASI $1D$ close to equilibrium plasma the Zakharov equations

$$
\begin{aligned}
&i\partial_t E - c_1 \nabla \wedge (\nabla \wedge E) + c_2 \nabla(\nabla \cdot E) = dnE \\
&\epsilon^2 \partial_{tt} n - c\Delta(n - |E|^2) = 0
\end{aligned}
$$

Eventually for E close to a solenoidal vector field and ϵ small enough one obtains the basic non linear Schrodinger equation.

$$
i\partial_t E - \Delta E + |E|^2 E = 0
$$

with solutions statistically described by a non linear (similar to the Kolmogorov) process.

For kinetic equations finding out the role of parameters that contribute to the appearance of turbulence, irreversibility and anomalous decay of energy or entropy is more evident. These observations are used for adaptation of the turbulent stress tensor. An Hamiltonian (reversible) kinetic equation.

$$\partial_t f + v \cdot \nabla_x f + Q(f) = 0$$

can be change into en energy or entropy dissipating equation

$$\partial_t f + v \cdot \nabla_x f + \tilde{Q}(f) = 0\,.$$

with $\tilde{Q}$ modified by the insertion of "weak turbulent effects.
In plasma physic the Penrose dispersion function discriminates between stable and unstable phenomena. Giving at least two examples for parabolic limit.
The Quasilinear Equation and the Landau and Balescu Lenhard equations.

$$
\begin{aligned}
&\partial_t F + \ v\cdot\nabla_x F_\epsilon + E[F]\cdot\nabla_v F = 0\\
&F(t,x,v)\geq 0 \quad \frac{d}{dt}\int\int F(t,x,v)dvdx = 0 \quad f = F - \int\int F(t,x,v)dvdx\\
&\langle f\rangle = 0 \quad E[F] = E[f] = \nabla_x(\Delta)^{-1}\int f(t,x,v)dv\\
&\frac{1}{2}\frac{d}{dt}\int\int |v|^2 F(t,x,v)dxdv + \int |E(x,t)|^2 dx = 0
\end{aligned}
\tag{1}
$$

with "convenient boundary conditions".
However at the scale of the phenomena in particular for weak turbulence, it is natural to consider the problem in $\Omega\subset\mathbb{R}^d_x \ \times\mathbb{R}^d_v$ with Ω being $\mathbb{R}^d$ or the torus $(\mathbb{R}/Z)^d$ with k denoting the Fourier variable.

Parabolic-diffusive regimes will be given by

$$\partial_t F_\epsilon + \frac{1}{\epsilon^2} v \cdot \nabla_x F_\epsilon + \frac{1}{\epsilon} E[F_\epsilon] \cdot \nabla_v F_\epsilon = 0$$

$$\Rightarrow \partial_t \langle F_\epsilon \rangle + \nabla_v \langle \frac{1}{\epsilon} E[f_\epsilon] \cdot f_\epsilon \rangle = 0$$

$$S_t^\epsilon F_0 = F_0(x - \frac{t}{\epsilon^2} v, v)$$

$$F^\epsilon(t) = S_t^\epsilon F_0^\epsilon - \frac{1}{\epsilon} \int_0^t S_{t-\sigma}^\epsilon E^\epsilon(\sigma) \cdot \nabla_v F^\epsilon(\sigma) d\sigma \,,$$

$$\langle \frac{E^\epsilon F^\epsilon}{\epsilon} \rangle = \int_0^{\frac{t}{\epsilon^2}} d\sigma \langle E^\epsilon(t, x + \sigma v) \otimes E^\epsilon(t - \epsilon^2 \sigma, x) \nabla_v F^\epsilon(t - \sigma \epsilon^2, x, v) \rangle$$

Formally this gives:

$$\partial_t \langle F \rangle - \nabla_v (\lim_{\epsilon \to 0} \langle E^\epsilon(t, x + \sigma v) \otimes E^\epsilon(t, x) \rangle \nabla_v \langle F(t) \rangle = 0 \,.$$

For a family of random electric fields E^ϵ the issue is the proof of the formula

$$\partial_t \lim_{\epsilon \to 0} \mathbb{E}[\langle F^\epsilon(t - \sigma\epsilon^2, x, v) \rangle] =$$
$$\nabla_v \lim_{\epsilon \to 0} \mathbb{E}[\langle E^\epsilon(t, x + \sigma v) \otimes E^\epsilon(t, x) \rangle \nabla_v \langle F(t) \rangle] =$$
$$\nabla_v \big(\lim_{\epsilon \to 0} \mathbb{E}[\langle E^\epsilon(t, x + \sigma v) \otimes E^\epsilon(t, x) \rangle] \nabla_v \mathbb{E}[\lim_{\epsilon \to 0} \langle \nabla_v F^\epsilon(t - \sigma\epsilon^2, x, v) \rangle] \big) \,.$$

With convenient hypothesis on the vector field E using a second order Duhamel formula this gives a diffusion. However de correlation for E is equivalent to de correlation for f hence classical result remain valid with convenient hypothesis on f . Now the issue is the propagation of chaos for f which would be the standard issue in plasma turbulence.

Projective dynamic is a standard way to approximate the solution of an non linéaire evolution equation of the form: $\partial_t F + H(F, x, t) = 0$ by different avatars of the equation:

$$\partial_t(\partial_t F) + A(t)(\partial_t F) = 0 \quad A(t) = \partial_F(H(F(t), t))$$

$$F(t) = F(0) + \int_0^t \partial_s(\partial_F(H(F(s), s))ds$$

With t slow variable and s fast variable:

$$e^{sA(t)} = \frac{1}{2i\pi}\int_\Gamma e^{s\lambda}(\lambda\mathbb{I} + A(t))^{-1}d\lambda\,.$$

For the Vlasov equation:

$$
\begin{aligned}
-A &= v\cdot\nabla_x F + \nabla_v(E(f)\nabla_v F(t,v))\\
-A &= v\cdot\nabla_x F - \nabla_v(E(f)\nabla_v H(|v|^2)) + \nabla_v F(t,v)
\end{aligned}
$$

In such cases A is a compact perturbation of the advection operator and for the last line if H is a monotone strictly decreasing function a perturbation of an anti adjoint operator for the norm

$$
\|F\|_H = \int dx\Big(\int dv \frac{|F|^2}{-H'(|v|^2)} + |E(f)|^2\Big)
$$

Convenient to consider also a more general potential that the Coulomb $\|x\|^{-1}$ $V(x)$ or $\widehat{V}(k)$ in Fourier variable. For the Vlasov with repulsive potential one has: $E(k) = -ik/|k|^2$.
For the resolvent equation, with Laplace and Fourier one has

$$\text{With} \quad A(t)f = v\cdot\nabla_x f - (\nabla_x \widehat{V} \star_x f)\nabla_v G(t,v)$$

$$(\lambda\mathbb{I} + A(t))f = g \Leftrightarrow ((\lambda + ik\cdot v) - ik\widehat{V}(k)\nabla_v G(t))\rho_k = S_k$$

$$f_k - i\frac{k\cdot\widehat{V}(k)\nabla_v G(t)\rho_k}{\lambda + ik\cdot v} = \frac{S_k}{\lambda + ik\cdot v}$$

$$\text{with} \quad \rho_k(f) = \int f_k(v)dv$$

$$\big(1 - i\widehat{V}(k)\int \frac{k\cdot\nabla_v G(t,v)}{\lambda + ik\cdot v}dv\big)\rho_k(f) = \int \frac{S_k}{\lambda + ik\cdot v}dv\,.$$

The function

$$D_k(k \cdot v, \lambda, \nabla_v F) = 1 - \widehat{V}_k \int_{\mathbb{R}^d} \frac{ik \cdot \nabla_v F(v)}{\lambda + ik \cdot v}\, dv$$

is called the Penrose dielectric-dispersion function. It is analytic in the half plane $\Re(\lambda) > 0$ and can be extended also as an analytic function in the half space $\Re(\lambda) > -a$ under some analytic regularity hypothesis on the profile $F(v)$. By duality with a $C^{0,\alpha}$ Holder function $v \mapsto \phi(k,v)$ the expression:

$$\int dk \int \frac{S_k(v)}{\lambda + ik \cdot v} \phi(k,v) dv$$

is also well defined over the complex plane. Dirac singularities appears on the imaginary axis and can also be computed with the Plemjel formula. Eventually observe the relation

$$D_k(v, \lambda, \nabla_v F) = 0 \Leftrightarrow D_{-k}(, v, \overline{\lambda}, \nabla_v F)$$

- $\lambda_k = 0$ in the region of $\Re\lambda > 0$ correspond to eigenvalue of finite multiplicity for the resolvent of the operator $(\lambda\mathbb{I} + A(t))$ and in the present case, eigenvalue for the operator $exps(\lambda_k(t))$. With such observation an approximation of the behaviour of the solution carrying the name of quasilinear approximation is proposed below.
- On the other hand in the region, $\Re\lambda < -a < 0$ for regular and small enough initial data (say in analytic or 3Besov space) this the stable regime. Mouhot Villani version of the Landau Damping the electric field $E(t)$ decays exponentially for $t \to 0$
- The behaviour of the solution in presence of zero of the Penrose function on the imaginary axis is not considered in the present talk. (Results can be obtained with the introduction of special solutions corresponding to $\Re(\lambda_m) \simeq 0$ involving so called. Langmuir wave.)
- Finally in the case of the stable regime no eigenmodes for $\Re\lambda > 0$ a finite time analysis (with limited regularity hypothesis) is the object of the Landau and Balescu Lenhard approximation. To be compared with Mouhot Villani

A said above this scenario correspond to the existence of zeroes , λ_m of the Penrose dielectric function with positive real part $\lambda_m(t) > 0$ i. e. unstable modes. Then for the solution of the Vlasov assume that in the parabolic scaling the solution can be written as:

$$F(t,v) = G_\epsilon(t,v) + f_\epsilon(t,x,v) \quad \text{with} \langle f_\epsilon \rangle = 0$$
$$\partial_t G_\epsilon + \nabla_v \frac{\langle E_\epsilon(f_\epsilon) f_\epsilon) \rangle}{\epsilon} = 0 \quad \int_0^{T^*} |\partial_t G_\epsilon(t,v)| dt \leq O(\epsilon)\,,$$
$$\epsilon^2 \partial_t f_\epsilon + v \nabla_x f_\epsilon + \epsilon \nabla_v (E(f_\epsilon) G_\epsilon) = \epsilon \big(\nabla_v \langle E(f_\epsilon) f_\epsilon) \rangle - \nabla_v (E(f_\epsilon), f_\epsilon) \big)$$

Next assume that the f is a small perturbation but with fast time oscillations:

$$f_\epsilon(x,v,t) = \epsilon g_\epsilon(x,v,t,\frac{t}{\epsilon^2})$$

with g_ϵ bounded in convenient spaces.

Then in the fast variable , for $s \mapsto g(x,s)$ one has:

$$\partial_s g_\epsilon + v\nabla_x g_\epsilon + \nabla_v (E(g_\epsilon)G_\epsilon(t)) = \epsilon^2(\nabla_v \langle E(g_\epsilon)g_\epsilon(t))\rangle - \nabla_v(E(g_\epsilon)g_\epsilon))$$

$$\partial_t G_\epsilon + \nabla_v \frac{\langle E_\epsilon(f_\epsilon)f_\epsilon)\rangle}{\epsilon} = 0 \quad f_\epsilon(x,v,t) = g_\epsilon(x,v,t,\frac{t}{\epsilon^2})$$

The above decomposition is the essential hypothesis. It gives replacing f by g_ϵ

$$\partial_t G_\epsilon + \epsilon\nabla_v \langle E_\epsilon(g_\epsilon)g_\epsilon)\rangle = 0$$

This implies the slow variation of the function $G(t,v)$ compatible with the hypothesis:

$$\text{for} \quad 0 < t < T^* \leq \infty \quad \int_0^{T^*} |\partial_t G(t,v)| ds \leq \epsilon$$

And eventually one assume that the spectra of $-A(t)$ for the linearised equation:

$$\partial_s g + A(t)g = 0 \Leftrightarrow \partial_s g + v \cdot \nabla_x g + \nabla_v (E(g)G(t)) = 0$$

for $0 < t < T^*$ contained in the half space $\Re\lambda > 0$ is the union of a non empty set of simple eigenvalues with eigenfunctions $(\lambda_m(t), f_m(t))$.
Therefore the main part (of order 1) of the solution is given by

$$g(s,t,v) = \sum_{\Re\lambda_m > 0} e^{s\lambda_m(t)} f_m(t)(v)$$

Using eigenmodes and conjugates,with the formula

$$E_k(s) = \frac{ik}{|k|^2} \int g_k(s,v)dv$$

one obtain for $G_\epsilon(t,v)$ the following asymptotic formula:

$$\text{For} \quad 0 < t < T^* \quad \partial_t G_\epsilon + \epsilon \nabla_v \big(\mathbf{D}(t,v) \nabla_v G_\epsilon \big) = O(\epsilon^2)$$

$$\mathbf{D}_G(t,v) = \sum_{\Re\lambda > 0} \epsilon^2 \nabla_v \cdot \left(\frac{k \otimes k : E(0,k(\lambda)) \otimes \overline{E}(0,k(\lambda)) e^{2\int_0^t \Re\lambda(s)ds}}{(k(\lambda)\cdot v - \Im\lambda)^2 + (\Re\lambda)^2} \right)$$

Then assume that for $0 < t \leq T^* \leq \infty$

$$\sigma(-A(t)) \cap \{\Re\lambda > 0\} = -v \cdot \nabla_x . + \nabla_v (E(.)G(t,v)) \cap \{\Re\lambda > 0\} \neq \emptyset$$

is composed of simple eigenvalues. Then consider $\tilde{G}$ the solution of

$$\partial_t \tilde{G}_\epsilon + \epsilon \nabla_v \left(\mathbf{D}_G(t,v) \nabla_v \tilde{G}_\epsilon \right) = 0\,.$$

with the same initial data. $(0, T^*)$ $\tilde{G}$ produce an approximation of order ϵ^2 of G_ϵ which can be use as a correction of the kinetic equation:

$$\partial_t G_\epsilon(t,x) - v \cdot \nabla_x G_\epsilon - \nabla_v (\mathbf{D}_G(t,v) \nabla_v G_\epsilon) = 0(\epsilon^2)$$

The conservation laws (mass , energy..) for the original Vlasov equation implies that all the eigenvalue do converge to the imaginary axis. Hence for the smallest one one has

$$\lim_{t\to T^*}\frac{e^{2\int_0^t \Re\lambda(s)ds}}{(k(\lambda)\cdot v-\Im\lambda)^2+(\Re\lambda)^2}=c(\delta(k\cdot v-\Im\lambda(T^*))+\delta(k\cdot v+\Im\lambda(T^*))$$

Hence a diffusion with Dirac profiles.
The above analysis plays an essential role in $1d$ due to existence of unstable eigenvalue (in particular with plateaux)-, Penrose diagram) .

In $3d$ this is not the case...A profile $G(|v|^2)$ then is always Penrose stable.

$$\partial_t F(t,v_1) = \nabla_{v_1} \cdot \int \mathcal{B}_F(v_1, v_1 - v_2)\Big[(\nabla_{v_1} - \nabla_{v_2})F(v_1)F(v_2)\Big]\, dv_2,$$

with diffusion matrix $\mathcal{B}_F(v_1, w)\,, \quad w = v_1 - v_2\,.$

$$\mathcal{B}_F(v_1,w) = \frac{1}{|D^F_{2,k}(ik\cdot v_1)|^2}\frac{c_V}{|w|}\Big(\mathbb{I} - \frac{w\otimes w}{|w|^2}\Big)$$

with

$$D^F_{2,k}(ik\cdot v_1) = \lim_{\Re\lambda\to 0_+}(1+\widehat{V}_k\int_{\mathbb{R}^d}\frac{ik\cdot\nabla_{v_2}F(v_2)}{\lambda - ik\cdot v_2}\, dv_2)$$

under the stability hypothesis opposite to the QL hypothesis:

$$\Re\lambda \geq 0 \Rightarrow |(1+\widehat{V}_k\int_{\mathbb{R}^d}\frac{ik\cdot\nabla_{v_2}F(v_2)}{\lambda - ik\cdot v_2}\, dv_2.)| > 0$$

$$\partial_t F(t,v) = \nabla_v \cdot \int \mathcal{B}_F(v, v-v_*)\Big[(\nabla_v - \nabla_{v_*})F(v)F(v_*)\Big]\, dv_*,$$

$$\partial_t \int_{\mathbb{R}^3_v} F \log F dv$$

$$+\frac{1}{2}\int\int FF_*(\frac{\nabla_v F}{F} - \frac{\nabla_v F_*}{F_*})(\mathcal{B}_F(v,v-v_*)(\frac{\nabla_v F}{F} - \frac{\nabla_v F_*}{F_*})dvdv_* = 0$$

It conserves mass momentum and energy and as an entropy.
It is the counter part of unstable QL approximation. In some cases valid for large time (Duerinck and Winter 2021, Guo). .
Here also the main obstruction is the

$$\lim_{t\to T^*} |(1+\widehat{V}_k \int_{\mathbb{R}^d} \frac{ik\cdot\nabla_{v_2}F(v_2)}{\lambda - ik\cdot v_2}\, dv_2)| = 0$$

The so called Landau equation corresponds to the "simplification":

$$|(1+\widehat{V}_k \int_{\mathbb{R}^d} \frac{ik\cdot\nabla_{v_2}F(v_2)}{\lambda - ik\cdot v_2}\, dv_2)| = 1$$

$$
\begin{aligned}
&\partial_t F_N = \Big\{ H_N, F_N \Big\}_N \\
&H_N = \frac{1}{2}\sum_{i=1}^{N} |v_i|^2 + \frac{1}{N}\sum_{1\le i<j\le N} V(x_i - x_j). \\
&\Big\{ f, g \Big\}_k = \sum_{i=1}^{k} (\nabla_{x_i} f \cdot \nabla_{v_i} g - \nabla_{v_i} f \cdot \nabla_{x_i} g)
\end{aligned} \tag{2}
$$

With $z_k = (x_k, v_k)$

$$
F_{N:k}(t, Z_k) = \int_{\mathbb{R}^{2(N-k)d}} F_N(t, Z_k, z_{k+1}, \cdots z_N)\, dz_{k+1}, \cdots dz_N,
$$

$$
\partial_t F_{N:1} + v_1 \cdot \nabla_{x_1} F_{N:1} = \frac{N-1}{N} \int_{\mathbb{R}^{2d}} \nabla_x V(x_1 - x_2) \cdot \nabla_{v_1} F_{N:2}(t, z_1, z_2)\, dz_2.
$$

With the cumulant hypothesis for $F_{N:3}$ up to $o(N^{-2})$

$$\int_{\mathbb{T}^3} g(s,x,v_1,v_2)dx = 0 \quad g(s,x,v_2,v_1) = g(s,-x,v_1,v_2)\,,$$
$$F_{N:2} = F(t,v_1)F(t,v_2) + g(x_1-x_2,v_1,v_2) + o(N^{-1})\,,$$
$$F_{N:3} = F(t,v_1)g(x,v_2,v_3) + F(t,v_2)g(x,v_3,v_1) + F(t,v_3)g(x,v_3,v_3)$$
$$+ F(t,v_1)F(t,v_2)F(t,v_3) + o(N^{-2})$$

With $\tilde{.}$ be omitted below: g changed into the fast variable

$$g(t,x,v_1,v_2) = \tilde{g}(s,t,x,v_1,v_2) = N^{-1}g(Nt,t,x,v_1,v_2)\,.$$

$$\rho_1[g](s,x,v_2) = \int_{\mathbb{R}^d} g(s,x,\eta,v_2)\;d\eta \quad \rho_2[g](s,x,v_1) = \int_{\mathbb{R}^d} g(s,x,v_1,\eta)\;d\eta,$$
$$E_1[g] := -\nabla_x V \star_x \rho_1[g], \qquad E_2[g] := -\nabla_x V \star_x \rho_2[g].$$

Denote by $\langle\cdot\rangle_x$ the average in x Up to a term of order $o(N^{-1})$

$$\partial_t F = \lim_{N\to\infty} \nabla_{v_1} \cdot \langle \nabla_x V \rho_2[g(Nt,t)]\rangle_x (t, v_1) \tag{3}$$

modulo a term of order N^{-2}:

$$\begin{aligned}\partial_s g(s,t) + \mathcal{L}_F g(s,t) &= \Big\{ V(x_1 - x_2), g(s,t) \Big\}_2 \\ &= \nabla_x V(x) \cdot (\nabla_{v_1} - \nabla_{v_2}) F(v_1,t) F(v_2,t),\end{aligned}$$

$$\mathcal{L}_F g = (v_1 - v_2)\cdot \nabla_x g + E_1[g]\cdot \nabla_{v_1} F(t,v_1) - E_2[g]\cdot \nabla_{v_2} F(t,v_2) \tag{4}$$

The exchange term

$$E_1[g](s,t)\cdot\nabla_{v_1}F(t,v_1)-E_2[g](s,t)\cdot\nabla_{v_2}F(t,v_2)$$

is ignored. Also ignored the initial value in the Duhamel formula (phase mixing or dispersion).

$$\partial_s g+(v_1-v_2)\cdot\nabla_x g=\nabla_x V(x)\cdot(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t),$$

With Fourier :

$$g_k(s,t,v_1,v_2)=\int_{-s}^{s} ik\widehat{V}_k e^{-iks\cdot(v_1-v_2)}\,ds\cdot(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t).$$

$$g_k(tN,t,v_1,v_2)=\int_{-Nt}^{Nt} ik\widehat{V}_k e^{-iks\cdot(v_1-v_2)}\,ds\cdot(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t).$$

Use $\quad \lim\frac{1}{2}\int_{-Nt}^{Nt} e^{iks\cdot w}\,ds\to\delta(k\cdot w).$

$$\rho_2[g_k](s,v_1) = \int_{\mathbb{R}^d} g_k(s,v_1,\eta)\, d\eta,$$

$$\partial_t F = \lim_{N\to\infty} \nabla_{v_1} \cdot \langle \nabla_x V \rho_2[g(Nt,t)]\rangle_x (t,v_1)$$

$$\partial_t F = \nabla_{v_1}\Big(\int_{-\infty}^{\infty} k\otimes k |\widehat{V}_k|^2 e^{-iks\cdot(v_1-v_2)} ds$$

$$\cdot (\nabla_{v_1} - \nabla_{v_2}) F(v_1,t)F(v_2,t) dv_2 dk\Big)$$

$$= \nabla_{v_1} \int \mathcal{B}_L(v_1-v_2)(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t)dv_2\,.$$

$$\mathcal{B}_L(w) = \frac{c_V}{|w|}\Big(\mathbb{I} - \frac{w\otimes w}{|w|^2}\Big)\,.$$

The derivation (from the above formulas) of the Landau is direct but is independent of the Penrose stability condition.

$$|D^F_{2,k}(ik\cdot v_1)| = |\lim_{\Re\lambda\to 0_+}(1+\widehat{V}_k\int_{\mathbb{R}^d}\frac{ik\cdot\nabla_{v_2}F(v_2)}{\lambda - ik\cdot v_2}\,dv_2)| > 0$$

In particular the term

$$E_1[g]\cdot\nabla_{v_1}F(t,v_1) - E_2[g]\cdot\nabla_{v_2}F(t,v_2)$$

is of the same (with respect to N) order as

$$\nabla_x V(x)\cdot(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t)$$

hence this term cannot be ignored for Vlasov or similar dynamic.

The analysis is similar, more elaborated: Start from the integrated form:

$$g_k(s,t,v_1,v_2)+$$
$$\int_{-s}^{s} e^{-ik\sigma\cdot(v_1-v_2)}[E_1[g]\cdot\nabla_{v_1}F(t,v_1)-E_2[g]\cdot\nabla_{v_2}F(t,v_2)](s-\sigma)d\sigma$$
$$=\int_{-s}^{s} ik\widehat{V}_k e^{-iks\cdot(v_1-v_2)}\,ds\cdot(\nabla_{v_1}-\nabla_{v_2})F(v_1,t)F(v_2,t).$$

to take in account the factor

$$|D^F_{2,k}(ik\cdot v_1)|^2$$

in the left hand side.

Wanted to show what can be described by “deterministic ” analysis versus where randomness seems compulsory.
In particular in the hierarchy of equations from particles to macroscopic the weak turbulence appeared already for macroscopic equation of plasma phyis (MHD). The counterpart is the derivation of Landau or Balescu Lenhard from propagation of chaos and cumulant formula.
The introduction of the Penrose dielectric function to discriminate between stable and unstable regime seems to play a crucial role. I did not consider the transition between stable and unstable...
Much more things to do.