# Radial magnetic drift effects on critical balance and secondary instability

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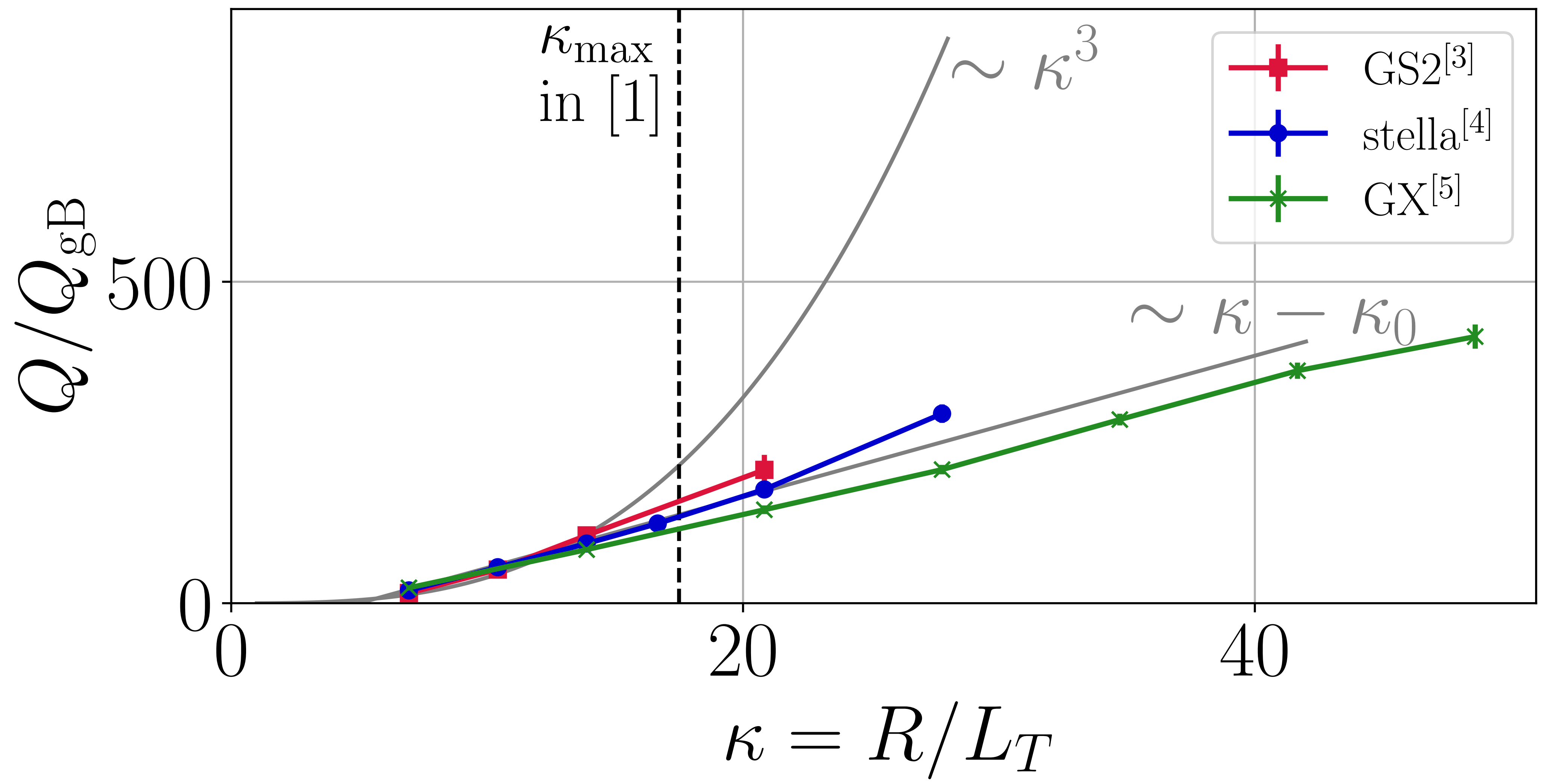
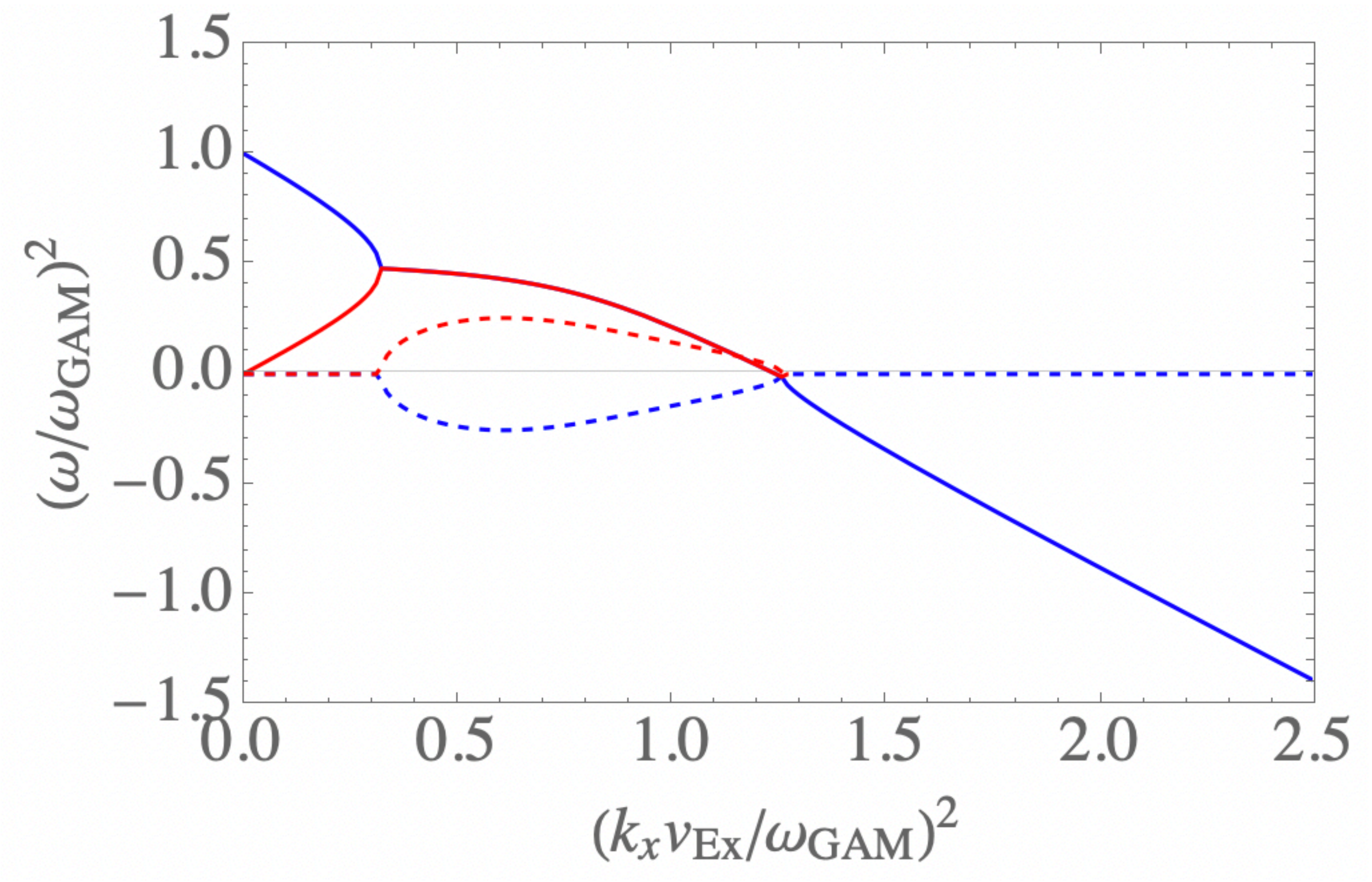
*14th Plasma Kinetics Working Meeting WPI Vienna, July 2023*




1. **Radial scale length in critical balance**

   Original theory[1] did not correctly account for anisotropy between binormal and radial length scales: $l_y/\rho_i \sim q\kappa$ while $l_x/\rho_i \sim q$ is independent of $\kappa$ and is set by zonal flow physics.

   (a) Review: critical balance for ITG in a torus

   (b) Revised scalings

   (c) Radial magnetic drift and critical balance

2. **Secondary theory including radial magnetic drift**

   New modes due to up-down asymmetric turbulent heat flux

   (a) Review: zonal flow linear physics and secondary theory

   (b) Revised dispersion relation

3. **Conclusion and future work**

   Potential implications for Dimits shift theories and stellarator turbulence optimisation

[1] Barnes, Parra & Schekochihin PRL 2011

- Gyrokinetic equation for ions ($h = \delta f + \Phi F_M$, $\Phi \equiv Ze\varphi/T$):

$$\partial_t(h - \langle\Phi\rangle_R F_M) + v_\parallel \nabla_\parallel h + \mathbf{v}_M \cdot \nabla h + \rho_i v_T \, \hat{b} \times \nabla \langle\Phi\rangle_R \cdot \nabla \left(F_M + h\right) = 0$$

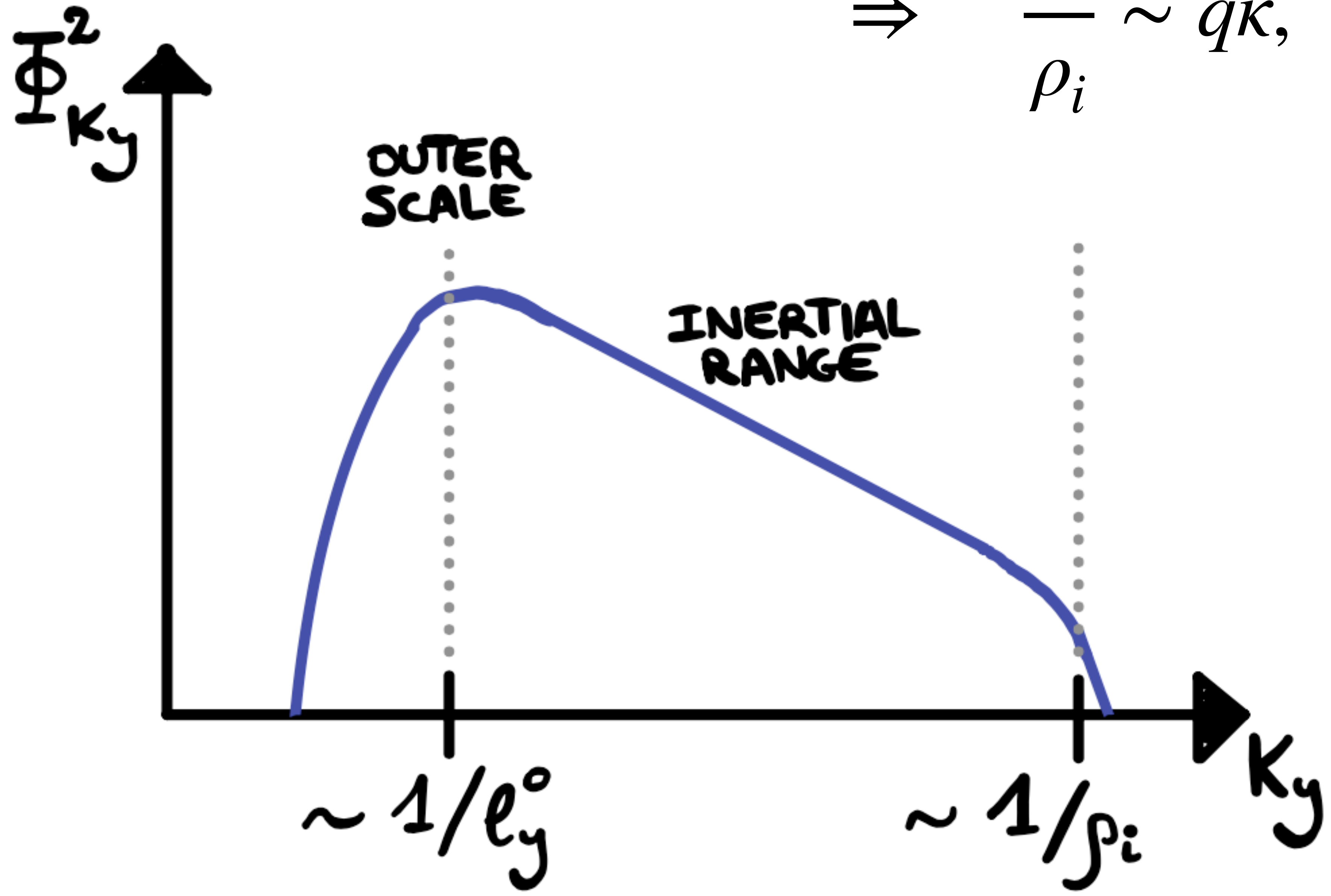
- **Critical Balance**[1]: parallel and nonlinear time scales balance

$$\frac{v_T}{l_\parallel} \sim \frac{\rho_i^2}{l_x l_y}\Omega_i \Phi_{\mathbf{l}}$$

- **Outer scale:** also balances diamagnetic drive (large $\kappa \equiv R/L_T$) & parallel scale set by connection length $l_\parallel^o \sim qR$,

$$\frac{v_T}{qR} \sim \frac{\rho_i^2}{l_x^o l_y^o}\Omega\Phi^o \sim \frac{\rho_i}{l_y^o}\frac{v_T}{L_T}$$

$$\Rightarrow \quad \frac{l_y^o}{\rho_i} \sim q\kappa, \qquad \Phi^o \sim \frac{l_x^o}{L_T}, \qquad \frac{Q}{Q_{\rm gB}} \sim \frac{\rho_i}{l_y^o}\left(\frac{R}{\rho_i}\Phi^o\right)^2 \sim \frac{\kappa}{q}\left(\frac{l_x^o}{\rho_i}\right)^2$$




[1] Barnes, Parra & Schekochihin PRL 2011 [2] Ghim *et al*. PRL 2013

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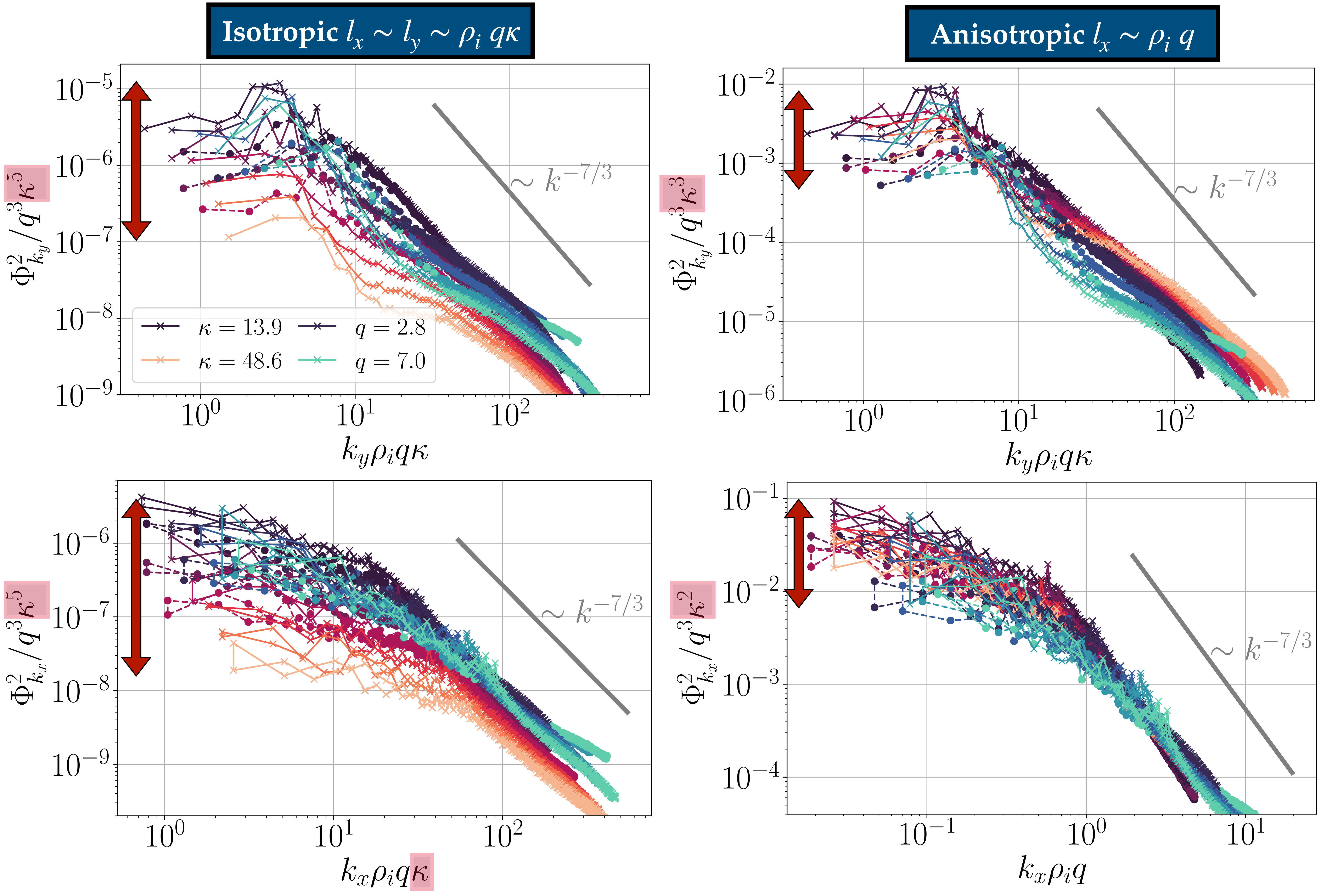
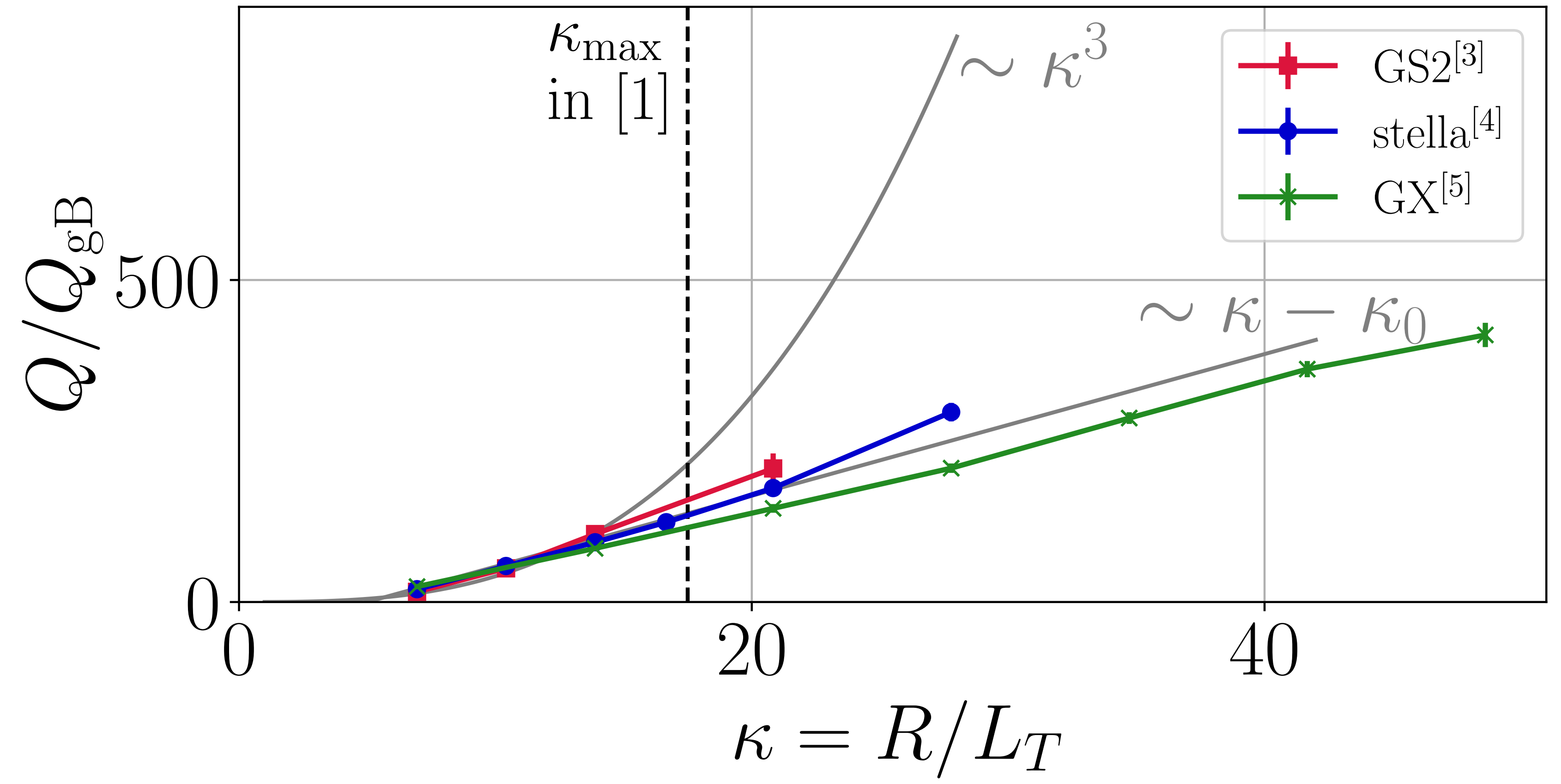
- **What sets the radial scale length?**
  - Barnes *et al.*[1]: perpendicular isotropy, $l_x^o \sim l_y^o \sim q\kappa\rho_i \;\Rightarrow Q/Q_{\mathrm{gB}} \sim q\kappa^3$
  - Ghim *et al.*[2]: also balance with $\omega_{Mx}$, $l_x^o \sim q\rho_i \;\Rightarrow Q/Q_{\mathrm{gB}} \sim q\kappa$ (grand critical balance)

[1] Barnes, Parra & Schekochihin PRL 2011 [2] Ghim *et al.* PRL 2013

Scans in safety factor $q$ and temperature gradient $\kappa$ of CBC using GS2[3], stella[4] and GX[5].



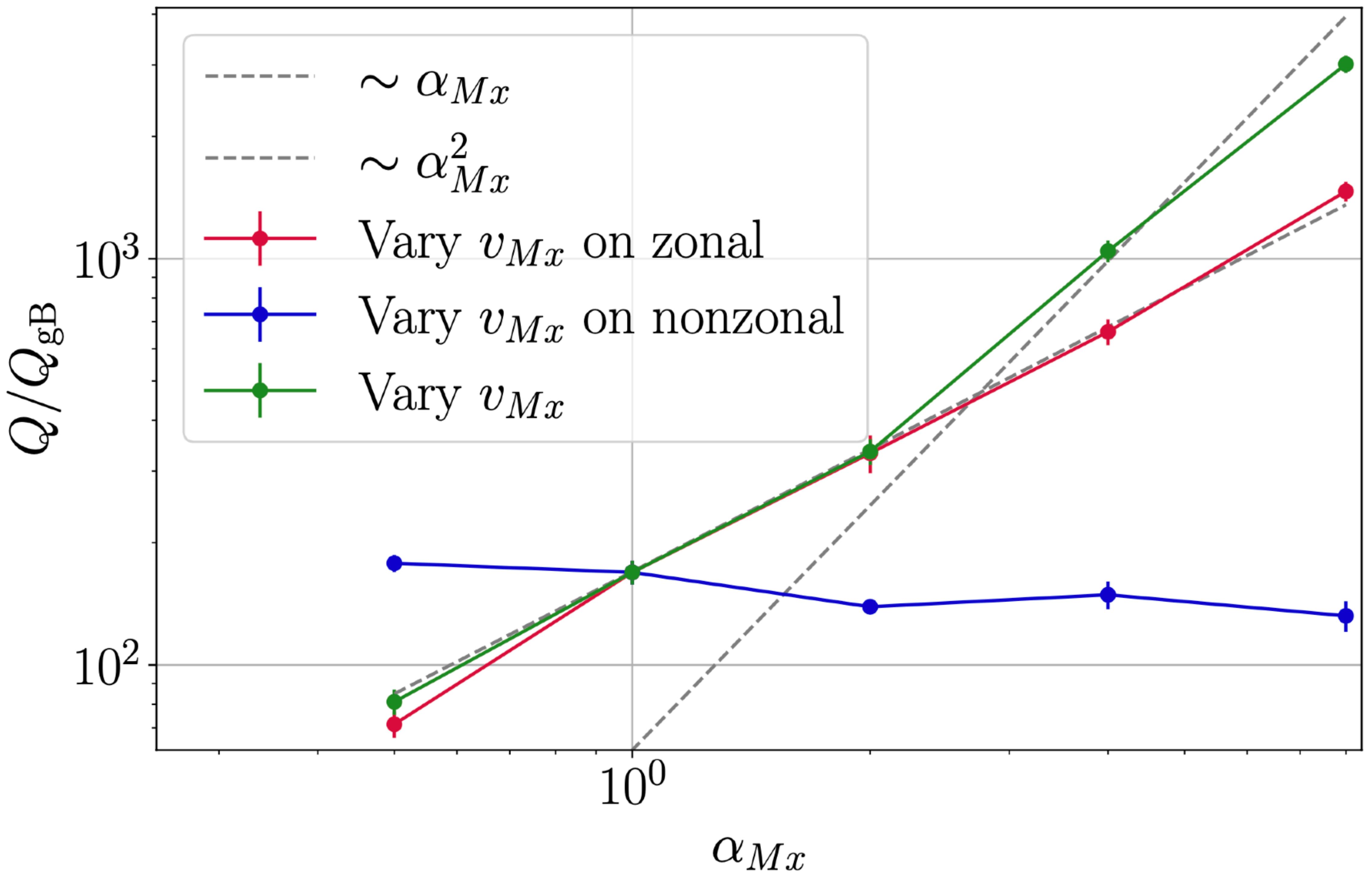




The amplitude at the outer scale and the radial length scale follow the anisotropic prediction (see curve overlap in $\kappa$ scan) and $q$-dependence is verified.

[3] Dorland *et al.* PRL 2000, [4] Barnes *et al.* JCP 2019, [5] Mandell *et al.* JPP 2018, 2023

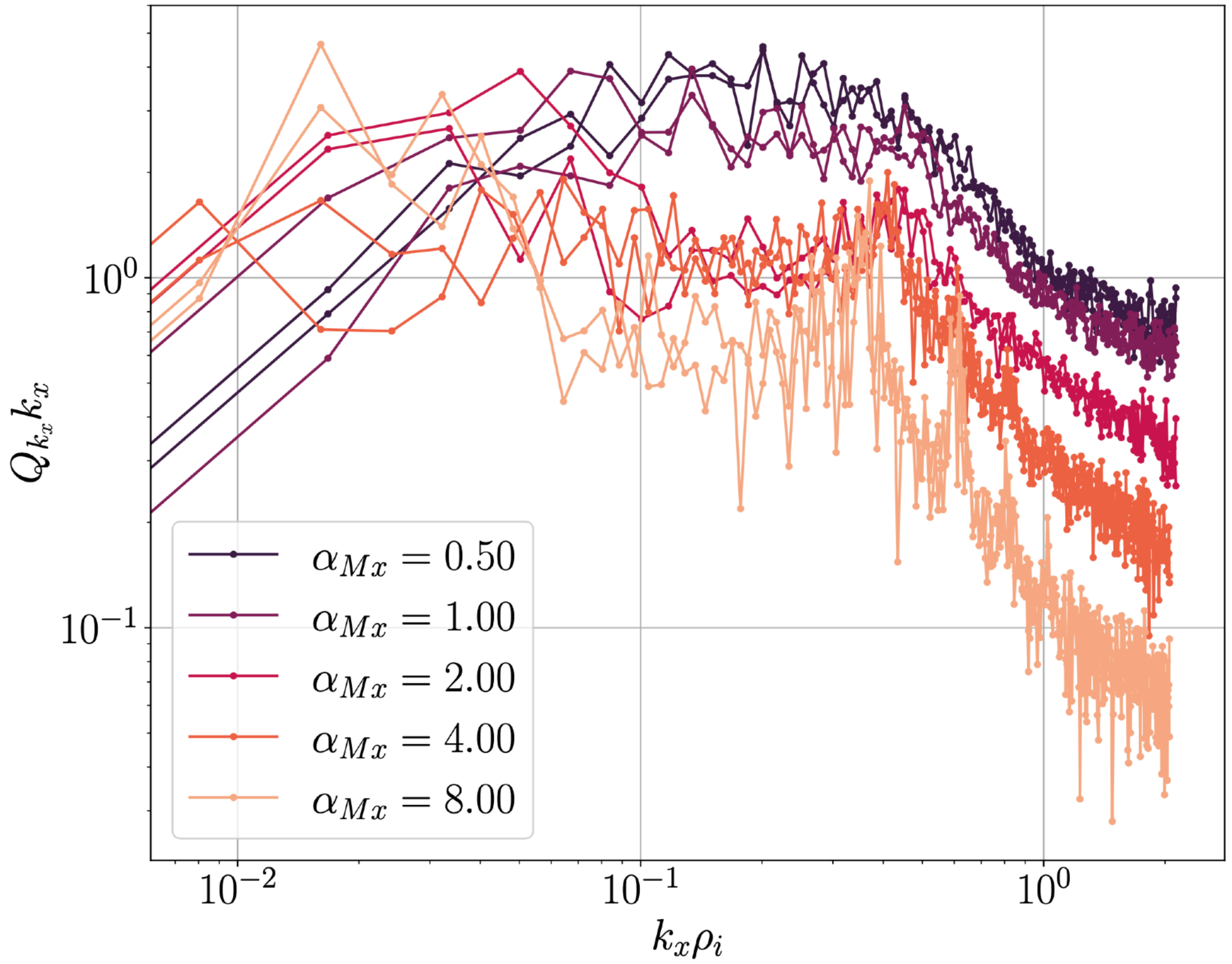
**So grand critical balance?**[2] $\omega_\parallel \sim \omega_{\rm NL} \sim \omega_{Mx} \Rightarrow Q \propto q\kappa\alpha_{Mx}^2$ where $\alpha_{Mx} = v_{Mx}/(\rho_i v_T/R)$

We separately vary the radial magnetic drift acting on zonal ($k_y = 0$) and nonzonal ($k_y \neq 0$) components.




**Disclaimer 1:** Large $v_{Mx}$ for both zonal and nonzonal goes back to grand critical balance?

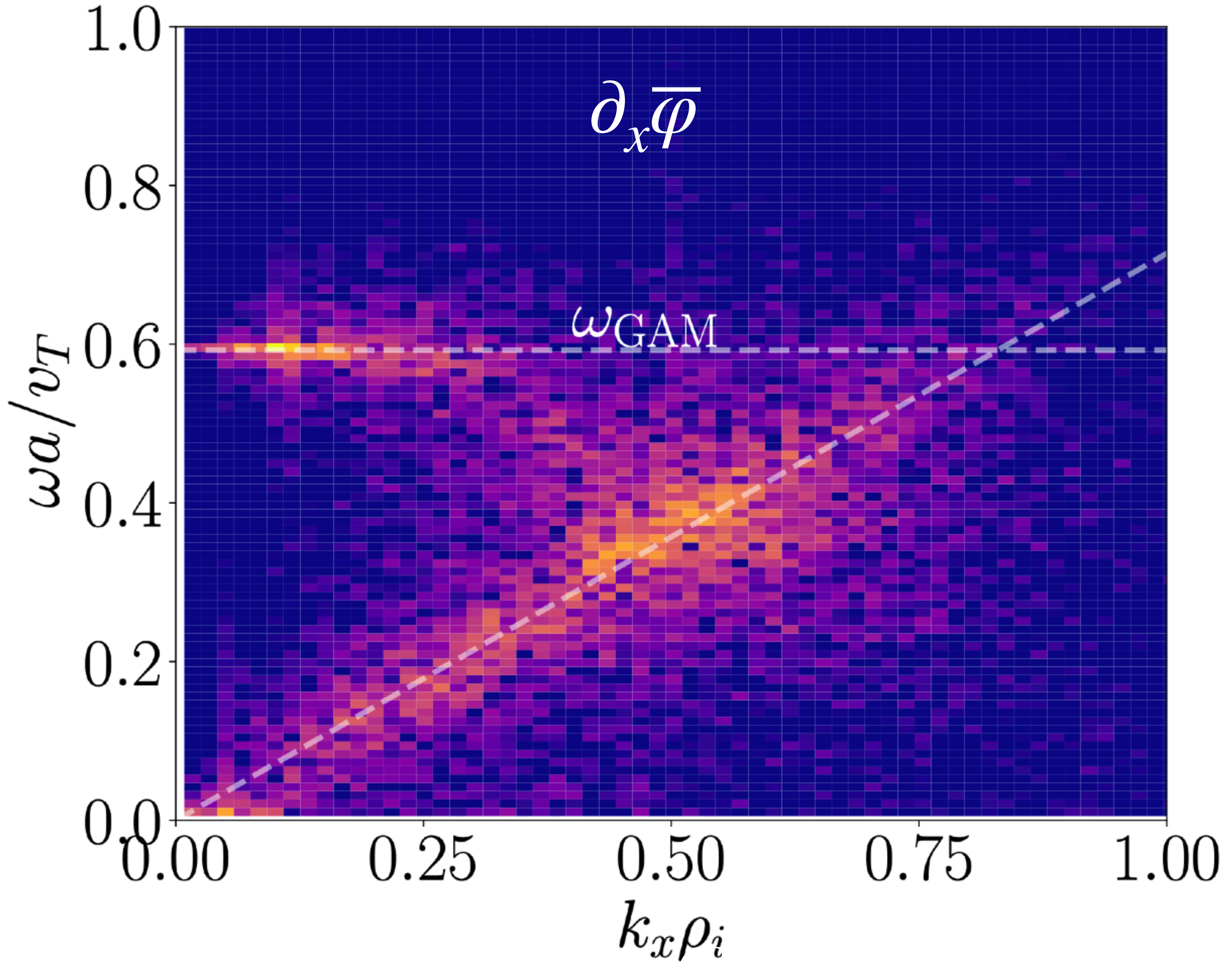
**Disclaimer 2:** $v_{Mx}^{NZ}$ scan non-obvious despite $Q \sim$ const.




→ Understanding of $v_{Mx}^{NZ}$ scan is work in progress.

What about the zonal flows and $v_{Mx}^{Z}$?

[2] Ghim *et al.* PRL 2013



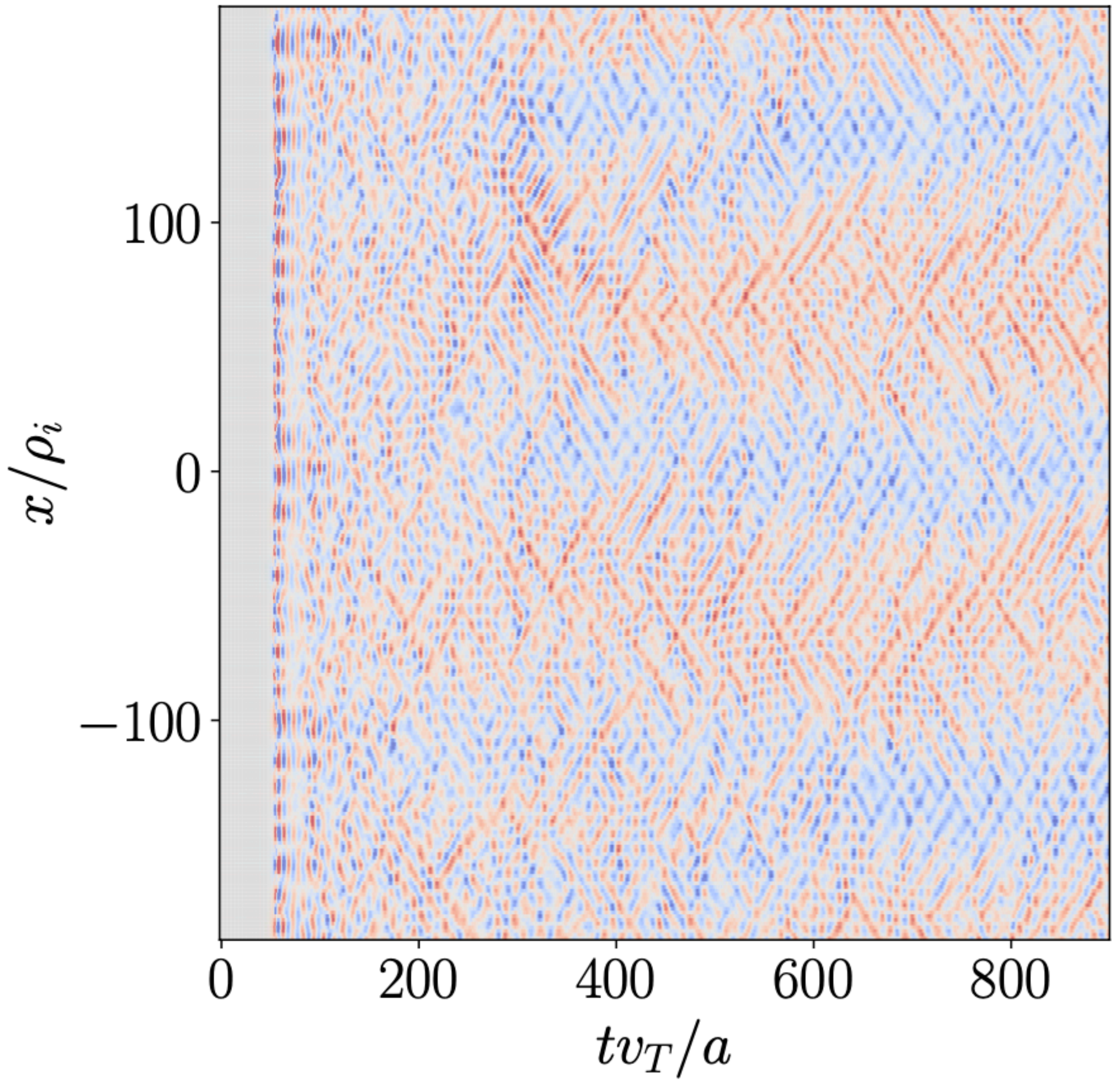

At $k_x\rho \ll 1$, GAM and stationary zonal flow, but also $\omega \sim k_x$ mode.

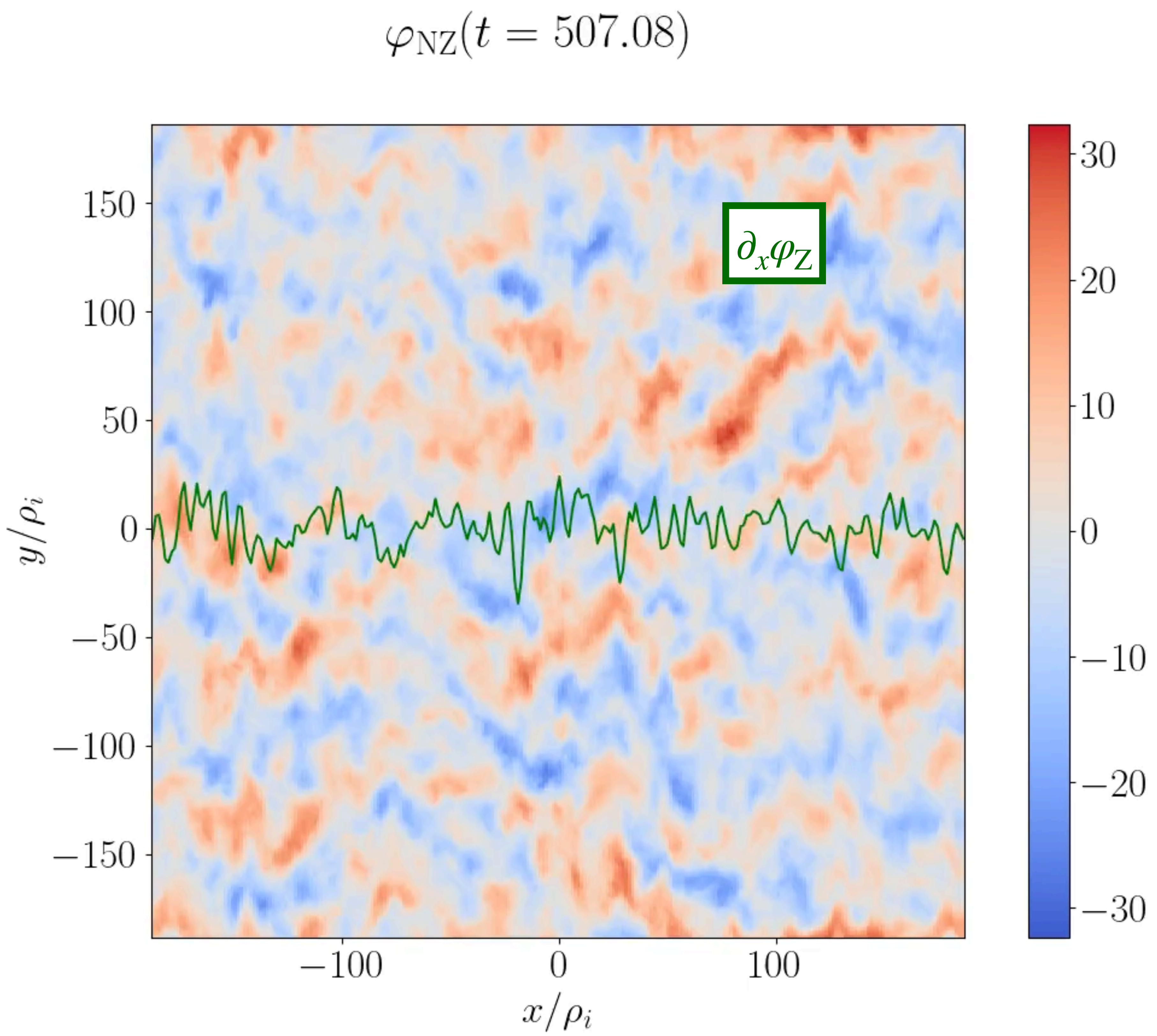
Potential connection with critical balance:

Can think of $\omega_\parallel \sim \omega_{NL}$ as $L_\parallel$ being limited by information propagation speed.

Similarly, $L_x$ would be set by mode speed.

Alternative interpretation: zonal shearing inefficient [6] for $\omega_{ZF} \gtrsim \omega_{NL}$



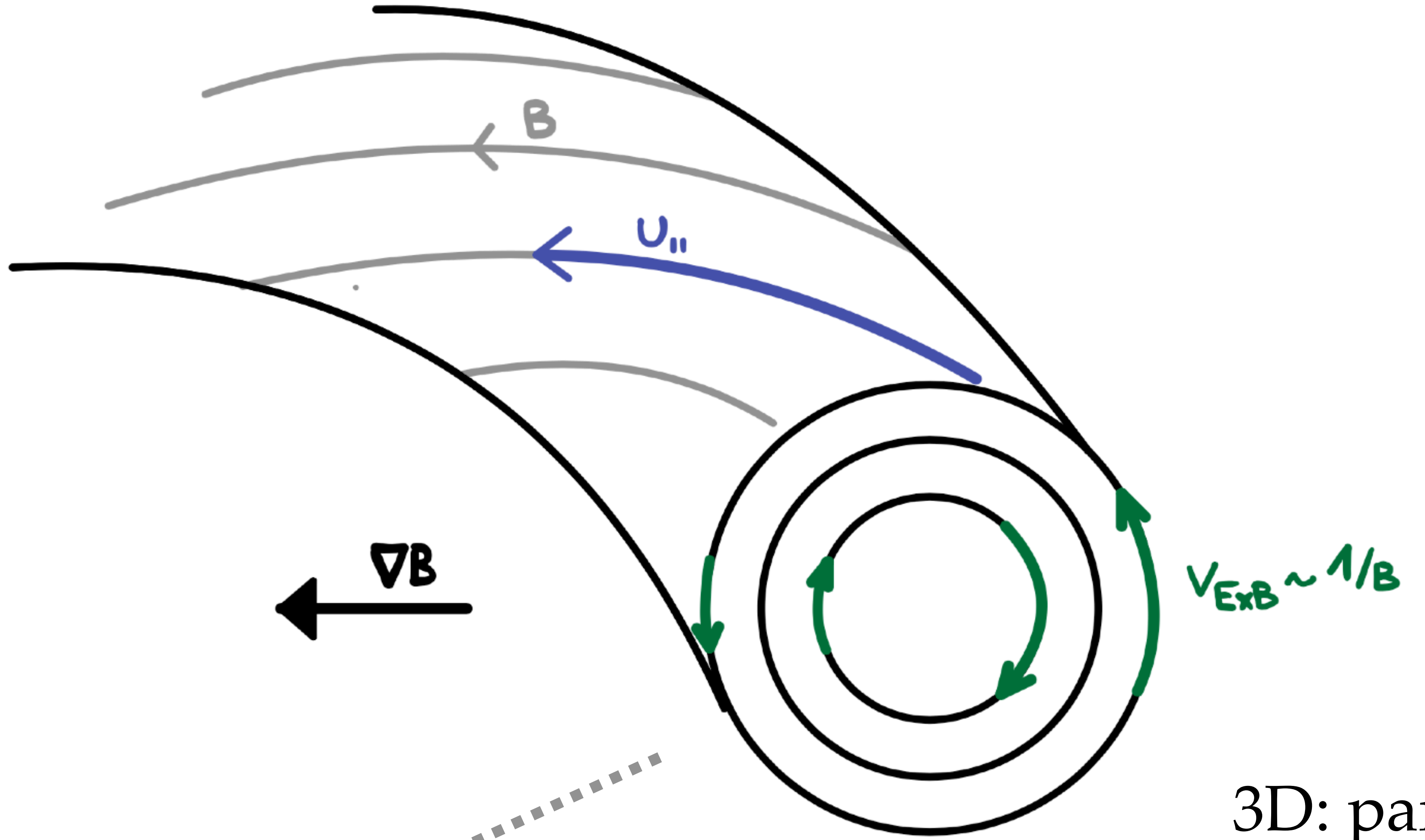




[6] Hahm *et al.* POP 1999

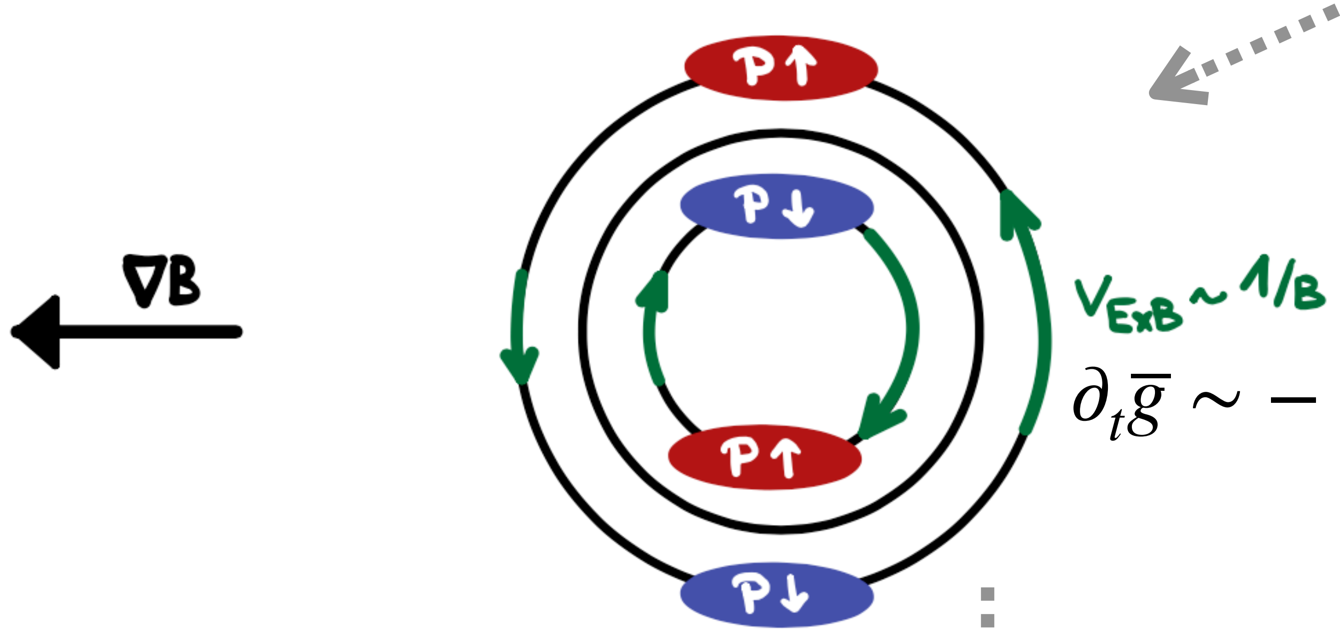
Vorticity equation for long wavelengths ($k_\perp \rho_i \ll 1$):

$$\partial_t \left[ \frac{-\nabla_\perp^2 \overline{\varphi}}{2} + \tau \left( \overline{\varphi} - \langle \overline{\varphi} \rangle_\parallel \right) \right] = - v_{Mx} \partial_x \left( \overline{\varphi} + \overline{P} \right) + \frac{1}{2} \overline{\nabla x \cdot \nabla \tilde{\varphi} \, \partial_y \left( \tilde{\varphi} + \tilde{P}_\perp \right)}$$

& GKE: $\partial_t \overline{g} = -(v_\parallel \nabla_\parallel + v_{Mx} \partial_x)(\overline{g} + \overline{\varphi} F_M) - \partial_x \overline{\tilde{v}_{Ex} \tilde{g}}$

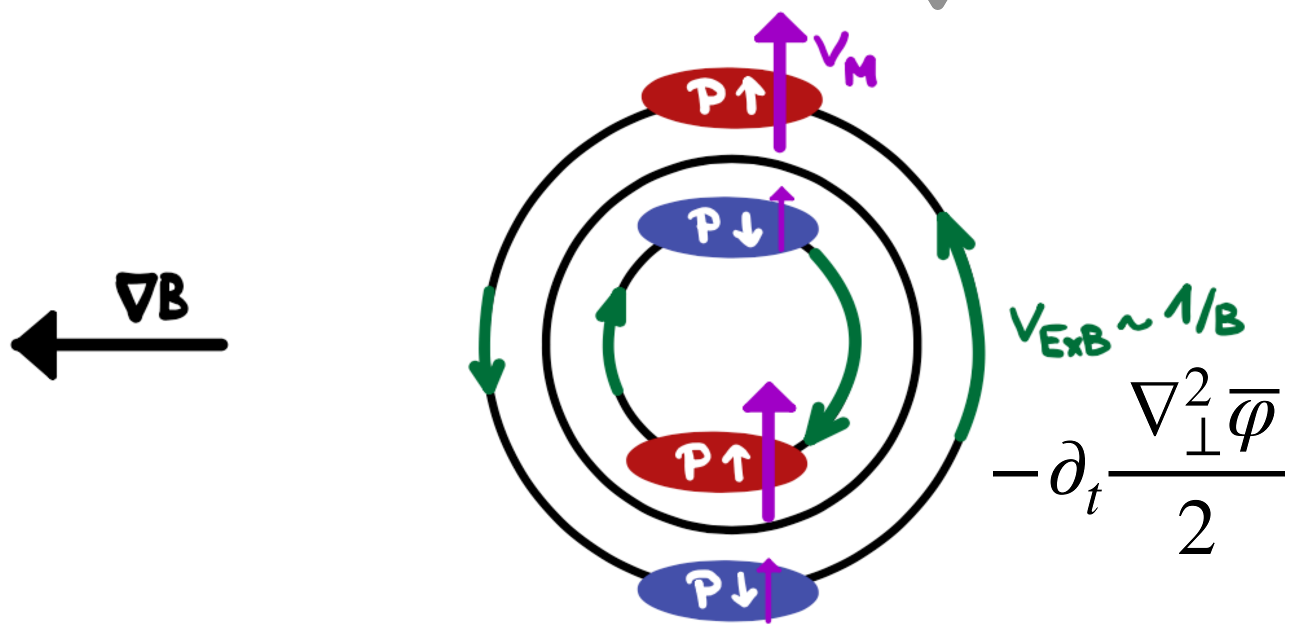



$\nabla \cdot v_{E\times B} \neq 0$




$\partial_t \overline{g} \sim - v_{Mx} \partial_x \overline{\varphi} F_M$

$\nabla \cdot J_{Mx} \neq 0$

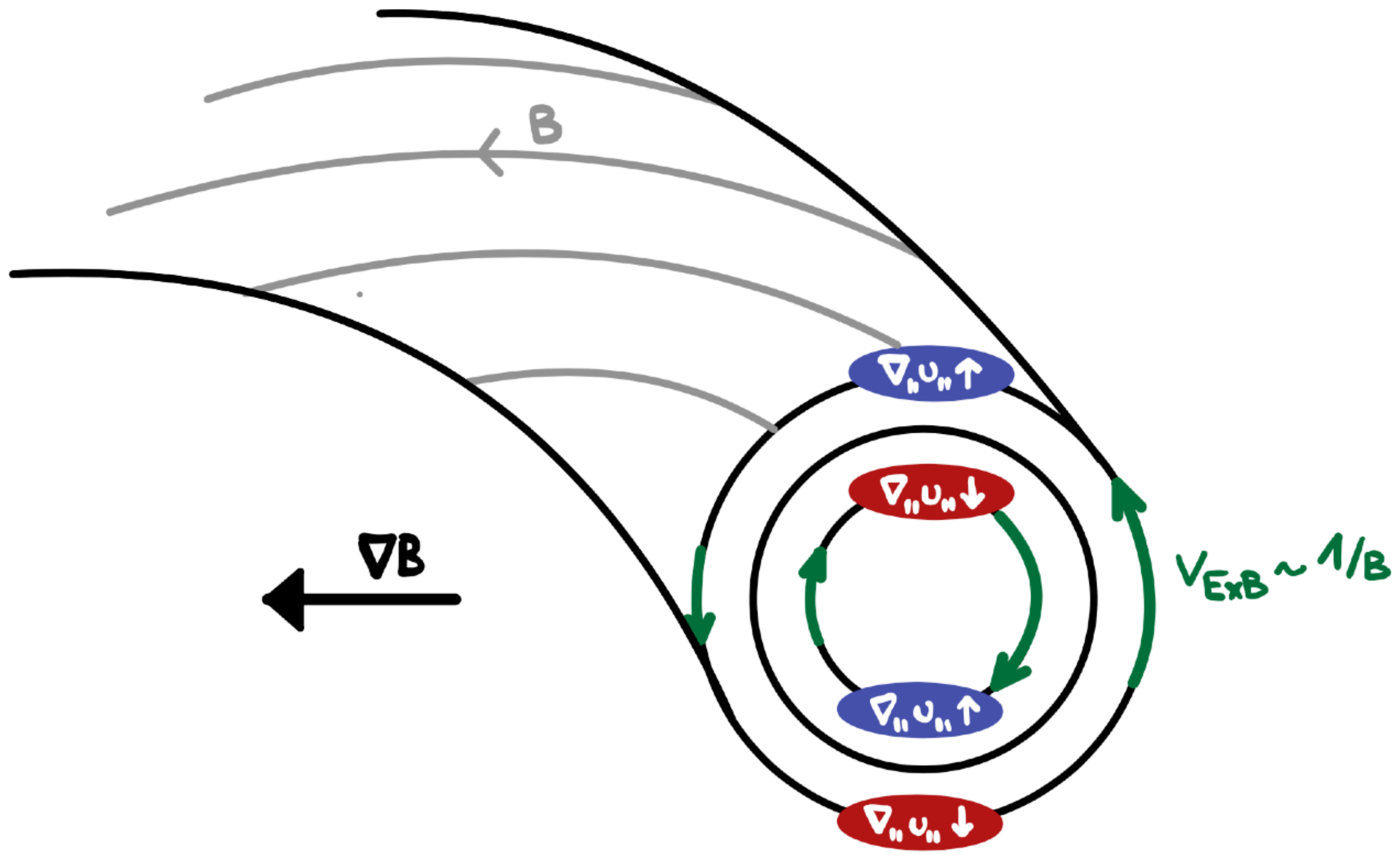



$-\partial_t \frac{\nabla_\perp^2 \overline{\varphi}}{2} \sim - v_{Mx} \partial_x \overline{P}$

Geodesic acoustic mode (GAM): $\omega = \frac{v_{Mx}}{\rho_i |\nabla x|} \sqrt{\frac{7}{4} + \frac{1}{\tau}}$

3D: parallel dynamics

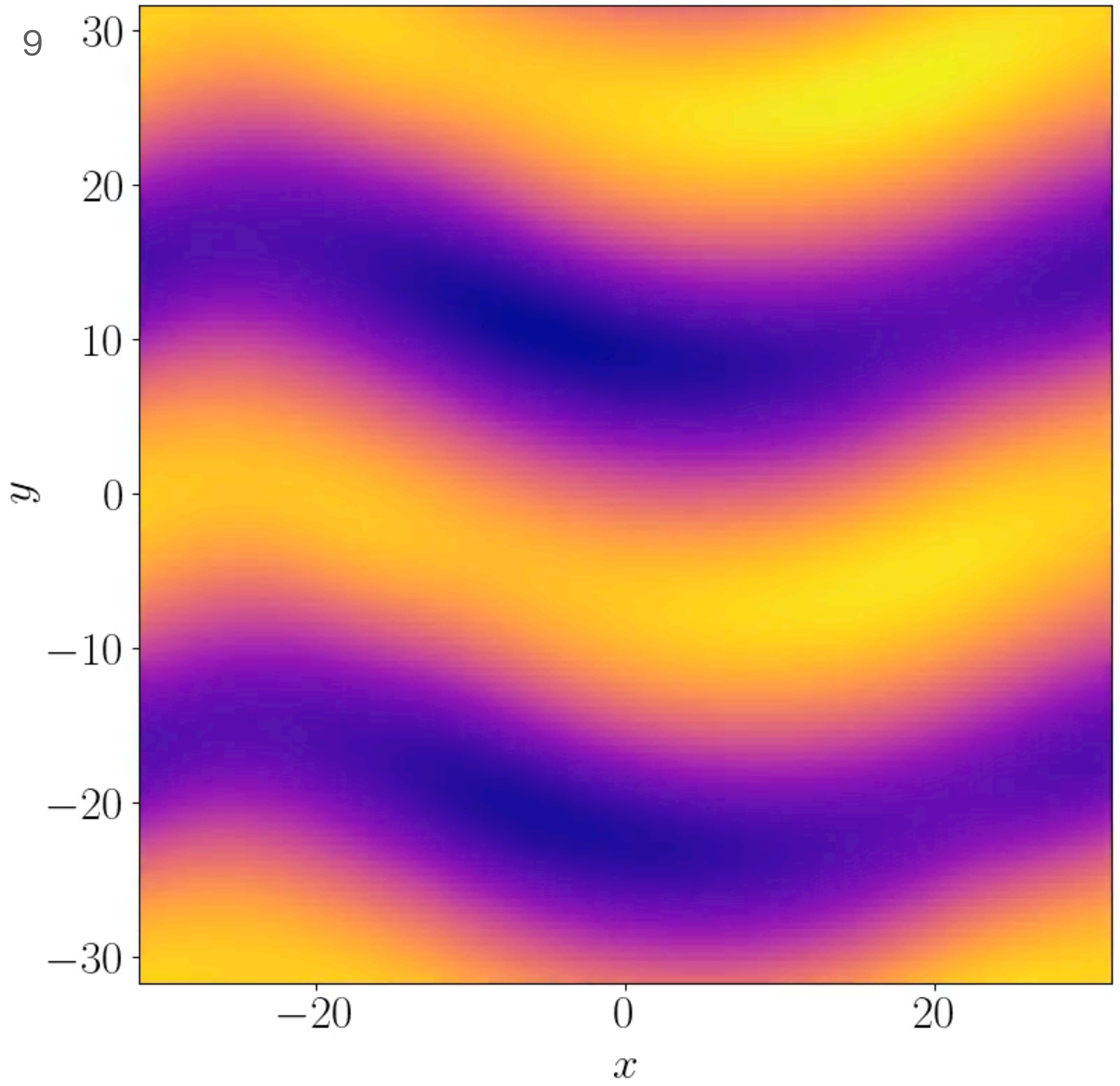
Can now avoid compression and have stationary flows (Pfirsch-Schlueter-like)




[7]: given $\delta f_i(t=0)$, what is $\mathscr{R} = \overline{\varphi}(t \to \infty)/\overline{\varphi}(t=0)$?

$\delta f_i(t=0) = F_M \delta n(x)/n_0 \Rightarrow \ \mathscr{R} = 1/\left(1 + 1.6\, q^2 \sqrt{R/r}\right)$

[7] Rosenbluth-Hinton PRL 1998






**Fast secondary** [8]: instability of primary (e.g. ITG) to zonal perturbation,

$g = g^P + g^S,\ \varphi = \varphi^P + \varphi^S$ with $g^S \ll g^P,\ \varphi^S = \tilde{\varphi}^S + \langle \varphi^S \rangle_\psi \ll \varphi^P$; neglect linear terms:

$$-i\omega g^S = -\left\{ \tilde{\varphi}^S + \langle \varphi^S \rangle_\psi, g^P \right\} - \left\{ \varphi^P, g^S \right\} = -\left\{ \langle \varphi^S \rangle_\psi, g^P \right\}$$

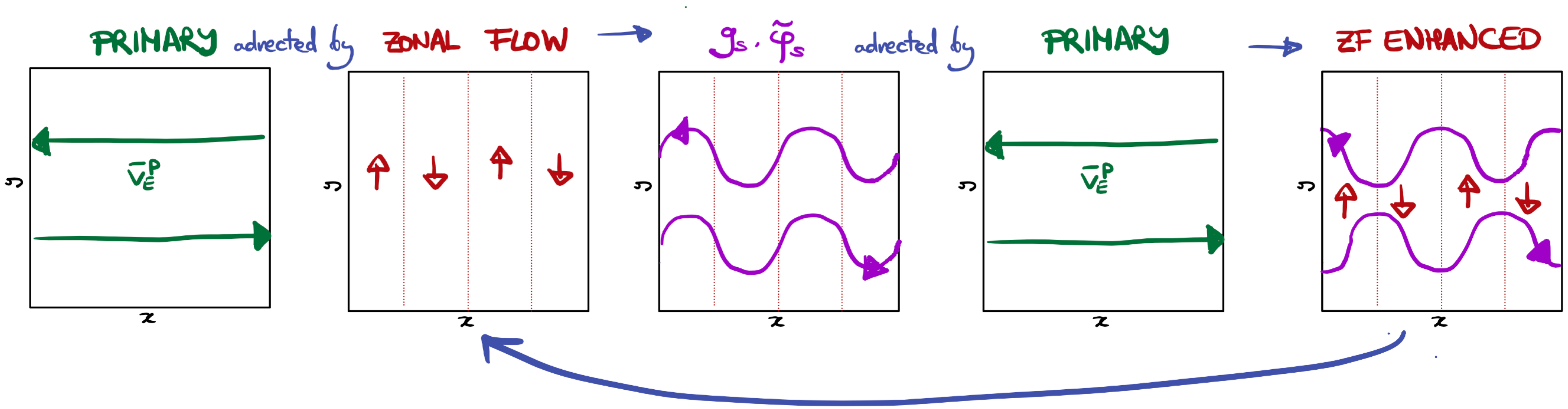
Shearing of primary by zonal flow

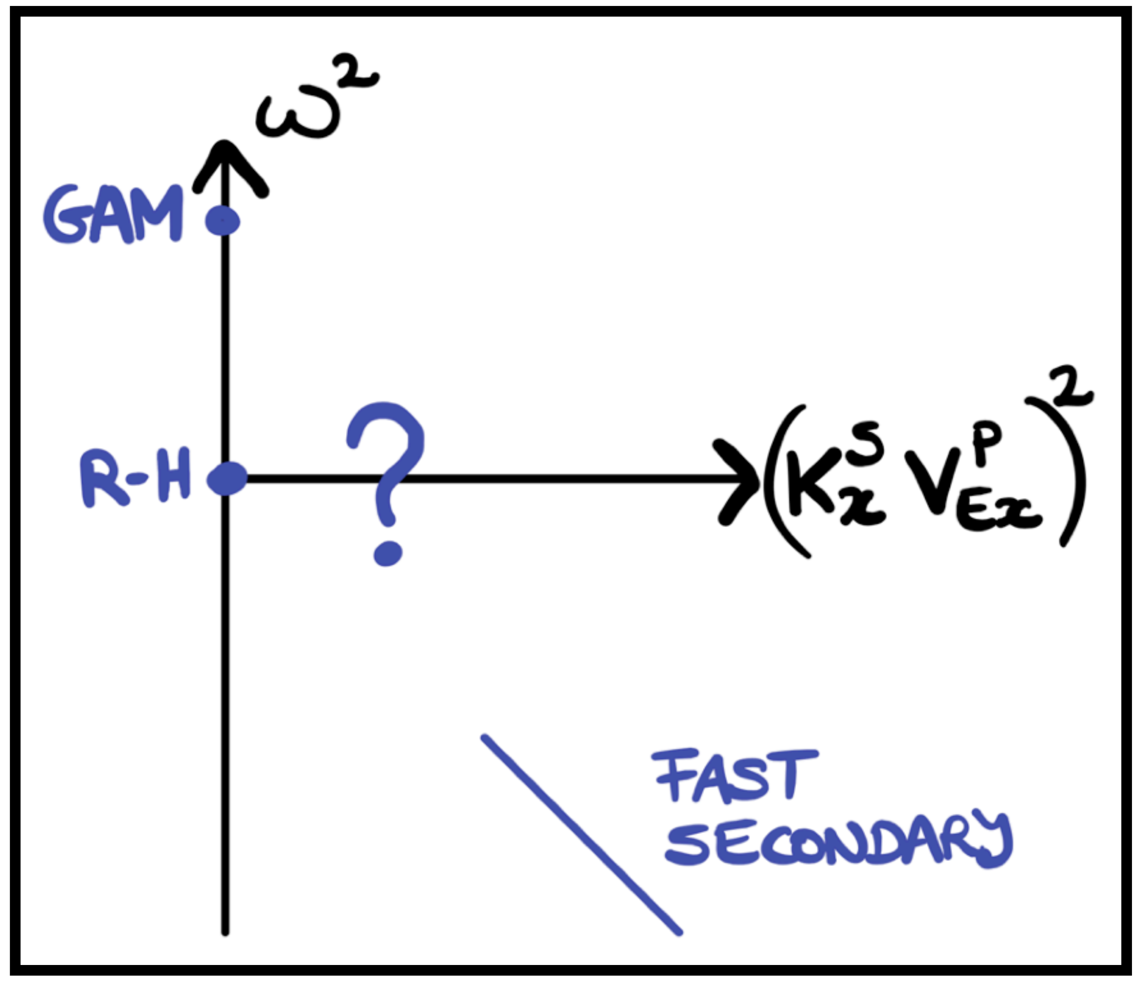
$$-i\omega \tilde{\varphi}^S = -\left\{ \langle \varphi^S \rangle_\psi, \varphi^P \right\}$$

$$-i\omega \left\langle \rho^2 |\nabla x|^2 \partial_x^2 \varphi^S \right\rangle_\psi = \partial_x^2 \left\langle \rho^2 \left[ \nabla x \cdot \nabla \varphi^P \partial_y \left( \tilde{\varphi}^S + P_\perp^S \right) + \nabla x \cdot \nabla \tilde{\varphi}^S \partial_y \left( \varphi^P + P_\perp^P \right) \right] \right\rangle_\psi$$

Transport of $E \times B$ and diamagnetic momentum

$$\Rightarrow \boxed{\omega^2 \left\langle |\nabla x|^2 \rho^2 \right\rangle_\psi \partial_x^2 \langle \varphi^S \rangle_\psi = \partial_x^2 \left( \left\langle |\nabla x|^2 \rho^2\, \partial_y \varphi^P \partial_y \left( \varphi^P + P_\perp^P \right) \right\rangle_\psi \partial_x^2 \langle \varphi^S \rangle_\psi \right)} \rightarrow \gamma \sim k_x^S v_{Ex}^P$$




[8] Rogers-Dorland PRL 2000

LW Vorticity: $-\left\langle \frac{|\nabla x|^2 \rho^2}{2} \partial_x^2 \partial_t \varphi \right\rangle_\psi = -\left\langle v_{Mx} \partial_x (P + \varphi) \right\rangle_\psi - \frac{1}{2} \left\langle \rho^2 \nabla x \cdot \nabla \varphi \partial_y (\varphi + P_\perp) \right\rangle_\psi$

Ordering $\frac{\omega_{\text{lin,GK}}}{\omega} \sim k_\perp \rho \ll 1 \sim \frac{k_x v_{Ex}^P}{\omega} \sim \frac{\omega_{\text{GAM}}}{\omega}$: linear term corrections to secondary enter through $v_{Mx}$ contribution.

*LO:* $-i\omega \tilde{\varphi}_{(0)}^S = -\left\{ \langle \varphi^S \rangle_\psi, \varphi^P \right\}$ $\quad -i\omega g_{(0)}^S = -\left\{ \langle \varphi^S \rangle_\psi, g^P \right\}$ No contribution to $v_{Mx}$ term in vorticity.

*NLO:* usual GAM contribution + $v_{Mx}$ acting on $g_{(0)}^S$ and $\tilde{\varphi}_{(0)}^S$

Dispersion relation: $$\omega^2 = -\gamma_S^2 + \omega_{\text{GAM}}^2 + \left\langle k_x^2 |\nabla x|^2 \rho^2 / 2 \right\rangle_\parallel^{-1} \left\langle \frac{\omega_{Mx}^2}{\omega(\omega + k_x v_{Ex}^P)} \left[ \frac{k_x \partial_y \chi^P}{\omega} + \frac{1}{\tau} \left( \frac{k_x \partial_y P^P}{\omega} \right)^2 \right] \right\rangle_\psi$$

$\chi^P \xrightarrow{\langle \varphi^S \rangle_\psi} \tilde{\chi}_{(0)}^S \xrightarrow{v_{Mx}} \tilde{P}_{(1)}^S(\theta) \xrightarrow[\varphi^P]{\nabla \cdot Q(\theta)} \overline{P}_{(1)}^S(\theta)$ $\qquad P^P \xrightarrow{\langle \varphi^S \rangle_\psi} \tilde{P}_{(0)}^S \xrightarrow{v_{Mx}} \tilde{\varphi}_{(1)}^S(\theta) \xrightarrow[P^P]{\nabla \cdot Q(\theta)} \overline{P}_{(1)}^S(\theta)$

Up-down asymmetric heat flux resulting from magnetic drift acting on sheared primary.

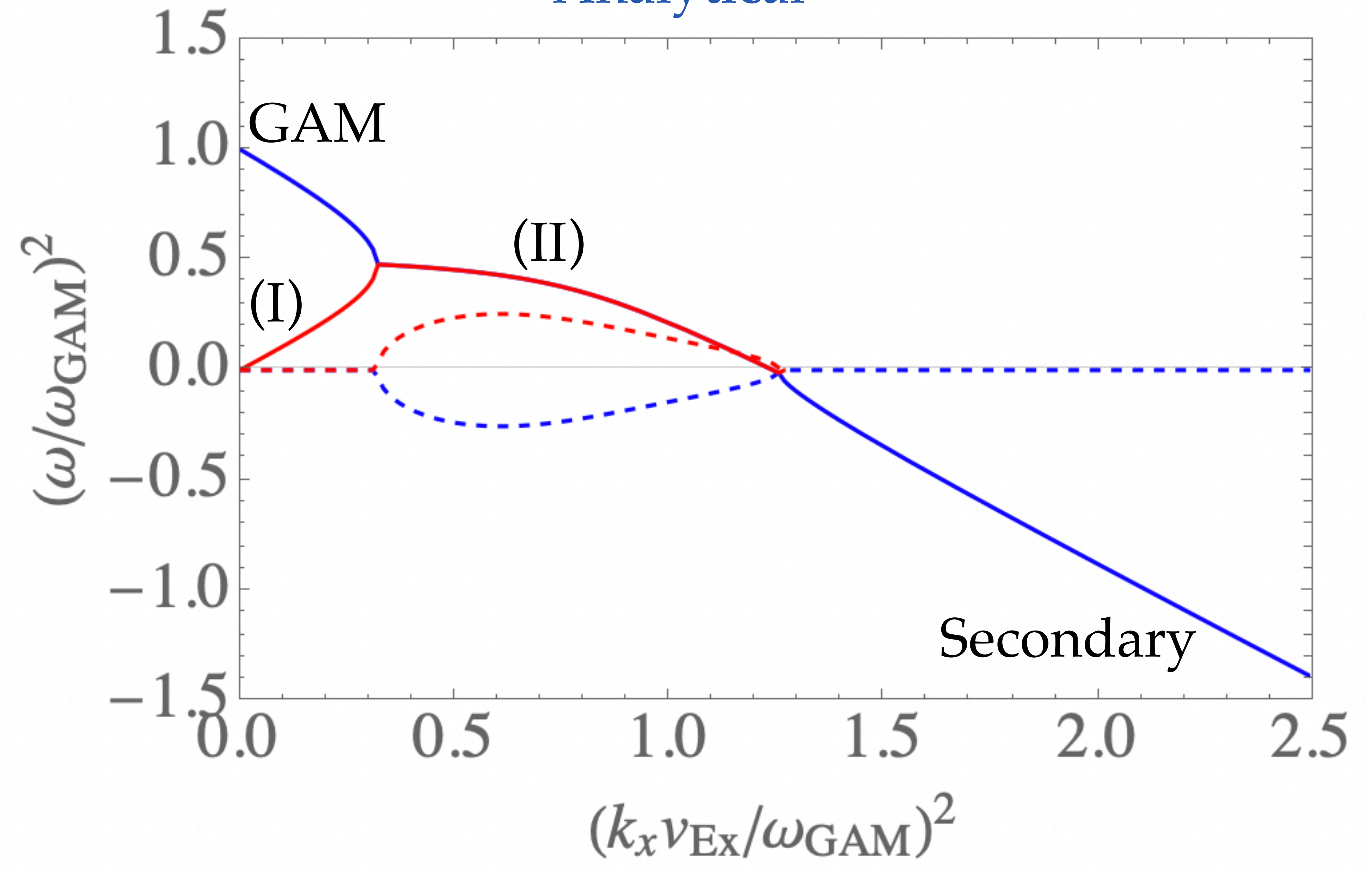
$(P_\perp, P, \chi) = \int d^3 v \, g \left( \frac{v_\perp^2}{v_T^2}, \frac{(v_\parallel^2 + v_\perp^2/2)}{v_T^2}, \frac{(v_\parallel^2 + v_\perp^2/2)^2}{v_T^4} \right)$

Need to evaluate flux-surface average, e.g. if primary is single harmonic in $y$ and $\theta$- independent

$$k_x v_{Ex}^P = \omega_{E0} \sin(k_P y), \quad k_x \partial_y P_P = \tau f_P \omega_{E0} \sin(k_p y + \Delta_P), \quad k_x \partial_y \chi_P = \tau f_\chi \omega_{E0} \sin(k_P y + \Delta_\chi),$$

$$\omega^2 = -(\ldots)\,\omega_{E0}^2 + \omega_{\mathrm{GAM}}^2 \left( 1 + (\ldots) \left( 1 - \frac{1}{\sqrt{1 - (\omega_{E0}/\omega)^2}} \right) + (\ldots) \frac{(\omega_{E0}/\omega)^2}{\sqrt{1 - (\omega_{E0}/\omega)^2}} \right)$$

Analytical

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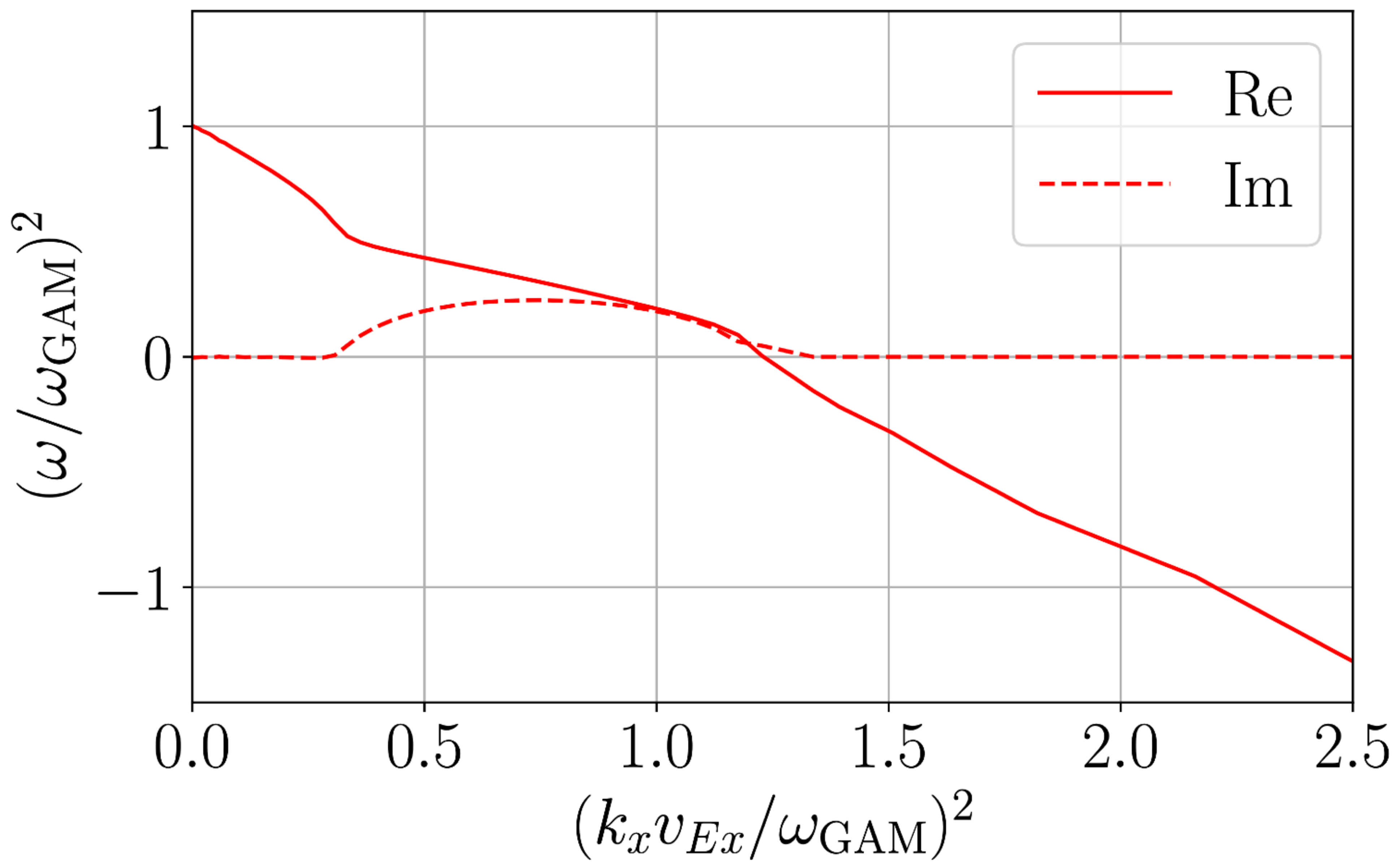


(I) $\omega \sim \omega_{E0} = k_x v_{Ex}^P$, antisymmetric heat flux balances $\nabla \cdot v_{E\times B} \neq 0$ s.t. $\langle v_{Mx} \partial_x (\varphi + P) \rangle_\psi = 0$

(II) Growing and propagating mode, interplay between all terms in vorticity equation.

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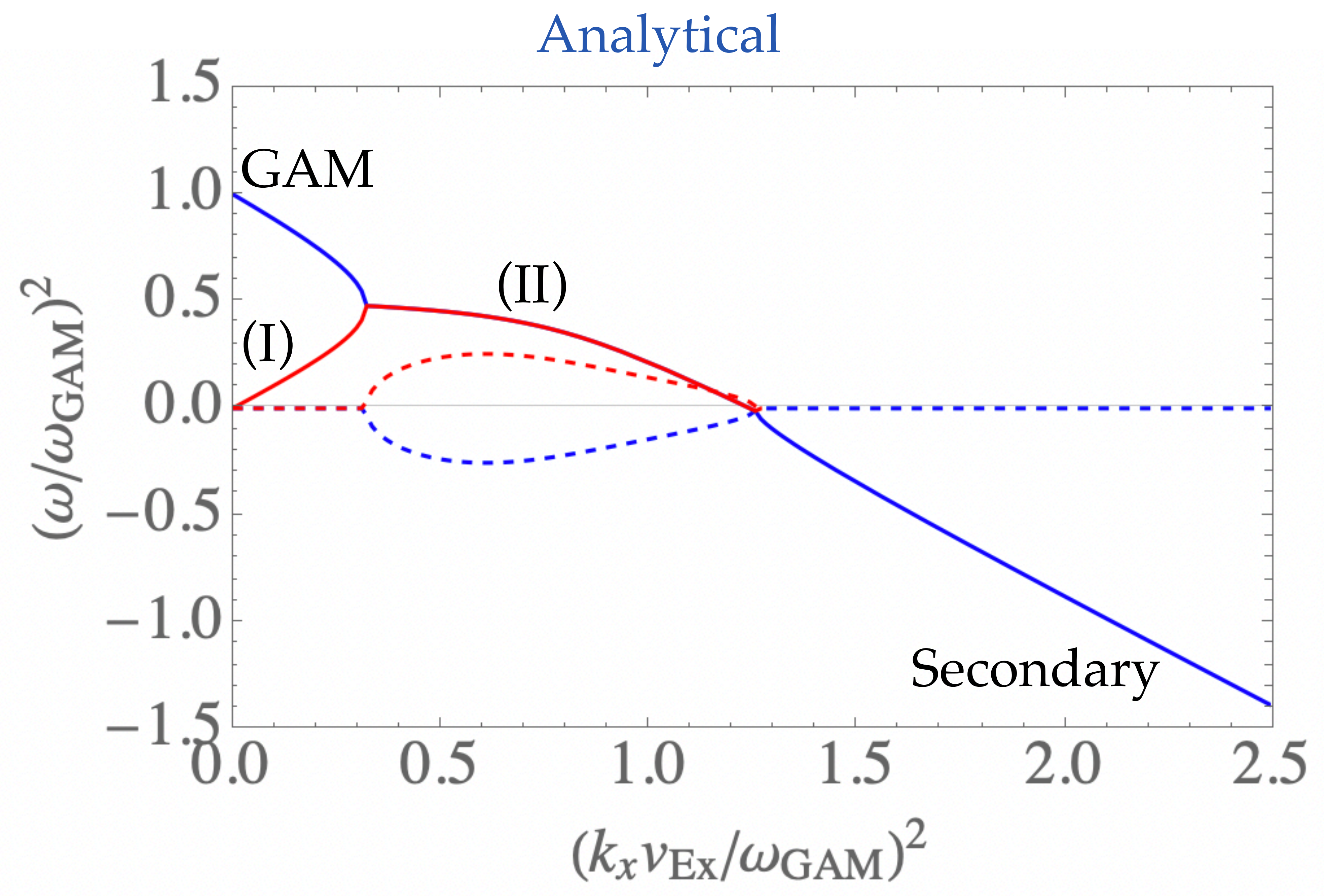
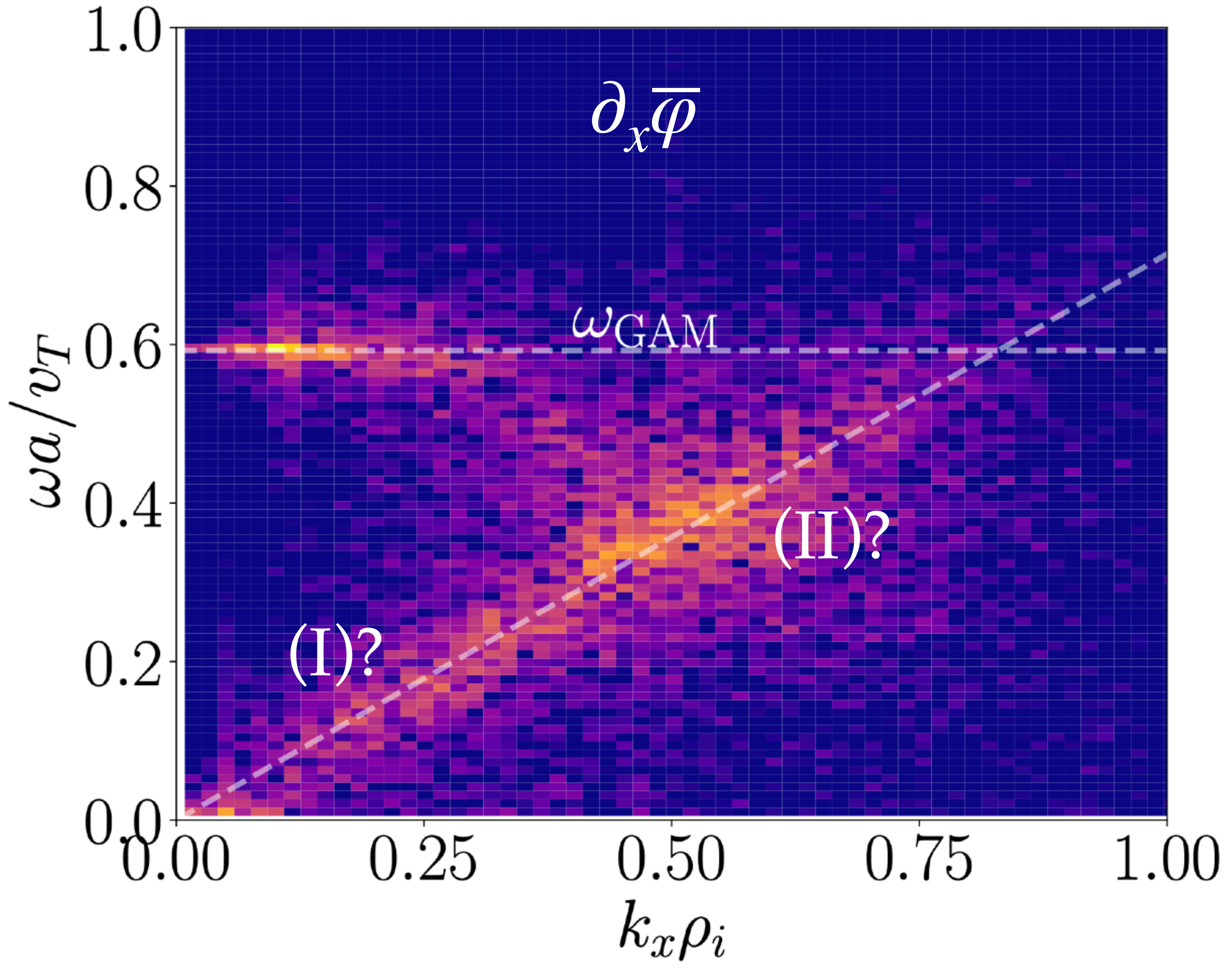
$$k_x v_{Ex}^P = \omega_{E0}\sin(k_P y), \quad k_x \partial_y P_P = \tau f_P \omega_{E0}\sin(k_p y + \Delta_P), \quad k_x \partial_y \chi_P = \tau f_\chi \omega_{E0}\sin(k_P y + \Delta_\chi),$$

$$\omega^2 = -(\ldots)\,\omega_{E0}^2 + \omega_{\mathrm{GAM}}^2\left(1 + (\ldots)\left(1 - \frac{1}{\sqrt{1-(\omega_{E0}/\omega)^2}}\right) + (\ldots)\frac{(\omega_{E0}/\omega)^2}{\sqrt{1-(\omega_{E0}/\omega)^2}}\right)$$

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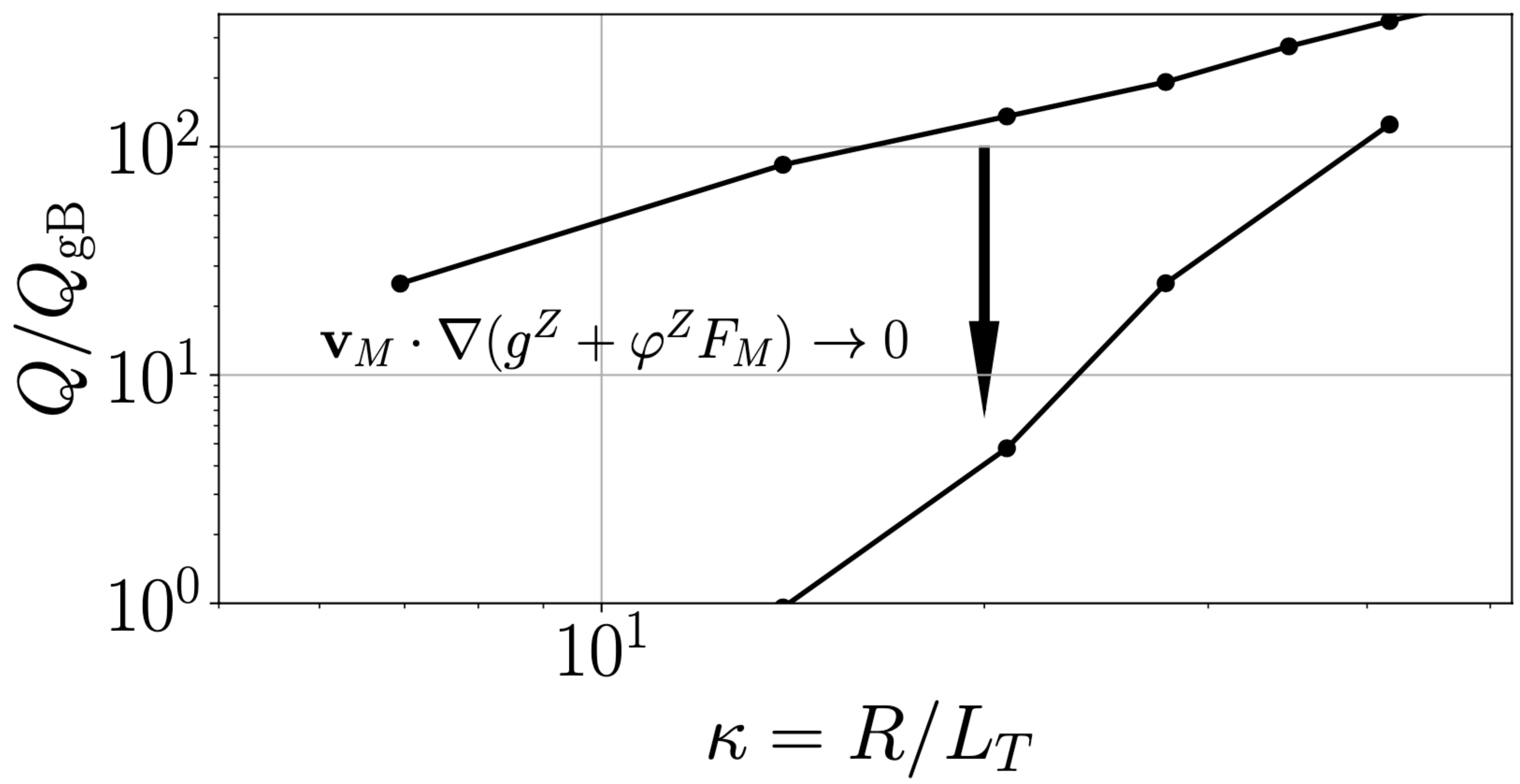
(II) Growing and propagating mode, interplay between all terms in vorticity equation.

**If $Q \propto v_{Mx}$, what happens as $v_{Mx} \to 0$?**

The heat flux of course eventually increases for large enough $R/L_T$, but the nonlinear upshift of $(R/L_T)_C$ is significant.




**Future work:**

- Can we obtain a revised theory of the Dimits shift?
    - So far, all in slab or Z-pinch geometry.
    - [Ivanov *et al.* JPP 2020, 2022]: Dimits shift related to secondary instability threshold. Need to take into account new mode?
- Can we use $v_{Mx}$ as a knob to optimize stellarators for reduced ITG turbulence? So far, optimisation of linear physics or DNS.