Particle orbits near rational flux surfaces in stellarators

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Outline

1. Motivation
2. Passing particles
3. Adiabatic invariant
4. Semi-trapped particles
5. Summary and future work

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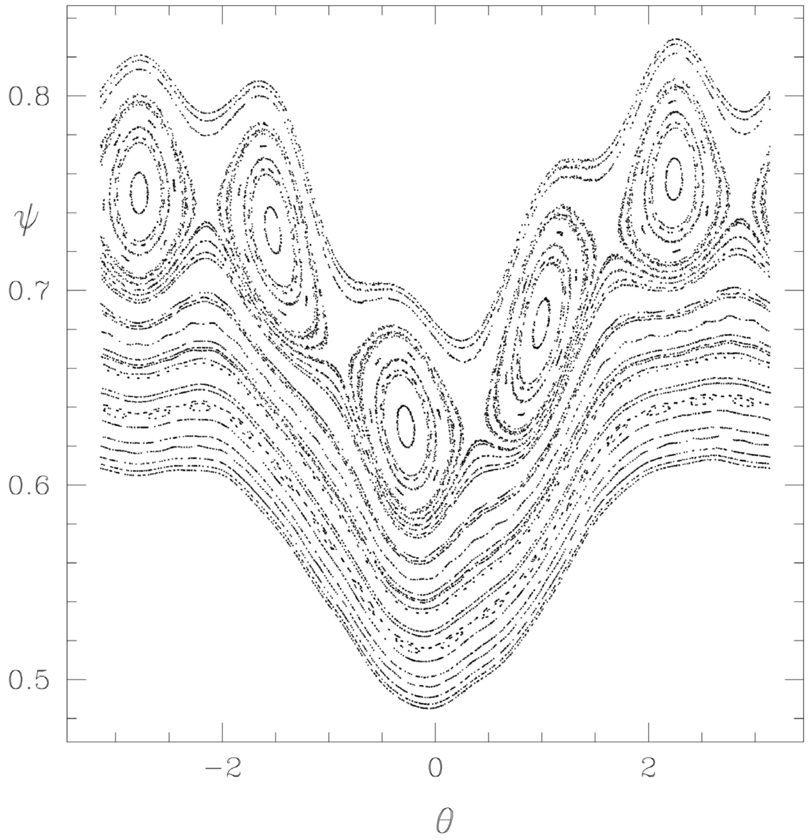
Motivation

- Collisionless guiding centre simulations show islands in the orbits of energetic particles.
  White (2022, Phys. Plasmas)
  White, Bierwage & Ethier (2022, Phys. Plasmas)
  Wobig & Pfirsch (2001, Plasma Phys. Control. Fusion)
- These islands exist around rational flux surfaces. The island width increases with energy, so they could lead to increased energetic particle transport.
- Aim: to understand particle orbits around rational flux surfaces and the properties of the islands in these orbits.



NCSX $\iota = 3/5$ resonance
Credit: Roscoe White

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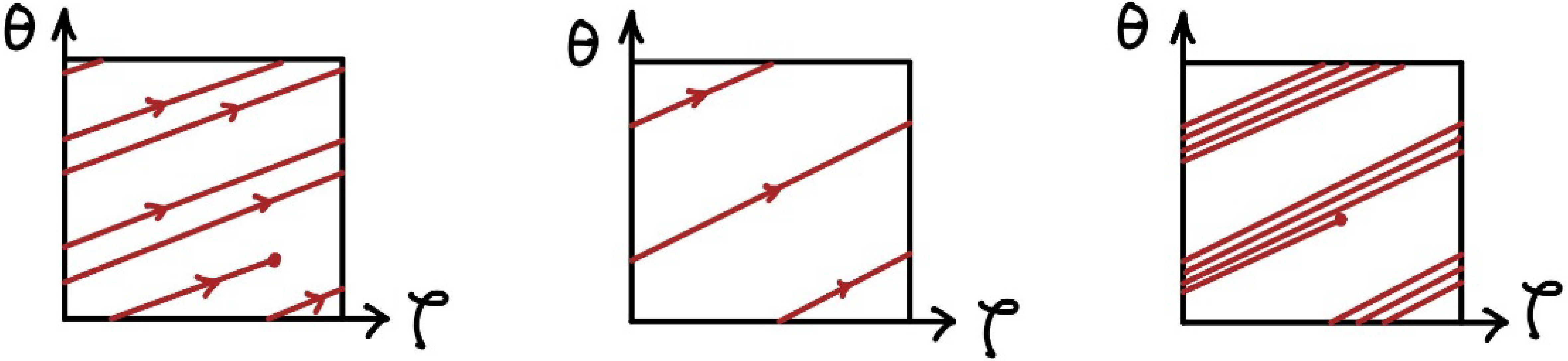
Passing Particle Orbits

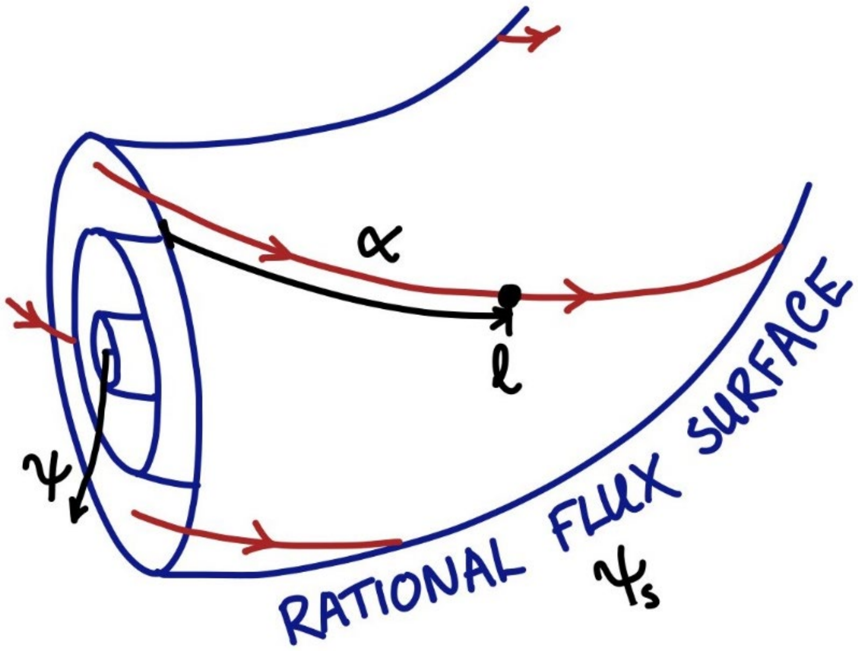
- On irrational flux surface, the radial drift of passing particles averages out and their orbit width is $\sim \rho_* L\,, \quad \rho_* = \rho/L\,.$
- On a rational surface, the particle does not sample the entire surface, so it can have a net radial drift.
- Just away from the rational surface, the particle takes a long time to cover the surface. Overall, it has a larger orbit width $\sim \sqrt{\rho_*}\,L\,.$
- We expand in $\sqrt{\rho_*} \ll 1\,.$



Guiding Centre Equations

- Coordinates: $(\psi, \alpha, l)\,, \quad \alpha = \theta - \iota(\psi_s)\zeta$
- Guiding centre equations:

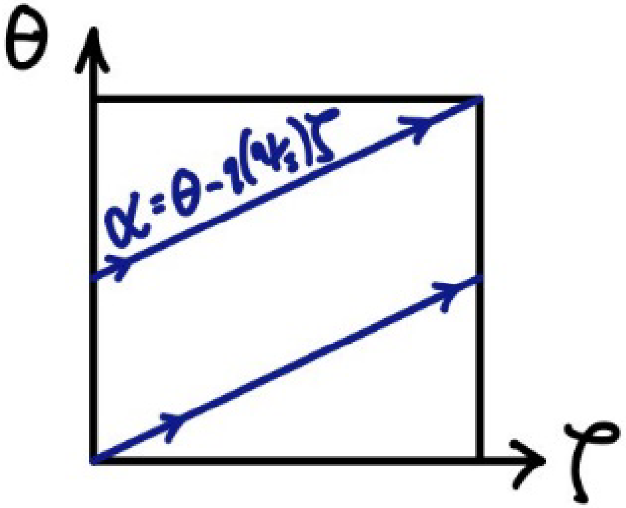
$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \boldsymbol{v}_{\mathrm{d}} \cdot \boldsymbol{\nabla}\psi\,, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \underbrace{v_{\parallel}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\alpha}_{\text{Shear}} + \underbrace{\boldsymbol{v}_{\mathrm{d}} \cdot \boldsymbol{\nabla}\alpha}_{\text{Poloidal drift}}$$

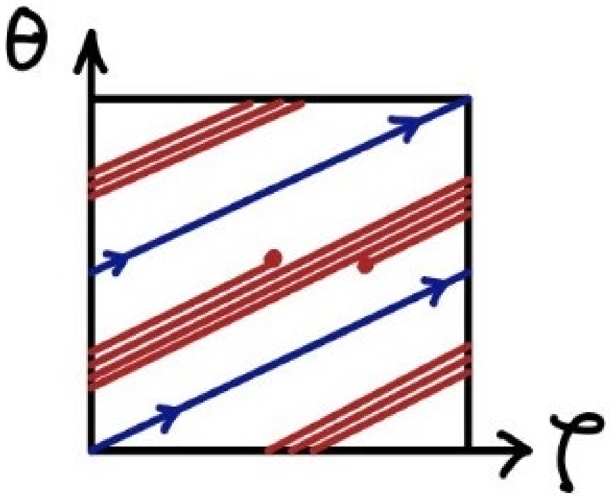


$$\boldsymbol{v}_{\mathrm{d}} = \frac{c}{ZeB}\hat{\boldsymbol{b}} \times \left(M v_{\parallel}^2 \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}} + \mu \boldsymbol{\nabla} B\right)$$

$$v_{\parallel} = \pm\sqrt{\frac{2}{m}(E - \mu B)}$$

Physical picture for shear term:





Guiding Centre Equations

- Order $\psi - \psi_s \sim \sqrt{\rho_*}\,\psi_s$ and Taylor expand shear term:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \boldsymbol{v}_\mathrm{d}\cdot\boldsymbol{\nabla}\psi\,, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = (\psi-\psi_s)\partial_\psi\left(v_\parallel\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\alpha\right)$$

- Time-average over fast transit motion along closed, rational-surface field lines:

$$\langle\ldots\rangle = \frac{1}{\tau_\mathrm{t}}\oint\frac{\ldots}{|v_\parallel|}\,\mathrm{d}l\,, \qquad \tau_\mathrm{t} = \oint\frac{1}{|v_\parallel|}\,\mathrm{d}l$$

$$\iota(\psi_s) = m/n$$

$$\left\langle\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle = \frac{c}{Ze\tau_\mathrm{t}}\partial_\alpha j\,, \qquad \left\langle\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right\rangle = \pm\frac{2\pi n}{\tau_\mathrm{t}}\iota'(\psi_s)(\psi-\psi_s)$$

$$j(\alpha) = \oint_s M|v_\parallel|\,\mathrm{d}l$$

- Characteristics:

$$J(\psi,\alpha) = j(\alpha) \mp \frac{Ze}{c}\pi n\iota'(\psi-\psi_s)^2 = \text{const.}$$

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Adiabatic Invariant

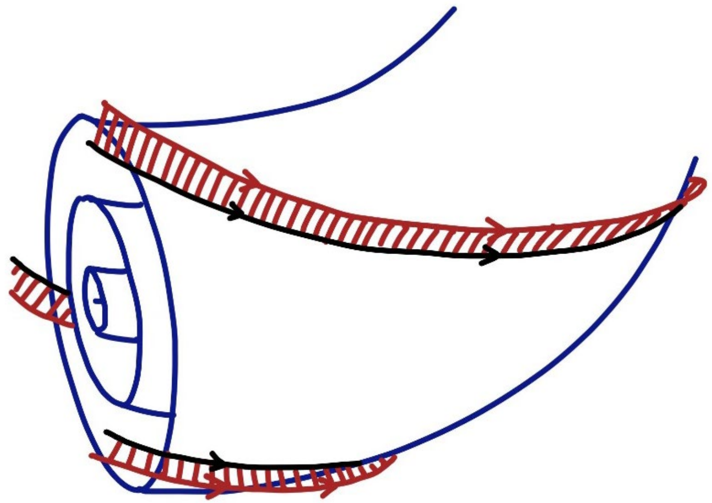
- "Transit invariant"

$$J(\psi,\alpha) = j(\alpha) \mp \frac{Ze}{c}\pi n \iota'(\psi-\psi_s)^2 = \text{const.} \qquad j(\alpha) = \oint_s M|v_\parallel|\,\mathrm{d}l$$

- Equivalent to usual formula for adiabatic invariants in Hamiltonian systems:

$$J = \oint \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{q}\,, \qquad \boldsymbol{p} = M\boldsymbol{v} + \frac{Ze}{c}\boldsymbol{A}$$

Hastie, Taylor & Haas (1967, Ann. Phys. (N. Y.))



- Determines island shape:

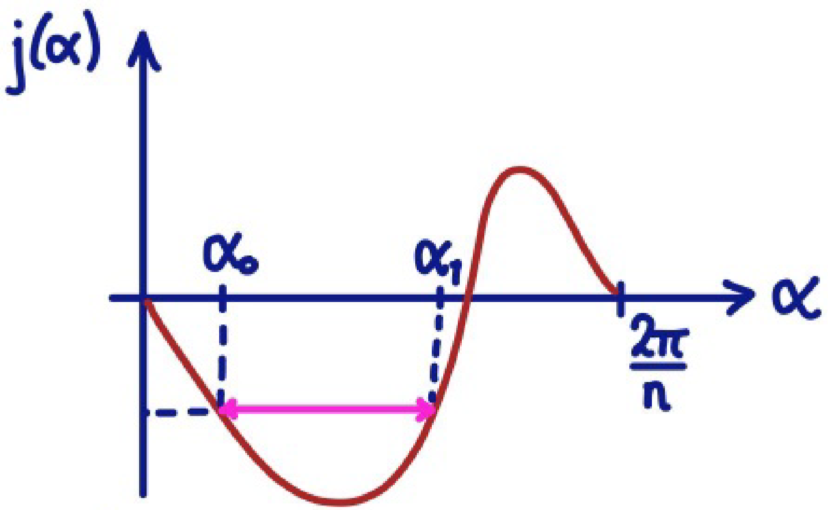
$$\psi-\psi_s = \pm\sqrt{\frac{c[j(\alpha)-j(\alpha_s)]}{\pi n Z e \iota'}}$$

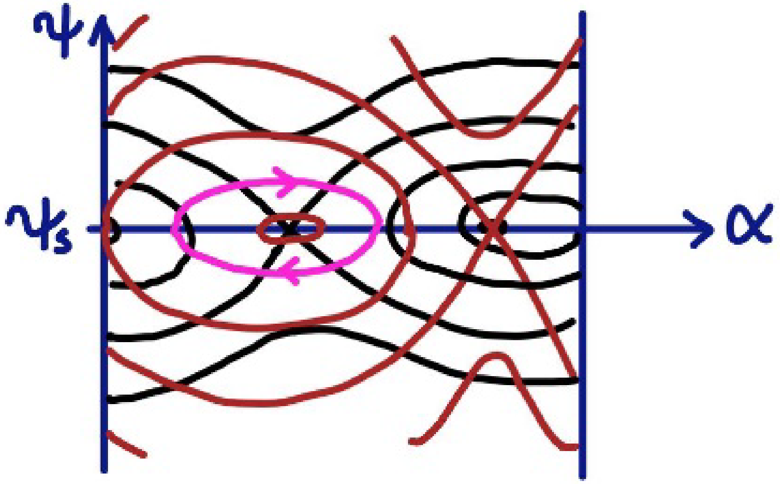
- Island width scaling: $\dfrac{\Delta\psi}{\psi_s} \sim \sqrt{\dfrac{\rho_*\delta}{\epsilon\,\Delta\iota}}$

Adiabatic Invariant

- Transit invariant determines island shape, analogous to phase plot for conservative system.

$$j(\alpha) \mp \frac{Ze}{c}\pi n \iota'(\psi - \psi_s)^2 = \text{const.} \quad \text{vs} \quad U(q) + \frac{1}{2M}p^2 = \text{const.}$$





- Note that co- and counter-passing particles have different islands.

Higher-Order Corrections

- $\sqrt{\rho_*}$ need not be extremely small for energetic particles.
- Observed islands shift away from the rational flux surface as the particle energy is increased – need higher-order formula to see this.
- Similar effect has been studied for runaway electrons: e.g. de Rover, Cardozo & Montvai (1996, Phys. Plasmas)

$$J = \bar{J} + \tilde{J}^{(1/2)}(\bar{\psi}, \bar{\alpha}, \bar{J}, \bar{\vartheta})$$

$$\psi = \bar{\psi} + \tilde{\psi}^{(1/2)}(\bar{\psi}, \bar{\alpha}, \bar{J}, \bar{\vartheta})$$

$$\alpha = \bar{\alpha} + \tilde{\alpha}^{(1/2)}(\bar{\psi}, \bar{\alpha}, \bar{J}, \bar{\vartheta})$$

$$\vartheta = \bar{\vartheta} + \tilde{\vartheta}^{(1/2)}(\bar{\psi}, \bar{\alpha}, \bar{J}, \bar{\vartheta})$$

- Phase space Lagrangian method (similar to gyrokinetics):

$$\mathcal{L} = \left(M v_\parallel \hat{\boldsymbol{b}} + \frac{Ze}{c} \boldsymbol{A} \right) \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - E$$

$$\downarrow$$

$$\mathcal{L} = \frac{\bar{J}\dot{\bar{\vartheta}}}{2\pi} + f(\bar{\psi}, \bar{\alpha}, \bar{J})\dot{\bar{J}} + \frac{Ze}{c}(\bar{\psi} - \psi_s)\,\dot{\bar{\alpha}} - E(\bar{\psi}, \bar{\alpha}, \bar{J})$$

Higher-Order Corrections

- Result:

$$\bar{J} = \overbrace{\oint_s M v_\parallel \,\mathrm{d}l}^{①} + \overbrace{(\psi - \psi_s)\partial_\psi \left(\oint_s M v_\parallel \,\mathrm{d}l \right)}^{② \;= -\frac{Ze\tau_\mathrm{t}}{c}\langle \boldsymbol{v}_\mathrm{d} \cdot \boldsymbol{\nabla}\alpha \rangle} \mp \overbrace{\frac{Ze}{c}\pi n \iota' (\psi - \psi_s)^2}^{③} \mp \overbrace{\frac{1}{3}\frac{Ze}{c}\pi n \iota'' (\psi - \psi_s)^3}^{④}$$

$$\underbrace{- \frac{Ze}{c} \oint_s \int_0^l \frac{\dot{\psi}(l')\dot{\alpha}(l'') - \dot{\psi}(l'')\dot{\alpha}(l')}{v_\parallel(l') v_\parallel(l'')} \mathrm{d}l'' \mathrm{d}l'}_{⑤} + \underbrace{\frac{1}{2}\frac{Ze}{c} \oint_s \int_0^{l'} \frac{\dot{\psi}(l')\dot{\alpha}(l'') - \dot{\psi}(l'')\dot{\alpha}(l')}{v_\parallel(l') v_\parallel(l'')} \mathrm{d}l'' \mathrm{d}l'}_{⑥}$$

- Island shift: $\dfrac{\Delta\psi}{\psi_s} = \dfrac{\rho_*}{\epsilon\, \Delta\iota}$

Numerical Comparison

- Island width scaling and island shape agrees well with simulations

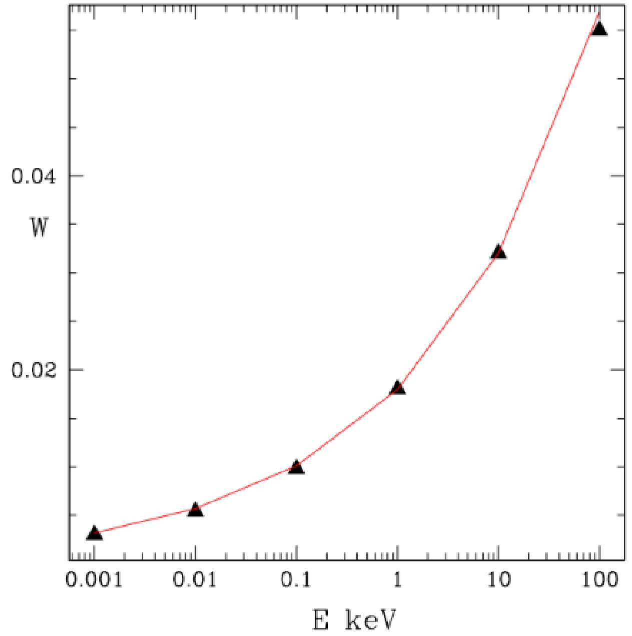


FIG. 9: Island width vs energy in NCSX. The line is $W = 0.018E^{1/4}$

Figure credit: Roscoe White

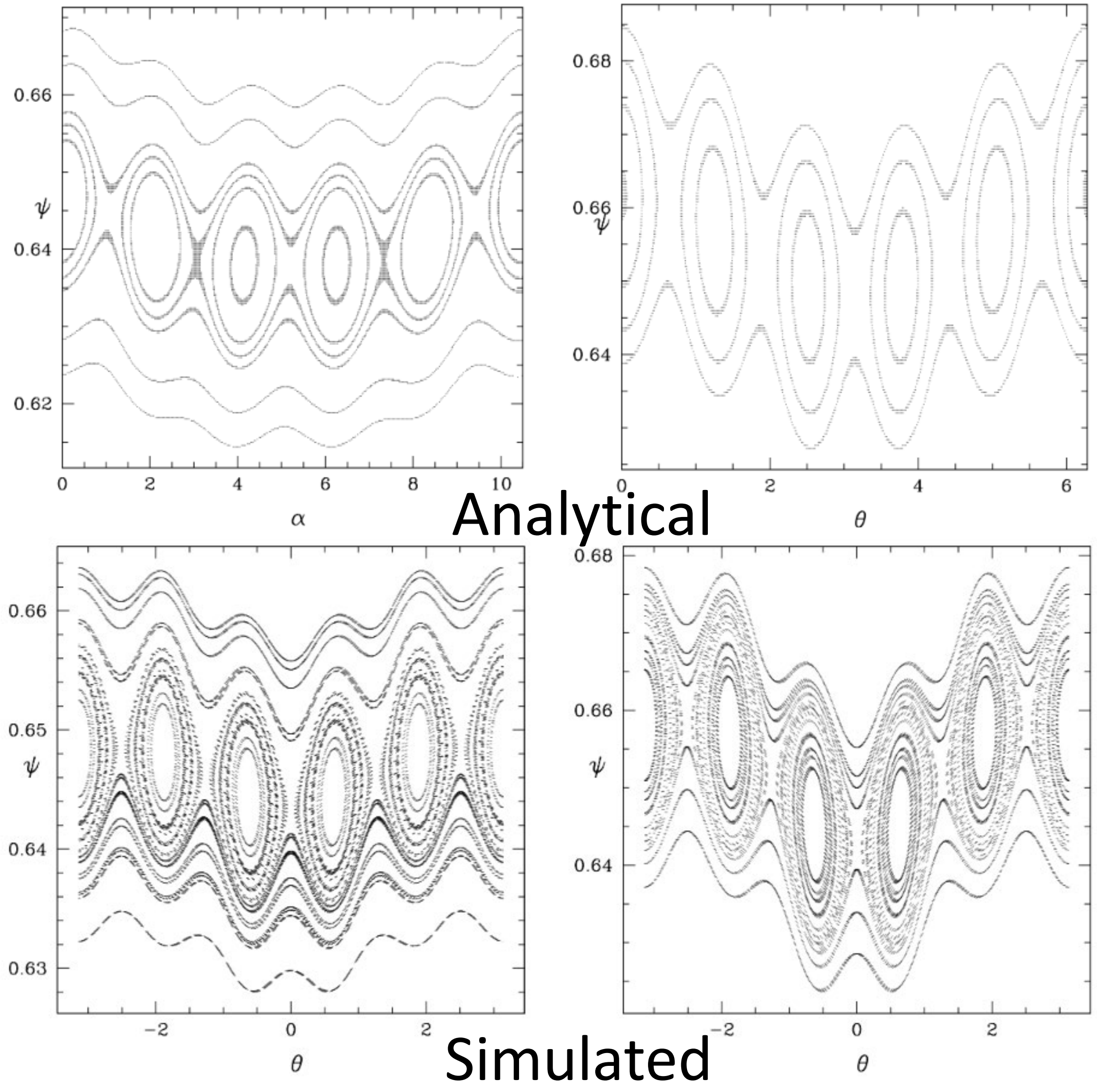


FIG. 8: Poincare plots for $E = 1$ keV and for $E = 10$ keV in NCSX

Resonance with Modulation of B

- On many surfaces, $j(\alpha)$ does not vary much and the islands are small.
- In Boozer coordinates:

$$j(\alpha) = \int_0^{2\pi n} \pm \frac{\sqrt{2M(E-\mu B)}}{\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\zeta}\, d\zeta = \int_0^{2\pi n} \underbrace{\pm\sqrt{2M(E-\mu B)}\,\frac{g+\iota I}{B}}\, d\zeta$$

Fourier series, assuming stellarator has N field periods $\longrightarrow$ $= \sum_{a,b} j_{ab}\, e^{i(a\theta + Nb\zeta)}$

$$\alpha = \theta - \frac{m}{n}\zeta = \text{const.} \longrightarrow = \sum_{a,b} j_{ab}\, e^{ia\alpha} e^{i(Nb + \frac{am}{n})\zeta}$$

This integrates to zero unless $Nb + a\frac{m}{n} = 0$.

Example: LHD, $N = 10$, $m/n = 1/2$ resonance would need $a = 20$.

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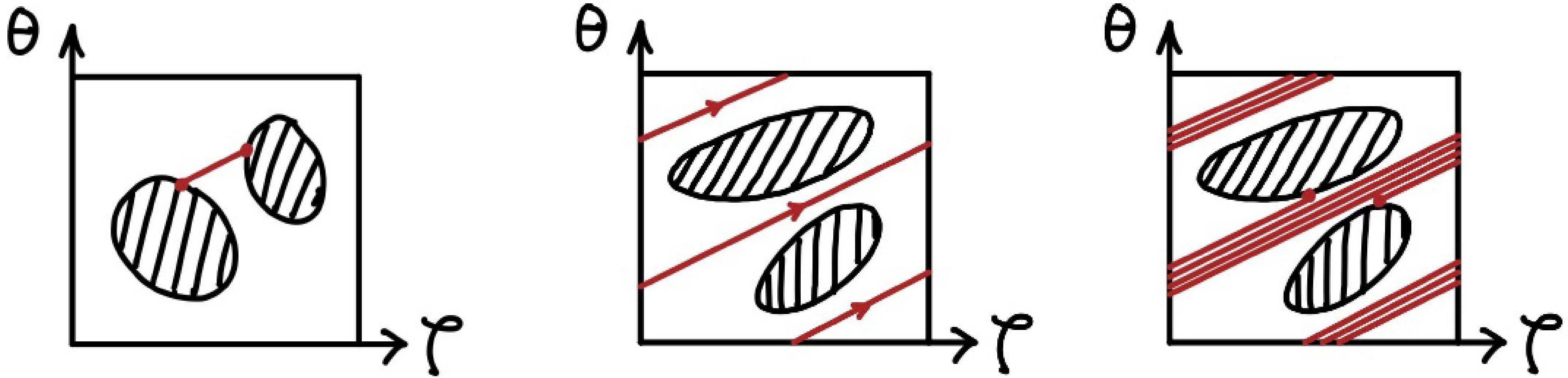
Trapped Particle Orbits

- Trapped particles can drift secularly because they do not sample the whole flux surface. Orbit determined by conservation of the bounce adiabatic invariant:

$$J_{\mathrm{b}} = 2\int_{l_L}^{l_R} m v_{\|}\,\mathrm{d}l$$

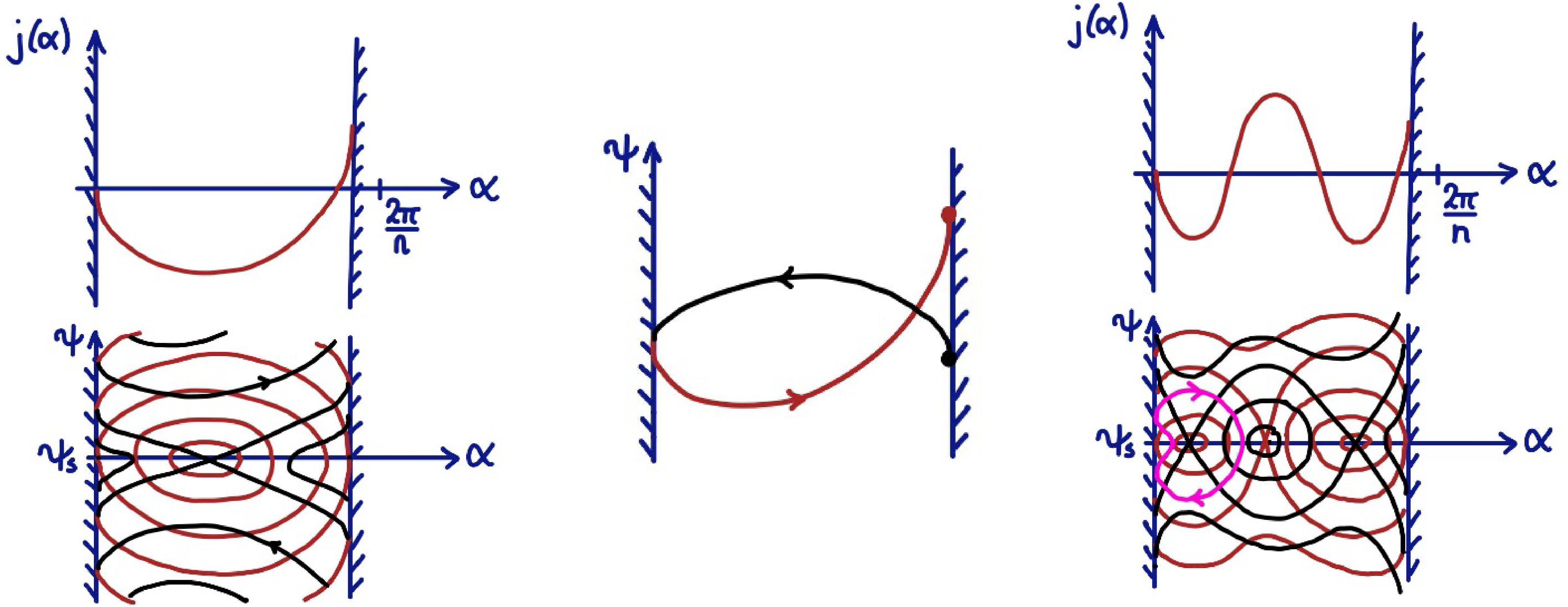
- Most trapped particles do not behave differently on rational surfaces.
- Trapped particles which make many toroidal transits before bouncing have diverging bounce period near rational surfaces, so must be described using the transit adiabatic invariant.



‘Semi-Trapped’ Particles

e.g. Kolesnichenko et. al. (2006, Nucl. Fusion)

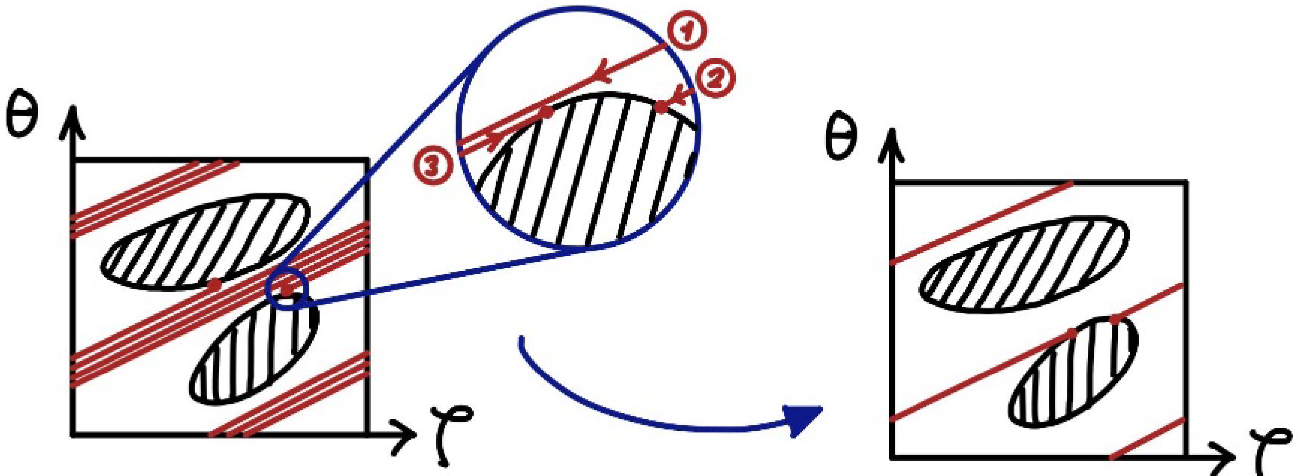
- Semi-trapped particles change from co-passing to counter-passing every time they bounce. This allows them to drift through the rational surface.
- However, there can be interesting orbits that are trapped at the rational surface.

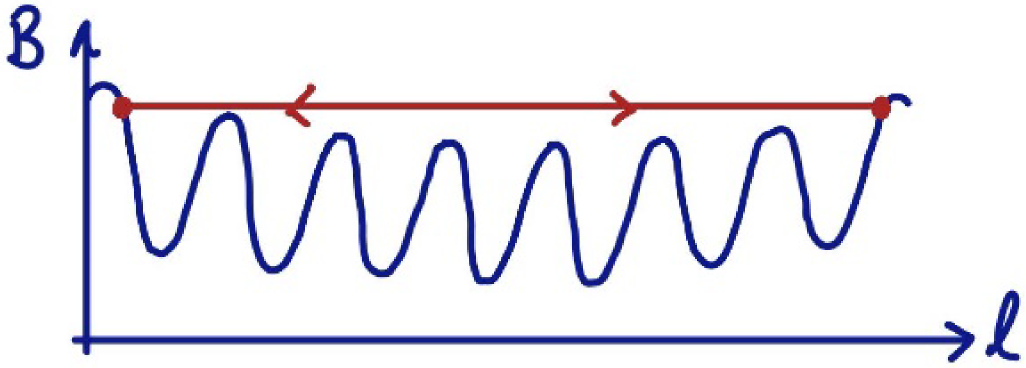


Transition Rate

- Near a bounce, the transit time diverges and adiabatic invariant theory doesn't immediately apply. But can show that the adiabatic invariant only changes by a small amount every bounce.
- Upon bouncing, there is a chance that the particle will transition and become trapped: $\psi_s \gg \psi - \psi_s \gg \rho_* \psi_s$

$$P = \begin{cases} \frac{c}{|\pi n Z e \iota'(\psi - \psi_s)|} \left(\partial_\alpha j_{\mathrm{sep}} \frac{\partial_\psi B_{\max}}{\partial_\alpha B_{\max}} - \partial_\psi j_{\mathrm{sep}} \right) & \text{if this is positive} \\ 0 \text{ otherwise} & \end{cases}$$





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Summary

- Near rational surfaces, passing and semi-trapped orbits exhibit islands and can be described using a transit adiabatic invariant.

- We can calculate the shapes and positions of these islands.

- Semi-trapped particles drift through the rational surface by alternating between co- and counter-passing orbits. There is a chance of a transition every time they bounce.

- Aim: theory of transport around rational flux surfaces / resonances, including the effect of collisions.