MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

Upper bounds on gyrokinetic instabilities

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EUROfusion

Gyrokinetics

Gyrokinetics has come to dominate large parts of theoretical plasma physics.

- Thousands of papers, millions (!) of lines of code.

Most of the literature treats particular linear instabilities and turbulence.

- Zoology: ITG, ETG, KBM, RBM, TEM, TIM, MTM ... (and branches thereof)
- Sensitive to details (geometry, collisions, impurities, beta, ...)

“All these bloody complications of plasma physics are the pain of our life.” A. Schekochihin 24.07.2023

What can be said <u>in general</u>?

- Except for obvious conservation laws etc.

The equations of flux-tube gyrokinetics

Consider the nonlinear gyrokinetic equation:

$$\frac{\partial g_{a\mathbf{k}}}{\partial t} + v_\parallel \frac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \frac{1}{B^2}\sum_{\mathbf{k}'} \mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'} g_{a\mathbf{k}''} = \sum_b \left[C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})\right] + \frac{e_a F_{a0}}{T_a}\left(\frac{\partial}{\partial t} + i\omega_{*a}^T\right)\chi_{a\mathbf{k}}$$

$$\chi_{a\mathbf{k}} = J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\phi_{\mathbf{k}} - v_\parallel A_{\parallel\mathbf{k}}\right) + J_1\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\frac{v_\perp}{k_\perp}\delta B_{\parallel\mathbf{k}}$$

where $\mathbf{B} = B\mathbf{b} = \nabla\psi\times\nabla\alpha,$

$$\mathbf{k} = k_\psi\nabla\psi + k_\alpha\nabla\alpha \qquad \omega_{da} = \mathbf{k}_\perp\cdot\mathbf{v}_{da} \qquad \mathbf{k}'' = \mathbf{k} - \mathbf{k}'$$

$$\omega_{*a} = \frac{k_\alpha T_a}{e_a}\frac{d\ln n_a}{d\psi} \qquad \omega_{*a}^T = \omega_{*a}\left[1 + \eta_a\left(\frac{m_a v^2}{2T_a} - \frac{3}{2}\right)\right] \qquad \Omega_a = \frac{e_a B}{m_a}$$

Field equations

Potentials (in Coulomb gauge, div A = 0) are found from Poisson‘s and Ampère‘s laws:

$$\sum_a \frac{n_a e_a^2}{T_a}\phi = \sum_a e_a \int g_a J_0 d^3v \qquad A_\parallel = \frac{\mu_0}{k_\perp^2}\sum_a e_a \int v_\parallel g_a J_0 d^3v \qquad \delta B_{\parallel \mathbf{k}} = -\frac{\mu_0}{k_\perp}\sum_a e_a \int v_\perp g_{a,\mathbf{k}} J_{1a} d^3v$$

Consider the entropy budget:

- multiply the gyrokinetic equation by $\frac{g_a^*}{F_{a0}}$ and take the real part,
- integrate over flux tube and velocity space:

$$\mathrm{Re}\sum_a T_a \left\langle \int (\cdots)\frac{g_a^*}{F_{a0}} d^3v \right\rangle$$

Entropy budget

This gives

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}}\left[C(\mathbf{k},t) + D(\mathbf{k},t)\right]$$

where

$$H(\mathbf{k},t) = \sum_a \left\langle T_a \int \frac{|g_{a,\mathbf{k}}|^2}{F_{a0}} d^3v - \frac{n_a e_a^2}{T_a} |\delta\phi_{\mathbf{k}}|^2 \right\rangle + \left\langle \frac{|\delta \mathbf{B}_{\mathbf{k}}|^2}{\mu_0} \right\rangle$$

$$C(\mathbf{k},t) = \mathrm{Re}\sum_{a,b} T_a \left\langle \int \frac{g_{a\mathbf{k}}^*}{F_{a0}} \left[C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{b\mathbf{k}})\right] d^3v \right\rangle \leq 0$$ (H-theorem)

$$D(\mathbf{k},t) = \mathrm{Im}\sum_a e_a \left\langle \int g_{a,\mathbf{k}} \omega_{*a}^T \bar{\chi}_{a,\mathbf{k}}^* d^3v \right\rangle$$ = entropy production by transport fluxes

Free energy

H(k,t) can be written

$$H(\mathbf{k},t) = U(\mathbf{k},t) - T_a S_a(\mathbf{k},t)$$ = Helmholtz free energy of fluctuations

where

$$U(\mathbf{k},t) = \left\langle \sum_a \frac{n_a e_a^2}{T_a}(1-\Gamma_{0a})|\delta\phi_{\mathbf{k}}|^2 + \frac{|\delta\mathbf{B}|^2}{\mu_0} \right\rangle \to \left\langle \sum_a \frac{m_a n_a k_\perp^2 |\delta\phi_{\mathbf{k}}|^2}{B^2} + \frac{|\delta\mathbf{B}|^2}{\mu_0} \right\rangle, \qquad k_\perp \rho_a \to 0$$

$$S_a(\mathbf{k},t) = -\left\langle \int \frac{|\delta F_{a,\mathbf{k}}|^2}{F_{a0}} d^3v \right\rangle \qquad \delta F_{a,\mathbf{k}} = g_{a,\mathbf{k}} - \frac{e_a J_{0a} \delta\phi_{\mathbf{k}}}{T_a} F_{a0}$$

$$-\int (F_0 + \delta F) \ln(F_0 + \delta F)\, d^3v = -\int \left[F_0 \ln F_0 + (1 + \ln F_0)\, \delta F + \frac{\delta F^2}{2F_0} \right] d^3v$$

Entropy production

In the relation

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}} \left[C(\mathbf{k},t) + D(\mathbf{k},t)\right]$$

D measures the production of free energy due to transport:

$$D = \mathrm{Re}\sum_a T_a \left\langle \int g_a \left(\mathbf{v}_E^* + \delta\mathbf{v}_d^* + v_\parallel \delta\mathbf{b}^*\right)\cdot\nabla F_{a0} d^3v \right\rangle = -\mathrm{Re}\sum_a \left\langle T_a \Gamma_a \frac{d\ln p_a}{d\psi} + q_a \frac{d\ln T_a}{d\psi}\right\rangle$$

$$\Gamma_a = \left\langle \int g_a \left(\mathbf{v}_E^* + \delta\mathbf{v}_d^* + v_\parallel \delta\mathbf{b}^*\right)\cdot\nabla\psi \; d^3v \right\rangle$$

$$q_a = \left\langle \int g_a \left(\frac{m_a v_\perp^2}{2} - \frac{5T_a}{2}\right) \left(\mathbf{v}_E^* + \delta\mathbf{v}_d^* + v_\parallel \delta\mathbf{b}^*\right)\cdot\nabla\psi \; d^3v \right\rangle$$

Bounds on instability growth rates

First consider linear instability with a single k. Thanks to the H-theorem

$$\gamma \leq \frac{D}{H}$$

where the quadratic form *D* is bounded from above and *H* from below.

- **Implies universal upper bounds on <u>all</u> gyrokinetic instabilities.**

Modes of optimal growth

The best possible upper bound is obtained by maximising

$$\Lambda[g] = \frac{D[g]}{H[g]}, \qquad g = \{g_a\}$$

The distribution function that maximises this ratio satisfies the eigenvalue problem

$$\frac{\delta D}{\delta g} - \Lambda \frac{\delta H}{\delta g} = 0$$

If the distribution function at t=0 is chosen in this way, the free energy will momentarily grow at the rate Λ.

Different from the usual gyrokinetic linear stability problem.

- Solutions correspond to “modes of optimal growth“ rather than linear eigenmodes.

Eigenmodes vs modes of optimal growth

An eigenmode is an exponentially growing solution to the linearised gyrokinetic equation

$$g_{\text{lin}} \sim e^{(\gamma - i\omega_r)t}$$

In contrast, a “mode of optimal growth“ maximises the ratio $\Lambda[g_{\text{opt}}] = \frac{D[g_{\text{opt}}]}{H[g_{\text{opt}}]}$

The instantaneous growth rate of the free energy is then maximised.

- This growth rate may, or may not, be sustainable. g_{opt} does not depend on time.
- It equals or exceeds the linear growth grate.

$$\Lambda[g_{\text{opt}}] \geq \gamma$$

- Can easily be computed: corresponds to (at most) a 6-dimensional matrix eigenvalue problem.
  - Size of matrix: 2 x number of fields

Example: hydrogen plasma with adiabatic electrons

In this case

$$\varphi = \frac{1}{n(1+\tau)} \int g J_0 d^3 v \qquad \left(\varphi = \frac{e\delta\phi_{\mathbf{k}}}{T_i}, \qquad g = g_{i\mathbf{k}} \right)$$

$$H = nT_i \left\langle \frac{1}{n} \int \frac{|g|^2}{F_{i0}} d^3 v - (1+\tau)|\varphi|^2 \right\rangle \qquad D = \frac{\eta_i \omega_{*i} T_i}{2i} \left\langle \int (\varphi^* g - \varphi g^*) \, x^2 J_{0i} d^3 v \right\rangle$$

where *D* and φ only depend on two moments of g:

$$K_j[g] = \frac{1}{n} \int g x^{2j} J_{0i} d^3 v, \qquad j = (0,1)$$

Therefore, begin by minimising H over all functions g with given values of these moments. Using Lagrange multipliers c0 and c1, we consider the functional

$$H[g] - 2c_0 K_0[g] - 2c_1 K_1[g]$$

Adiabatic electrons, cont‘d

It follows that the minimising function is of the form

$$g = \left(c_0 + c_1 x^2\right) J_{0i} F_{i0}$$

Hence

$$D = \frac{nT_i G(b_i)}{2i(1+\tau)} \left(c_0^* c_1 - c_0 c_1^*\right)$$

$$H = nT_i \left[G_0 \left(1 - \frac{G_0}{1+\tau}\right) c_0 c_0^* + G_1 \left(1 - \frac{G_0}{1+\tau}\right) (c_0^* c_1 + c_0 c_1^*) + \left(G_2 - \frac{G_1^2}{1+\tau}\right) c_0 c_0^*\right]$$

where

$$G_j(b_i) = \frac{1}{n} \int F_{i0} x^{2j} J_{0i}^2 d^3v, \qquad G(b) = G_0(b_i) G_2(b_i) - G_1^2(b_i)$$

and the problem has been reduced to finding the minimum ratio of two quadratic forms in c_0 and c_1.

Bound on instabilites with adiabatic electrons

Upper bound on any instability with adiabatic electrons:

$$\gamma \leq \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau-G_0(b_i)]}} \qquad b_i = k_\perp^2 \rho_i^2, \quad \tau = \frac{T_i}{T_e}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i \Gamma_0(b_i)\Gamma_1(b_i) - b_i^2 \Gamma_1^2(b_i^2) \qquad \Gamma_n(b) = I_n(b)e^{-b}$$

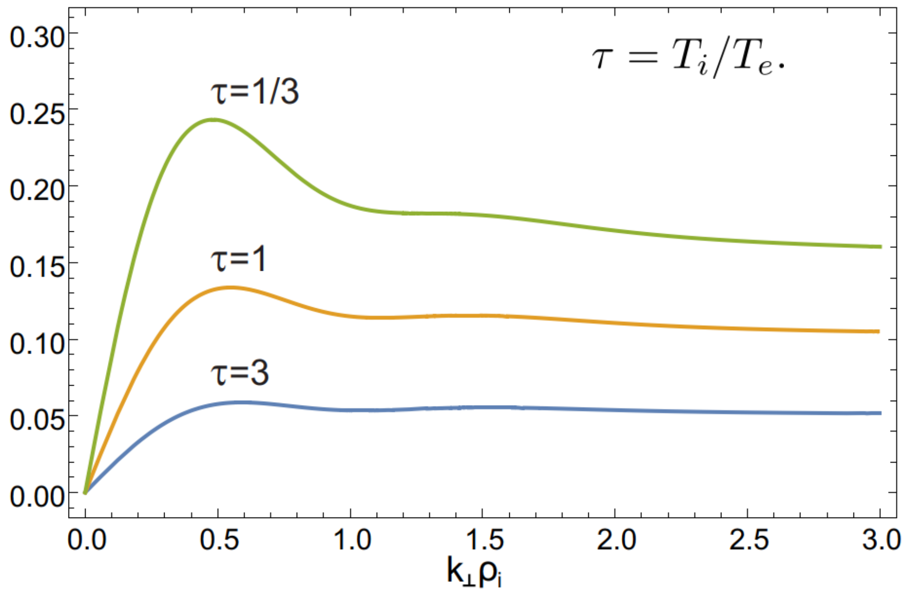
Valid for ITG and trapped-ion instabilities with adiabatic electrons in any magnetic geometry and for any collisionality.

Of order

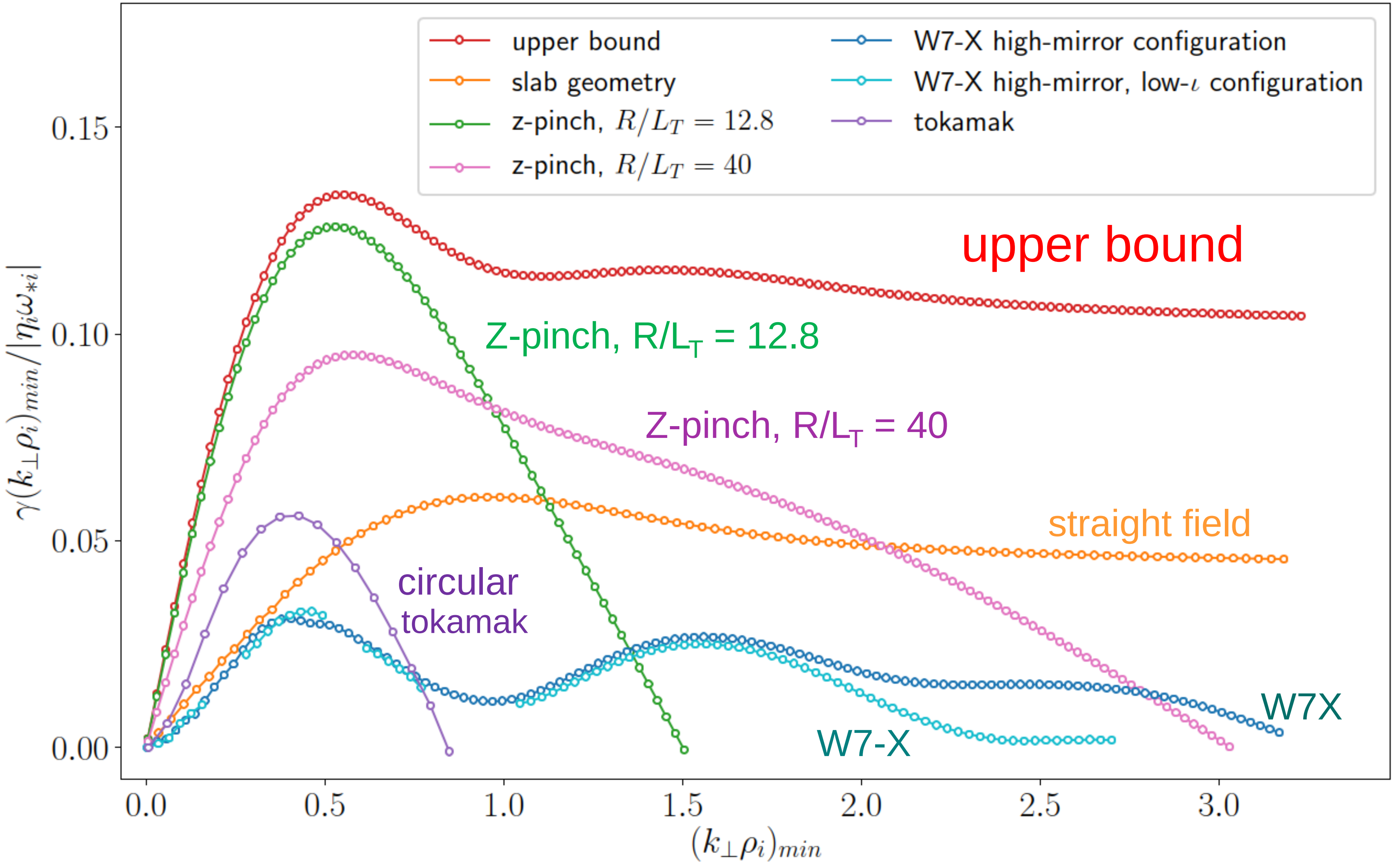
$$\gamma_{\max} \sim \frac{k_\perp \rho_i}{\sqrt{\tau(1+\tau)}} \cdot \frac{v_{Ti}}{L_\perp}, \qquad k_\perp \rho_i \leq 1$$

$$\gamma_{\max} \sim \frac{v_{Ti}}{(1+\tau)L_\perp}, \qquad k_\perp \rho_i \geq 1$$

Normalised upper bound on growth rate vs wave number



Comparison with numerical simulations: adiabatic electrons

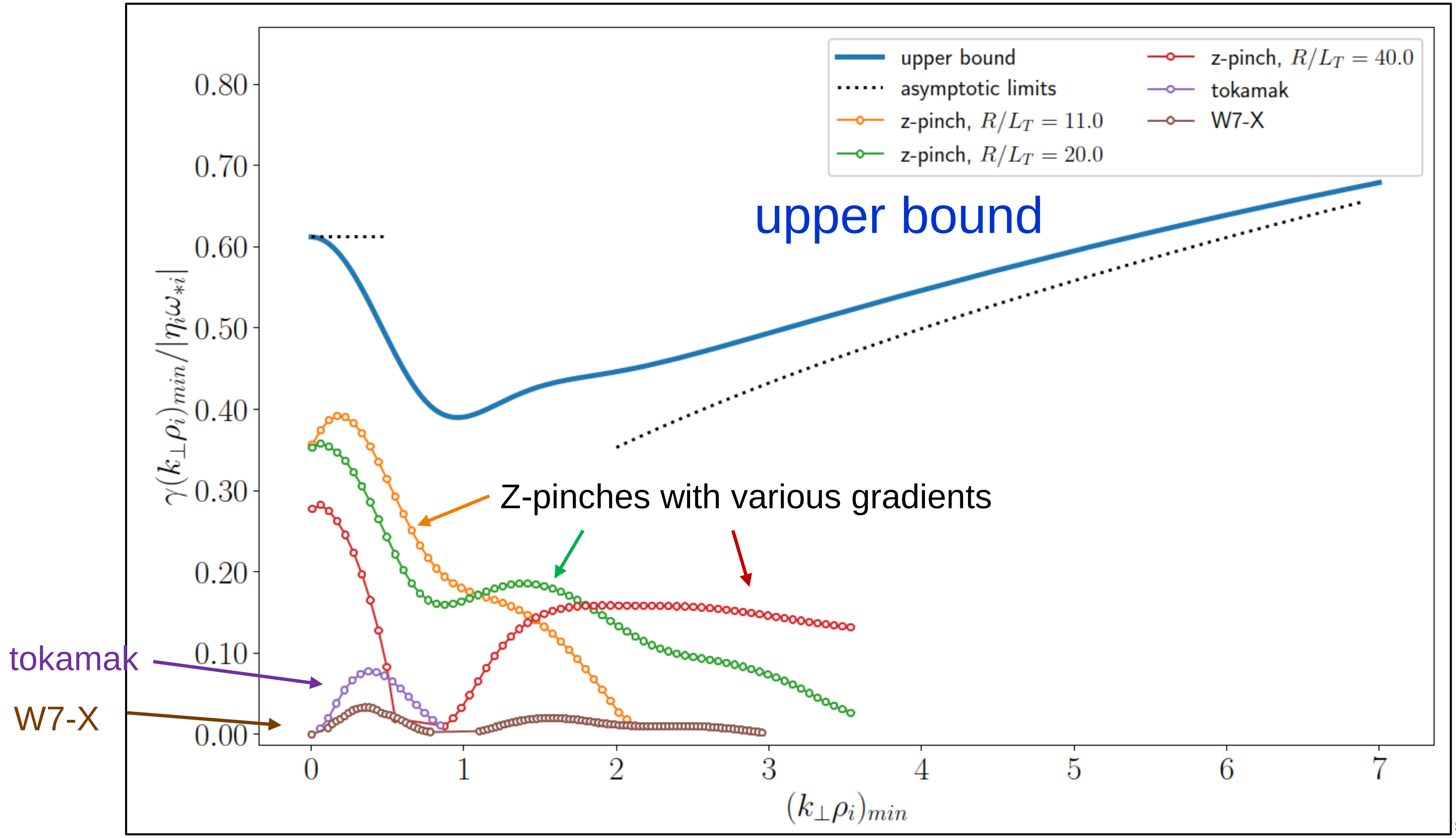


Stella calcuations by

Linda Podavini

Kinetic electrons

When kinetic electrons are included, the bound no longer vanishes in the limit $k_\perp \to 0$



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Bounds on electromagnetic instabilities

Electromagnetic terms arise that are proportional to

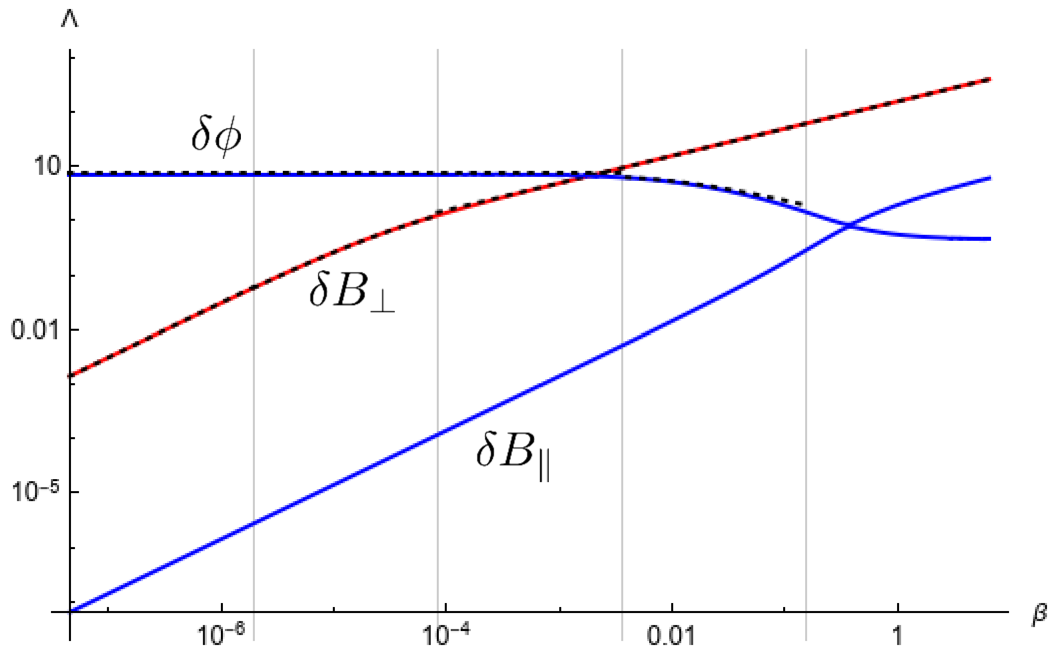
$$\beta_e = \frac{2\mu_0 n_e T_e}{B^2}$$

Can be calculated from a matrix eigenvalue problem.

- Terms from parallel magnetic fluctuations relatively unimportant if $\beta_e << 1$.

Collisions can only lower the bounds.

Upper bound in the limit $k_\perp \rho_e \ll k_\perp \rho_i \ll 1$



Bounding the bounds

A non-optimal bound is

$$\frac{\gamma}{|\omega_{*e}|} \leq \sqrt{\frac{\tau(\Gamma_{0i}+\tau)}{(1+\tau)(1-\Gamma_{0i})}} \left(\sqrt{\tau M(\eta_i, b_i)} + \sqrt{1+\frac{3\eta_e^2}{2}} \right) + \beta_e \sqrt{\frac{1+2\eta_a+7\eta_e^2/2}{2b_e\left(\beta_e+2b_e\right)}}, \qquad k_\perp \rho_e \ll 1$$

$$\frac{\gamma}{|\omega_{*e}|} \leq \frac{\tau}{1+\tau} \sqrt{\frac{1-\eta_e+5\eta_e^2/4}{2\pi b_e(l_0)}}, \qquad k_\perp \rho_e \gg 1$$

$$M(\eta, b) = \left(1+\frac{3\eta^2}{2} - 2\eta(1+\eta)b + 2\eta^2 b^2\right)\Gamma_0(b) + \eta b\left(2+\eta-2\eta b\right)\Gamma_1(b)$$

Nonlinear growth

Since

$$D(\mathbf{k}, t) \leq \gamma_{\text{bound}}(\mathbf{k}) H(\mathbf{k}, t).$$

the growth of the sum

$$H_{\text{tot}}(t) = \sum_{\mathbf{k}} H(\mathbf{k}, t),$$

is bounded by

$$\frac{dH_{\text{tot}}}{dt} \leq 2 \sum_{\mathbf{k}} \gamma_{\text{bound}}(\mathbf{k}) H(\mathbf{k}, t) \leq 2\gamma_{\max} H_{\text{tot}}$$

$$\boxed{\gamma_{\text{bound}}(\mathbf{k}) < \gamma_{\max} \quad \text{for all } \mathbf{k}}$$

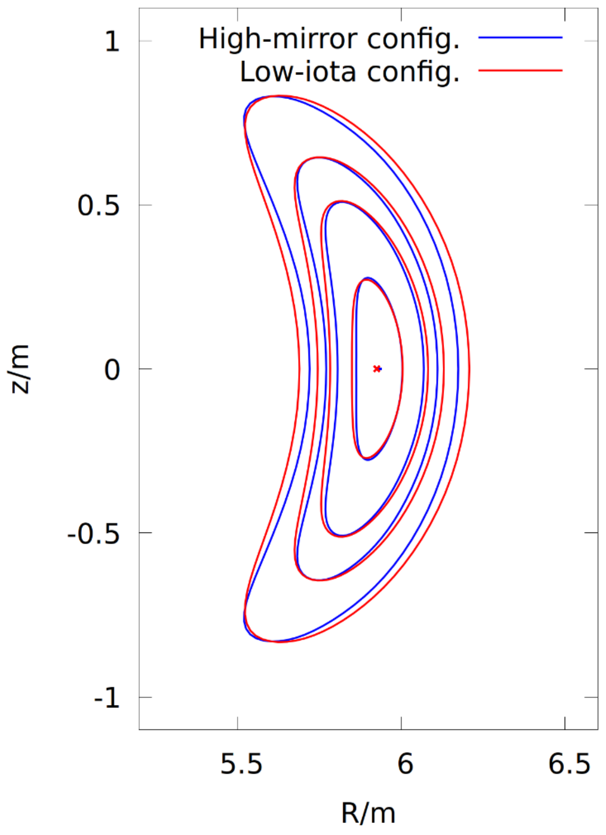
- In the absence of collisions, the free energy can grow momentarily at any rate up to this bound.
- If the plasma is linearly stable, this growth is followed by damping.

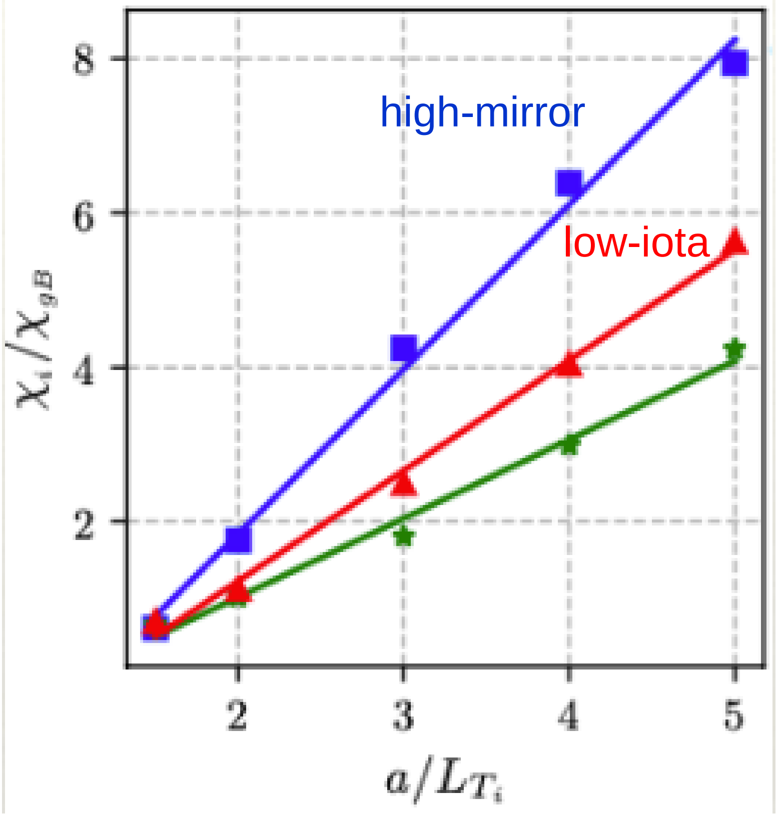
Dependence on geometry

- These bounds are general and thus insensitive to magnetic-field geometry.
  - …except for depence on

$$\eta_i \omega_{*i} \sim \frac{k_\perp |\nabla T_i|}{eB}$$

- Discriminates between configurations with different flux-surface compression.
  - Example: low-iota and high-mirror configurations in W7-X.





Stroteich, Xanthopoulos, Plunk and Schneider (2022)

More dependence on geometry with different choice of energy.

Extensions

1. At low beta, the electrostatic energy satisfies

$$E = \left\langle (\tau + 1 - \Gamma_0) \frac{n_i e_i^2}{T_i} |\delta\phi|^2 \right\rangle$$

$$\frac{d}{dt} \sum_{\mathbf{k}} E = 2 \sum_{\mathbf{k}} K, \qquad K = -\mathrm{Re}\, e_i \left\langle \int \overline{\delta\phi}^* \left(v_\parallel \frac{\partial}{\partial l} + i\omega_d \right) g d^3 v \right\rangle$$

and one can consider the growth of

$$\tilde{H} = H - \Delta E$$

where Δ is a free parameter to be optimised over. The result is an upper bound that depends on the geometry of the magnetic field.

2. If the electrons are fast, $\omega \ll k_\parallel v_{Te}$, we can constrain their distribution function by $\nabla_\parallel g_e = 0$.

Conclusions

- **Rigorous upper bounds can be derived on the growth rates of gyrokinetic instabilities.**
  - Valid both linearly and nonlinearly.
- **These bounds apply for any magnetic geometry (flux-tube), any collisionality, and for any number of species.**
- **Apply to all branches of the ITG, ETG, TEM, TIM, KBM, and MTM instabilities.**
  - For ion-scale instabilities

$$\gamma \leq \frac{C v_{Ti}}{L_\perp}, \qquad C(\beta, k_\perp \rho_i, \eta_i, \eta_e, T_i/T_e, \mathbf{B}) = \mathcal{O}(1)$$

- **The bounds reflect dependencies on gradients, temperatures, and wave numbers derived in a large number of special cases derived over the years.**

References

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