Linear equations for stellarator local MHD equilibria around irrational & rational flux surfaces

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Problems with stellarator MHD equilibria

- Stellarator rational flux surfaces pose a fundamental problem
  - Perpendicular current given by force balance

$$\mathbf{J}_\perp = \frac{c}{B^2}\mathbf{B} \times \nabla P$$

    - Recall pressure is a flux function: $P = P(\rho)$
  - Part of the parallel current is given by conservation of current

$$\nabla \cdot \left(\frac{J_\| \mathbf{B}}{B}\right) + \nabla \cdot \mathbf{J}_\perp = 0 \Rightarrow \mathbf{B} \cdot \nabla \left(\frac{J_\|}{B}\right) = -\frac{2cP'}{B^3}(\mathbf{B} \times \nabla B) \cdot \nabla \rho$$

  - On a rational flux surface, we can eliminate the term with $J_\|$

$$\oint \mathbf{B} \cdot \nabla \left(\frac{J_\|}{B}\right) \frac{\mathrm{d}l}{B} = 0 \Rightarrow P' \oint \frac{\mathrm{d}l}{B^4}(\mathbf{B} \times \nabla B) \cdot \nabla \rho = 0$$

- This condition need not be satisfied when $P' \neq 0$: charge accumulates in some magnetic field lines!

Local MHD equilibria for stellarators

- Local MHD equilibrium equations can be used to calculate, with low computational cost, the effect on the magnetic field $\mathbf{B}$ of changes in the rotational transform ι, the pressure gradient P', the shape of the flux surface…
  - Widely used for gyrokinetic simulations of tokamak turbulence [Miller et al PoP 95]

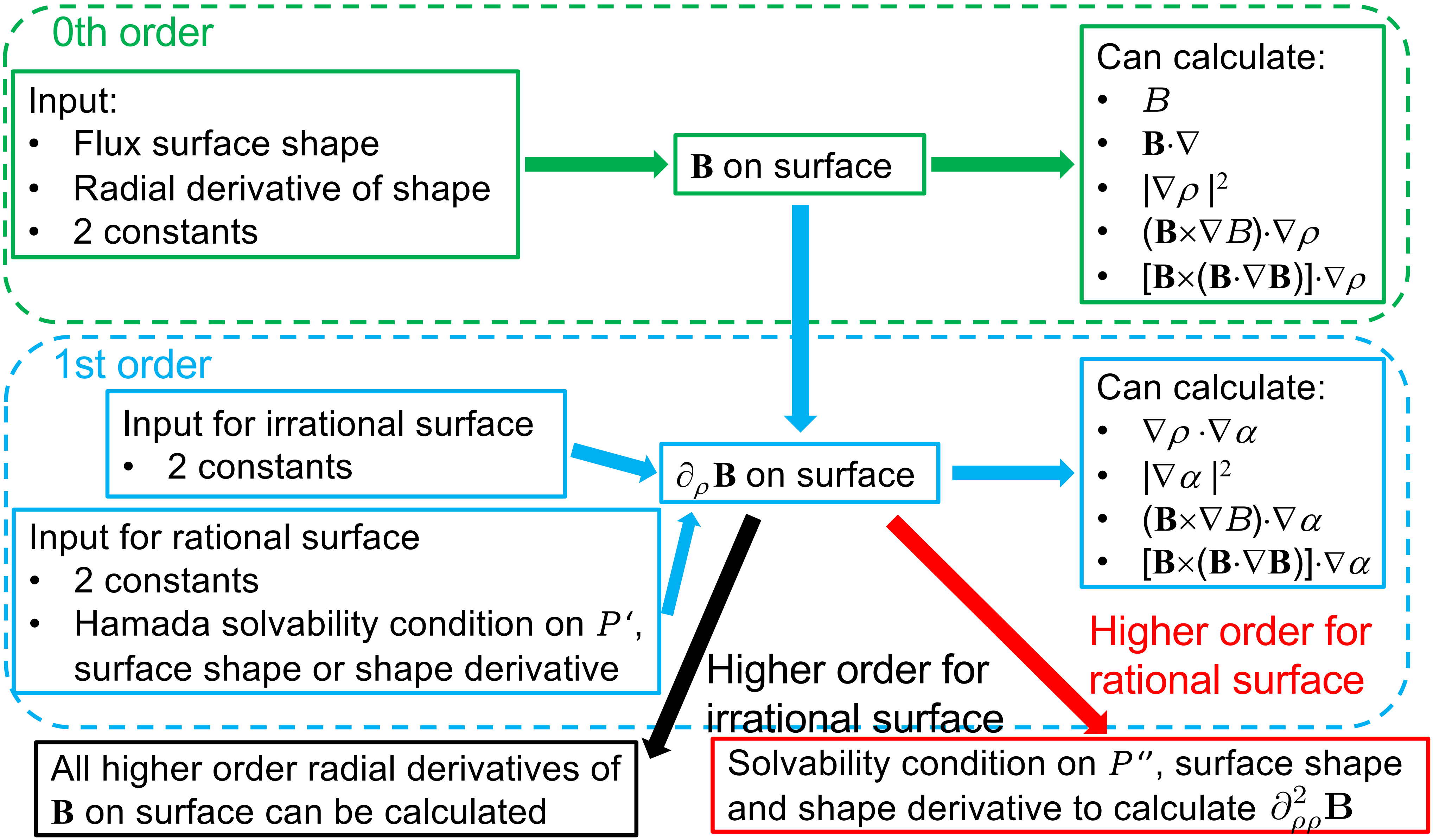
- Local MHD equilibrium equations provide understanding of global MHD equilibria with flux surfaces

- For stellarators, there exist two sets of nonlinear local MHD equilibrium equations: [Hegna PoP 00] and [Candy & Belli JPP 15]

Purpose of the work

- To derive linear equations for stellarator local MHD equilibria
  - Based on [Boozer PoP 02]
- To use local MHD equilibria equations to obtain geometric coefficients for drift kinetic and gyrokinetic equations
  - Inputs
    - Flux surface shape
    - Shape of contiguous flux surfaces
    - 4 constants = values and values of derivatives of flux functions at flux surface of interest
- To show that the coefficients are independent of the choice of coordinates to describe the flux surface
- To determine the conditions necessary to be able to solve the local MHD equilibrium equations on a rational flux surface

Sketch of the calculation



Calculating B on surface $\rho = \rho_a$

- Inputs
  - Flux surface shape $\mathbf{x}_a(u, v)$
    - u = poloidal angle and v = toroidal angle, but can be generalized
  - Radial derivative of flux surface shape
    - Only projection on normal to flux surface matters:

$$\partial_\rho \mathbf{x}(\rho_a, u, v) \cdot \hat{\mathbf{n}}$$

    - Giving shape derivative equivalent to giving Jacobian of (ρ, u, v) transformation

$$\mathcal{J} := \partial_\rho \mathbf{x} \cdot (\partial_u \mathbf{x} \times \partial_v \mathbf{x}) = |\partial_u \mathbf{x} \times \partial_v \mathbf{x}|\, \partial_\rho \mathbf{x} \cdot \hat{\mathbf{n}}$$

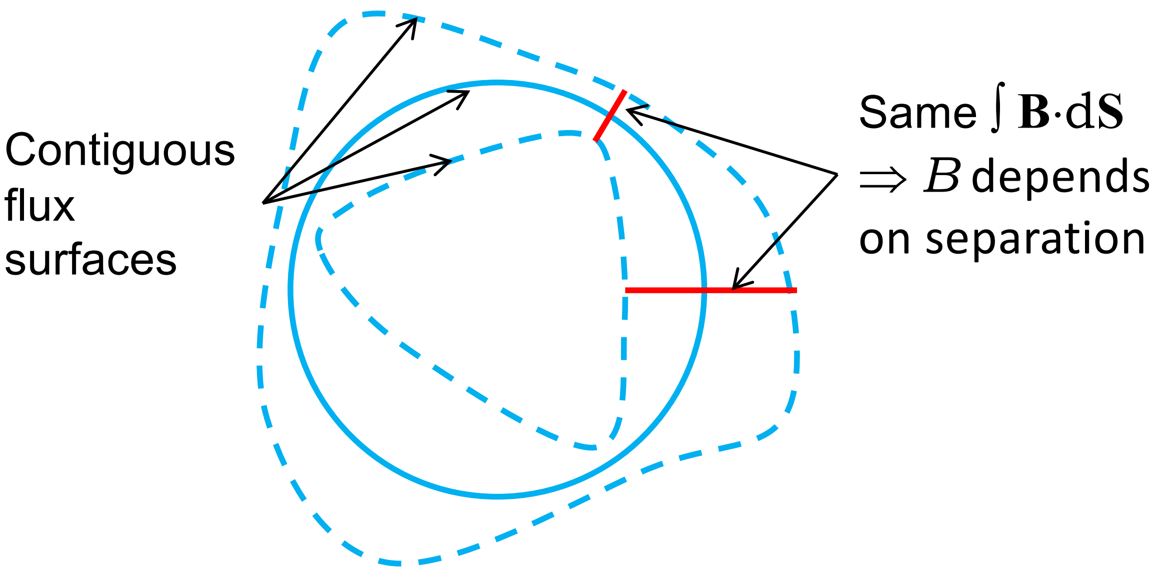
  - Two constants, e.g. magnetic field strength $\Psi_t'(\rho_a)$ and $\iota(\rho_a)$
    - $2\pi\Psi_t(\rho)$ = toroidal flux within flux surface ρ
- [Boozer PoP 02] shows how to solve for **B** in both contravariant and covariant form on $\rho = \rho_a$

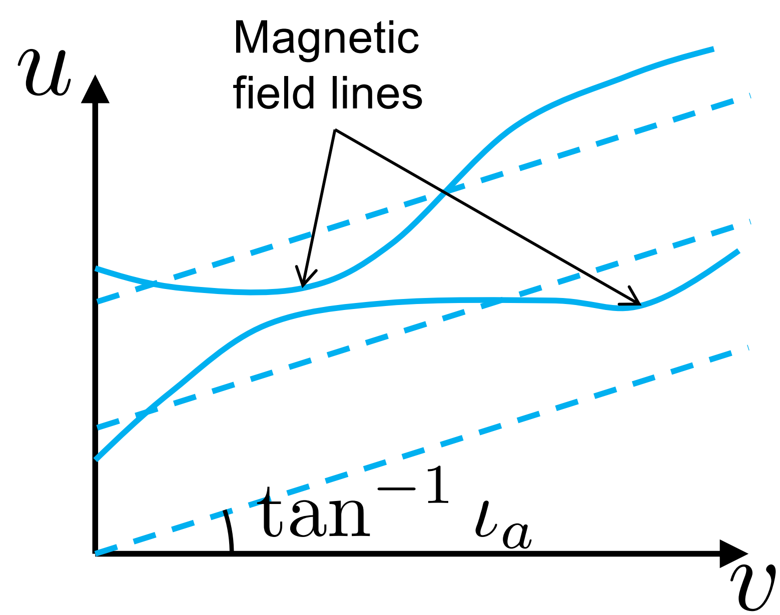
Contravariant B on surface $\rho = \rho_a$

- Form of $\mathbf{B}$ satisfies $\nabla \cdot \mathbf{B} = 0$

$$\mathbf{B}_a = \frac{\Psi_t'(\rho_a)}{\mathcal{J}_a} \left[(1 + \partial_u \nu_a)\ \partial_v \mathbf{x}_a + (\iota(\rho_a) - \partial_v \nu_a)\ \partial_u \mathbf{x}_a \right]$$

  - $\nu_a(u, v)$ is an unknown function that determines direction of $\mathbf{B}$
  - Magnitude B = overall constant $\Psi_t'(\rho_a)$ + separation between contiguous flux surfaces $\mathcal{J}_a$





Contravariant B on surface $\rho = \rho_a$

- Form of $\mathbf{B}$ satisfies $\nabla\cdot\mathbf{B} = 0$

$$\mathbf{B}_a = \frac{\Psi_t'(\rho_a)}{\mathcal{J}_a}\left[\left(1 + \partial_u \nu_a\right)\,\partial_v \mathbf{x}_a + \left(\iota(\rho_a) - \partial_v \nu_a\right)\,\partial_u \mathbf{x}_a\right]$$

- Impose zero radial current density: $\mathbf{J}\cdot\nabla\rho = 0$

$$\begin{pmatrix} \partial_u & \partial_v \end{pmatrix} \frac{1}{\mathcal{J}_a} \begin{pmatrix} |\partial_v \mathbf{x}_a|^2 & -\partial_u \mathbf{x}_a \cdot \partial_v \mathbf{x}_a \\ -\partial_u \mathbf{x}_a \cdot \partial_v \mathbf{x}_a & |\partial_u \mathbf{x}_a|^2 \end{pmatrix} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} \nu_a$$

$$= -\partial_u \left(\frac{|\partial_v \mathbf{x}_a|^2 + \iota(\rho_a)\partial_u \mathbf{x}_a \cdot \partial_v \mathbf{x}_a}{\mathcal{J}_a}\right) + \partial_v \left(\frac{\partial_u \mathbf{x}_a \cdot \partial_v \mathbf{x}_a + \iota(\rho_a)|\partial_u \mathbf{x}_a|^2}{\mathcal{J}_a}\right)$$

  - Linear elliptic partial differential equations solvable with periodic boundary conditions

First set of geometric coefficients

- Knowing $\mathbf{B}$ on a flux surface, one can calculate
  - B, needed to calculate trapping, velocity-space volume, etc
  - $\mathbf{B}\cdot\nabla v$ = projection of parallel motion in relevant coordinates
  - $|\nabla\rho|^2$ = inverse of flux surface separation
  - $(\mathbf{B}\times\nabla B)\cdot\nabla\rho$ and $[\mathbf{B}\times(\mathbf{B}\cdot\nabla\mathbf{B})]\cdot\nabla\rho$ = projection of magnetic drifts in the radial direction
- These coefficients can be explicitly shown to be independent of choice of u and v if $\mathcal{J}_a$ is rescaled appropriately

$$\widetilde{u}(u,v), \widetilde{v}(u,v) \Rightarrow \widetilde{\mathcal{J}}_a = \frac{\mathcal{J}_a}{\partial_u\widetilde{u}\,\partial_v\widetilde{v} - \partial_v\widetilde{u}\,\partial_u\widetilde{v}}$$

- Still need coefficients related to $\mathbf{k}_\perp = k_\rho\nabla\rho + k_\alpha\nabla\alpha$

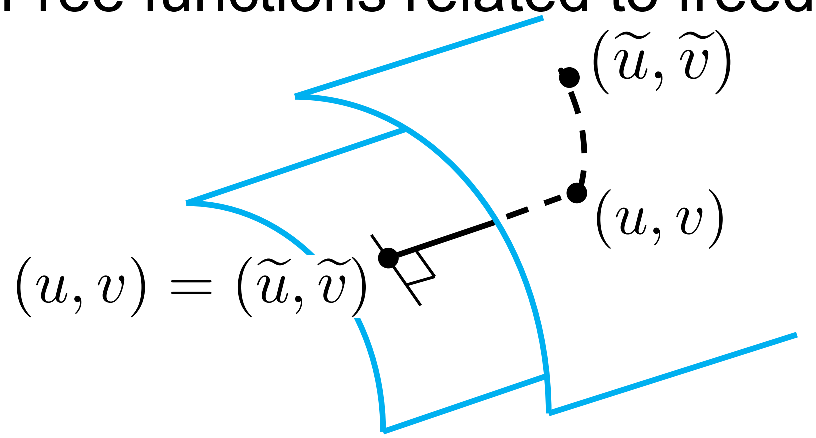
$$\alpha := u - \iota(\rho)v + \nu(\rho,u,v) \Rightarrow \nabla\alpha = \nabla\rho\,[\partial_\rho\nu(\rho,u,v) - \iota'(\rho)v] + \ldots$$

$\partial_\rho \mathbf{B}$ and $\partial_\rho \mathcal{J}$ on surface $\rho = \rho_a$

- [Boozer PoP 02] only calculates $\partial_\rho \mathbf{B}$ (in covariant form), but $\partial_\rho \mathcal{J}$ is needed as well for geometric coefficients
- Input: two other constants, e.g. pressure gradient $P'(\rho_a)$ and magnetic shear $\iota'(\rho_a)$
- Need all components of $\partial_\rho \mathbf{x}_a$: two free functions A^u and A^v

$$\partial_\rho \mathbf{x}(\rho_a, u, v) = \frac{\mathcal{J}_a \hat{\mathbf{n}}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|} + A^u(u,v)\,\partial_u \mathbf{x}_a + A^v(u,v)\,\partial_v \mathbf{x}_a$$

  - Free functions related to freedom of choice of coordinates u and v



$$\mathbf{x}(\rho_a + \Delta\rho, \widetilde{u}, \widetilde{v}) = \mathbf{x}(\rho_a + \Delta\rho, u, v) + \Delta\rho\,(A^u \partial_u \mathbf{x}_a + A^v \partial_v \mathbf{x}_a)$$

$\partial_\rho \mathbf{B}$ and $\partial_\rho \mathcal{J}$ on surface $\rho = \rho_a$

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  - Free functions related to freedom of choice of coordinates u and v
- Can make results independent of A^u and A^v using

$$\mathbf{B}'_a := \left. \frac{\nabla\rho \cdot \nabla \mathbf{B}}{|\nabla\rho|^2} \right|_{\rho=\rho_a} = \partial_\rho \mathbf{B}(\rho_a, u, v) - A^u(u,v)\, \partial_u \mathbf{B}_a - A^v(u,v)\, \partial_v \mathbf{B}_a$$

and other similar definitions for radial derivatives of $\mathcal{J}$, ν…

Contravariant formulation for $\partial_\rho \mathbf{B}$

- Contravariant form of $\partial_\rho \mathbf{B}$

$$\mathbf{B}'_a = \frac{\Psi'_t(\rho_a)}{\mathcal{J}_a}\left[\partial_u \nu'_a \, \partial_v \mathbf{x}_a + (\iota'(\rho_a) - \partial_v \nu'_a)\, \partial_u \mathbf{x}_a\right] + \left(\frac{\Psi''_t(\rho_a)}{\Psi'_t(\rho_a)} - \frac{\mathcal{J}'_a}{\mathcal{J}_a}\right)\mathbf{B}_a$$

$$+\mathbf{B}_a \cdot \nabla_S \left(\frac{\mathcal{J}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|}\hat{\mathbf{n}}_a\right)$$

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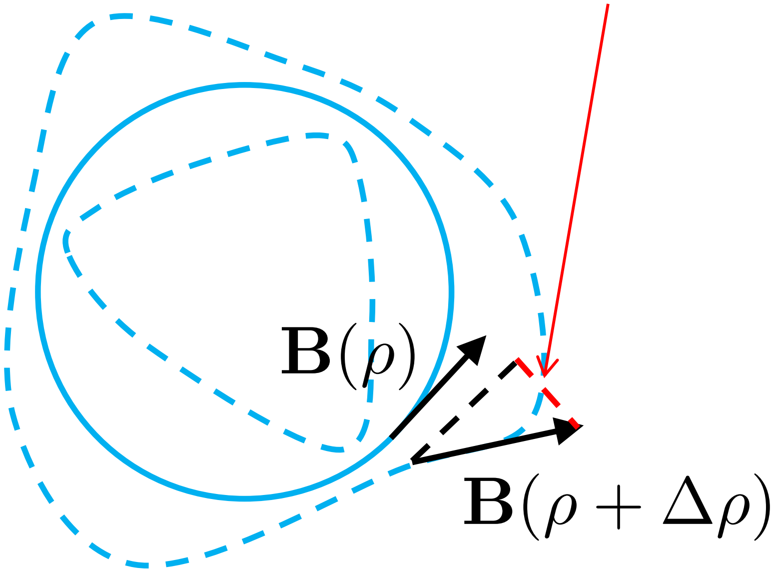

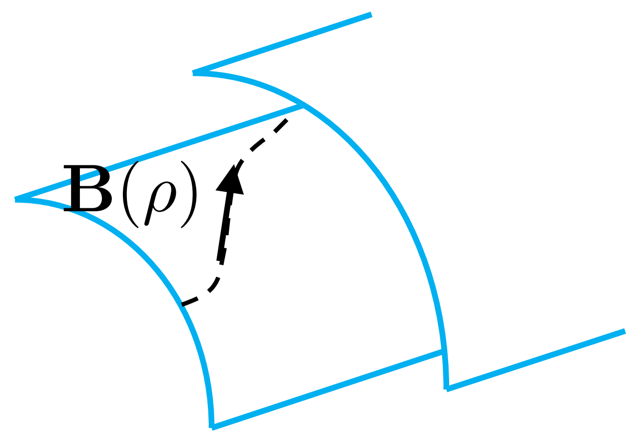

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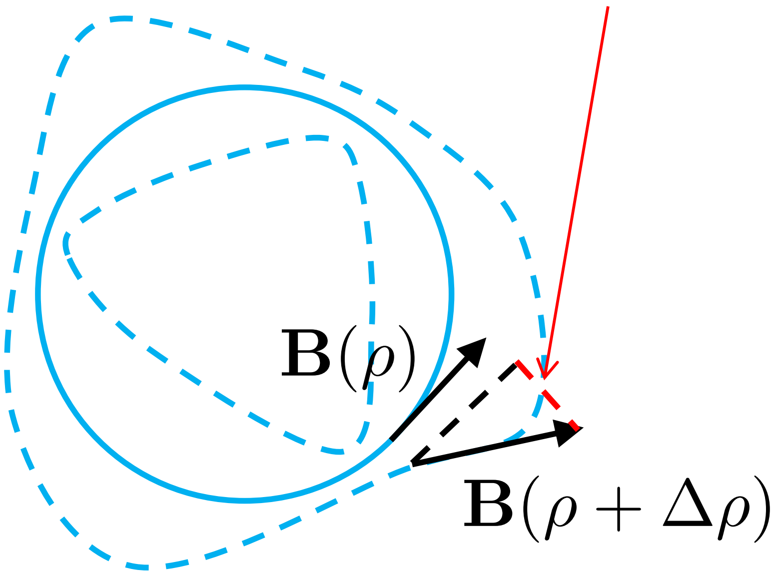

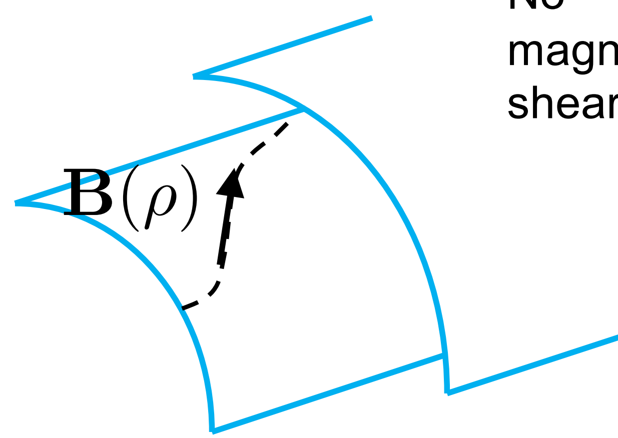

No magnetic shear!

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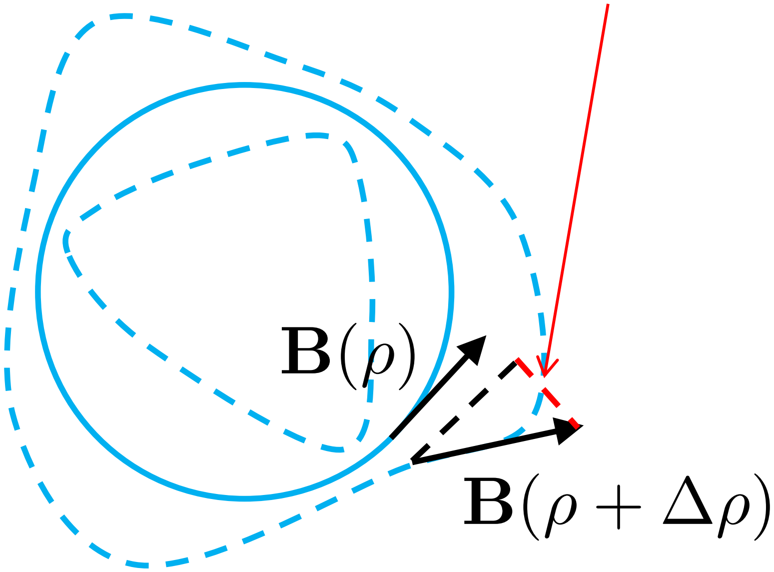

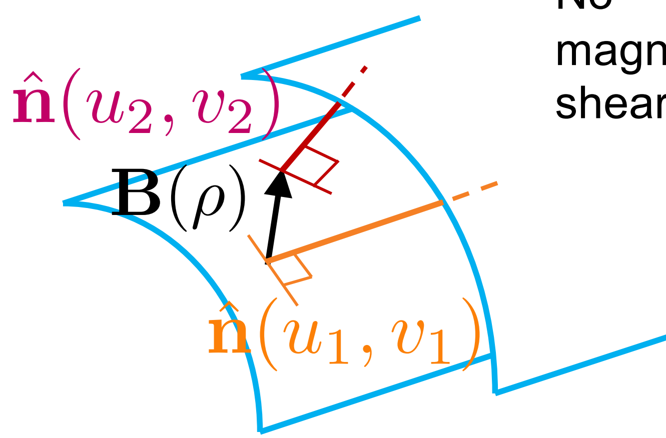

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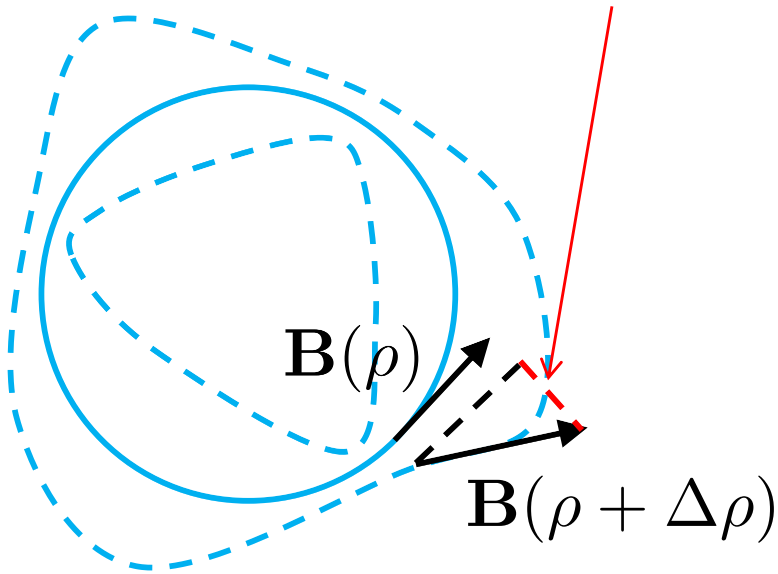

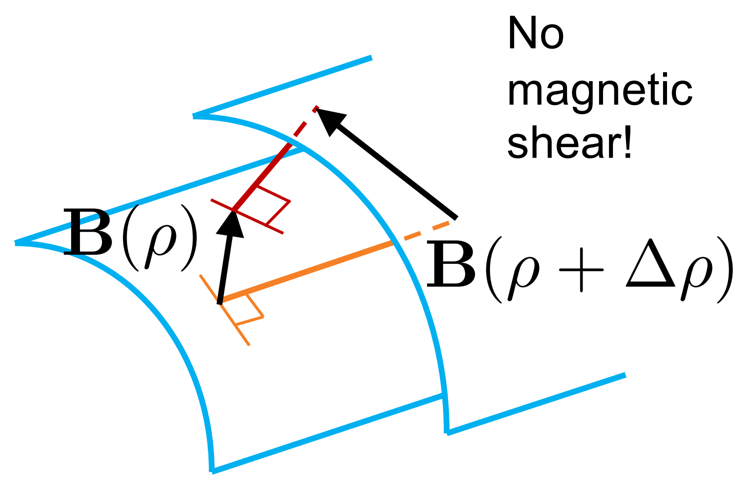

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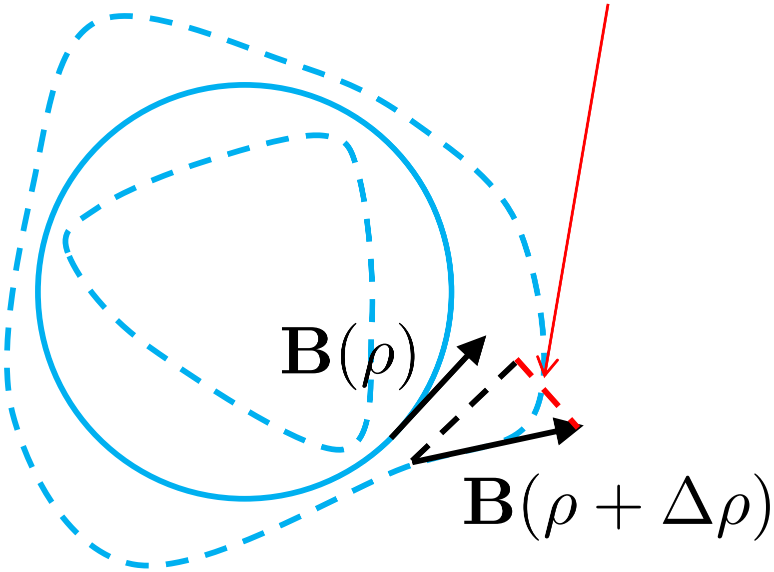

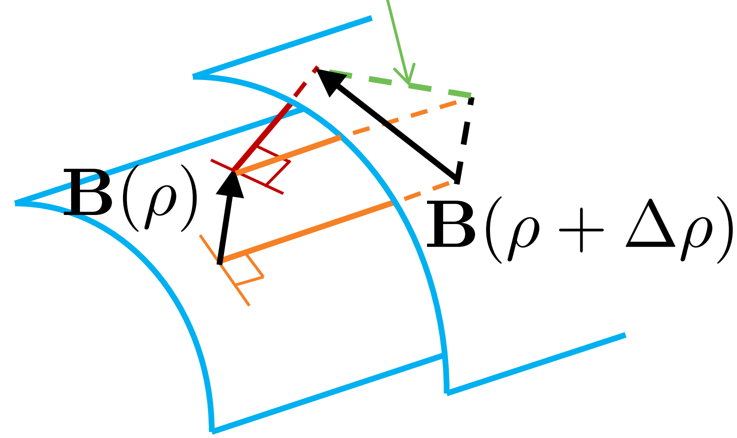

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- Solve for all unknows using MHD equilibrium equations
  - Solve for $\mathcal{J}_a'$ using radial force balance

$$\hat{\mathbf{n}} \cdot \nabla \left(\frac{B^2}{8\pi} + P\right) = \frac{1}{4\pi}\mathbf{B} \cdot \nabla \mathbf{B} \cdot \hat{\mathbf{n}}$$

  - Solve for ν_a' using Ampere's law assuming $J_{||}$ is known
  - Solve for $J_{||}$ using current conservation

$$\mathbf{B} \cdot \nabla \left(\frac{J_{||}}{B}\right) + \nabla \cdot \left(\frac{c}{B^2}\mathbf{B} \times \nabla P\right) = 0$$

Equations for $\partial_\rho \mathbf{B}$ and $\partial_\rho \mathcal{J}$

- Two equations to be integrated along magnetic field lines
  - Radial force balance

$$\frac{\Psi_t''(\rho_a)}{\Psi_t'(\rho_a)} - \frac{\mathcal{J}_a'}{\mathcal{J}_a} = \frac{2\mathcal{J}_a \mathbf{B}_a \cdot \nabla_S \mathbf{B}_a \cdot \hat{\mathbf{n}}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a| B_a^2} - \frac{4\pi P'(\rho_a)}{B_a^2} - \frac{\Psi_t'(\rho_a)}{\mathcal{J}_a B_a^2}\left[\partial_u \nu_a' \, \partial_v \mathbf{x}_a + (\iota'(\rho_a) - \partial_v \nu_a') \, \partial_u \mathbf{x}_a\right] \cdot \mathbf{B}_a$$

  - Current conservation

$$\mathbf{B}_a \cdot \nabla_S \left(\frac{J_{\|a}}{B_a}\right) = \frac{2c|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a| P'(\rho_a)}{\mathcal{J}_a B_a^3} (\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \nabla_S B_a$$

  - Parallel Ampere's law

$$\mathbf{B}_a \cdot \nabla_S \nu_a' = \frac{\iota'(\rho_a)(\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \partial_u \mathbf{x}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|} - \frac{4\pi \mathcal{J}_a^2 B_a J_{\|a}}{c\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} + \frac{\mathcal{J}_a^2 \left[\mathbf{B}_a \cdot \nabla_S \hat{\mathbf{n}}_a \cdot (\mathbf{B}_a \times \hat{\mathbf{n}}_a) + (\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \nabla_S \hat{\mathbf{n}}_a \cdot \mathbf{B}_a\right]}{\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2}$$

Solutions on irrational flux surfaces

- $J_{||a}$ known up to constant K_a

$$J_{||a}(u,v) = K_a B_a(u,v) + J_{PS,a}(u,v)$$

  - Pfirsch-Schlüter current $J_{PS,a}$ = particular solution with

$$\langle J_{PS,a} B_a \rangle = 0$$

    - $\langle \ldots \rangle$ = flux surface average

- Constant K_a determined by solvability condition of parallel Ampere's law: $\langle \mathbf{B}_a \cdot \nabla_S \nu_a' \rangle = 0 \Rightarrow$

$$K_a = \left\langle \frac{\mathcal{J}_a^2 B_a^2}{\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} \right\rangle^{-1}$$

$$\times \left[\frac{c}{4\pi} \left\langle \frac{\iota'(\rho_a)(\mathbf{B}_a \times \hat{\mathbf{n}}_a)\cdot \partial_u \mathbf{x}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|} \right\rangle - \left\langle \frac{\mathcal{J}_a^2 B_a J_{PS,a}}{\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} \right\rangle \right.$$

$$\left. + \frac{c}{4\pi} \left\langle \frac{\mathcal{J}_a^2 [\mathbf{B}_a \cdot \nabla_S \hat{\mathbf{n}}_a \cdot (\mathbf{B}_a \times \hat{\mathbf{n}}_a) + (\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \nabla_S \hat{\mathbf{n}}_a \cdot \mathbf{B}_a]}{\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} \right\rangle \right]$$

Effect of geometry on rotational transform

Final set of geometric coefficients

- With $\partial_\rho \mathbf{B}$ and $\partial_\rho \mathcal{J}$ on a flux surface, one can finally calculate
  - $\nabla\rho \cdot \nabla\alpha$ and $|\nabla\alpha|^2$ = effect of magnetic shear on $k_\perp{}^2$
  - $(\mathbf{B}\times\nabla B)\cdot\nabla\alpha$ and $[\mathbf{B}\times(\mathbf{B}\cdot\nabla\mathbf{B})]\cdot\nabla\alpha$ = projection of magnetic drift on binormal direction and effect of magnetic shear on magnetic drift

- These coefficients are independent of functions A^u and A^v

$$\mathbf{k}_\perp = \frac{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|}{\mathcal{J}_a}\left[k_\rho - k_\alpha(\iota'(\rho_a)v - \nu_a')\right]\hat{\mathbf{n}}_a + \frac{\mathcal{J}_a k_\alpha}{\Psi_t'(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|}\mathbf{B}_a \times \hat{\mathbf{n}}_a$$

  - ν_a' is obtained up to a constant, and that constant determines the zero for k_ρ

What about rational flux surfaces?

- Going back to 1st slide, current conservation is solvable
  - $P'(\rho_a) = 0$, or
  - $\oint \frac{\mathrm{d}l}{B}$ is independent of $\mathbf{B}$ line in which it is evaluated

[Hamada NF 62]

- When this condition is satisfied (e.g. we choose $P'(\rho_a) = 0$ or an appropriate $\mathcal{J}_a$), $J_{\|a}$ obtained up to function of the field line
  - Use α to distinguish between lines

$$J_{\|a} = \widetilde{K}_a(\alpha) B_a(u,v) + J_{PS,a}(u,v)$$

  - $J_{PS,a}$ dependence on α given by continuity with J_{PS} in irrational flux surfaces $\Rightarrow$ $J_{PS,a}$ must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\oint\left[\frac{(\nabla\rho\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\nabla\rho)\times\nabla\rho}{|\nabla\rho|^4 B}\cdot\left(\frac{J_{PS}\mathbf{B}}{B}+\frac{cP'}{B^2}\mathbf{B}\times\nabla\rho\right)-\frac{cP''}{B}\right]\mathrm{d}l\right\}=0$$

Calculating the total current

- Parallel Ampere's law has infinitely many solvability conditions in irrational flux surfaces

$$\oint \mathbf{B}_a \cdot \nabla_S \nu'_a \frac{\mathrm{d}l}{B_a} = 0 \Rightarrow$$

$$\widetilde{K}_a(\alpha) = \left(\oint \frac{\mathcal{J}_a^2 B_a}{\Psi'_t(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} \mathrm{d}l \right)^{-1}$$
$$\times \oint \left[\frac{c}{4\pi} \frac{\iota'(\rho_a)(\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \partial_u \mathbf{x}_a}{|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a| B_a} - \frac{\mathcal{J}_a^2 J_{PS,a}}{\Psi'_t(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2} \right.$$
$$\left. + \frac{c}{4\pi} \frac{\mathcal{J}_a^2 [\mathbf{B}_a \cdot \nabla_S \hat{\mathbf{n}}_a \cdot (\mathbf{B}_a \times \hat{\mathbf{n}}_a) + (\mathbf{B}_a \times \hat{\mathbf{n}}_a) \cdot \nabla_S \hat{\mathbf{n}}_a \cdot \mathbf{B}_a]}{\Psi'_t(\rho_a)|\partial_u \mathbf{x}_a \times \partial_v \mathbf{x}_a|^2 B_a} \right] \mathrm{d}l$$

- This would seem to be the end of the calculation, but if we try to find $\partial^2_{\rho\rho}\mathbf{B}$, there will be trouble…

Second order radial derivatives of $\mathbf{B}$

- Form of the current conservation equation

$$\mathbf{B}_a \cdot \nabla_S Q_a = S_a$$

- Obtain equation for radial derivative of $J_{||}$ by differentiating the current conservation equation

$$B_a\,\partial_l\left(Q_a' + \frac{\Psi_t''(\rho_a)}{\Psi_t'(\rho_a)}Q_a\right) + B_a\Big[\left(-\iota'(\rho_a)\partial_\alpha v + \partial_\alpha \nu_a'\right)\partial_l Q_a$$

$$+\left(\iota'(\rho_a)\partial_l v - \partial_l \nu_a'\right)\partial_\alpha Q_a\Big] = S_a' + \frac{S_a\,\mathcal{J}_a'}{\mathcal{J}_a}$$

- Dividing by B_a and integrating along each line

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\oint\left(\iota'(\rho_a)\partial_l v - \partial_l\nu_a'\right)Q_a\,\mathrm{d}l\right] = \oint\left(S_a' + \frac{S_a\,\mathcal{J}_a'}{\mathcal{J}_a}\right)\mathrm{d}l$$

- Condition on $J_{||a}$! $\Rightarrow \iota'(\rho_a)\dfrac{\mathrm{d}\widetilde{K}_a}{\mathrm{d}\alpha}\oint \mathrm{d}v = 0$

New condition for rational flux surfaces

- For flux surfaces to exist around a rational flux surface

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\left(\oint \frac{B}{|\nabla\rho|^2}\,\mathrm{d}l\right)^{-1}\left(\frac{c\iota'\Psi_t'}{4\pi}\oint \mathrm{d}v - \oint \frac{J_{PS}}{|\nabla\rho|^2}\,\mathrm{d}l\right.\right.$$
$$\left.\left. + \frac{c}{4\pi}\oint \frac{[\mathbf{B}\cdot\nabla\hat{\mathbf{n}}\cdot(\mathbf{B}\times\hat{\mathbf{n}}) + (\mathbf{B}\times\hat{\mathbf{n}})\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{B}]}{B|\nabla\rho|^2}\,\mathrm{d}l\right)\right] = 0$$

Related to condition in [Weitzner PoP 16]

- Condition requires choosing correct $P''(\rho_a)$, $\mathbf{x}_a$ and/or $\mathcal{J}_a$

- Without this condition, current in each line would give different "local" magnetic shears when they should all give the same global magnetic shear ι'

Conclusions

- We have derived a system of linear equations for stellarator local MHD equilibria
  - One 2D elliptic partial differential equation
  - Two equations that require integrating along magnetic field lines
- We can calculate the geometric coefficients needed for drift kinetic and gyrokinetic equations using the local MHD equations that we have derived
- We have explicitly shown these coefficients to be independent of the choice of coordinates for the flux surface
- For rational flux surfaces, two solvability conditions must be met, and one of these conditions is new

Summary of rational surface conditions

- Hamada condition: $P' \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\oint \frac{\mathrm{d}l}{B}\right) = 0$
- New condition

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\left(\oint \frac{B}{|\nabla\rho|^2}\,\mathrm{d}l\right)^{-1}\left(\frac{c\iota'\Psi_t'}{4\pi}\oint \mathrm{d}v - \oint \frac{J_{PS}}{|\nabla\rho|^2}\,\mathrm{d}l\right.\right.$$

$$\left.\left. + \frac{c}{4\pi}\oint \frac{[\mathbf{B}\cdot\nabla\hat{\mathbf{n}}\cdot(\mathbf{B}\times\hat{\mathbf{n}}) + (\mathbf{B}\times\hat{\mathbf{n}})\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{B}]}{B|\nabla\rho|^2}\,\mathrm{d}l\right)\right] = 0$$

  - Pfirsch-Schlüter current J_{PS} defined by

$$\mathbf{B}\cdot\nabla\left(\frac{J_{PS}}{B}\right) = \frac{2cP'}{B^3}(\mathbf{B}\times\nabla\rho)\cdot\nabla B, \quad \langle J_{PS}B\rangle = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\oint\left[\frac{(\nabla\rho\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\nabla\rho)\times\nabla\rho}{|\nabla\rho|^4 B}\cdot\left(\frac{J_{PS}\mathbf{B}}{B} + \frac{cP'}{B^2}\mathbf{B}\times\nabla\rho\right) - \frac{cP''}{B}\right]\mathrm{d}l\right\} = 0$$