Formulation of a self-consistent reduced transport theory for discrete near-threshold modes — with applications to the dynamics of fast ions in tokamaks and dark matter in galaxies —

Vinícius N. Duarte[1]

in collaboration with Jeff Lestz[2], Chris Hamilton[3], Nick Pham[1], and Nikolai Gorelenkov[1]

[1]Princeton Plasma Physics Laboratory, Princeton, NJ, USA

[2]General Atomics, San Diego, CA, USA

[3]Institute for Advanced Study, Princeton, NJ, USA

Based on VND et al, *Phys. Rev. Lett.* **130**, 105101 (2023)

14th Plasma Kinetics Working Meeting, Vienna, July 25th 2023

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Motivation and main results

- **Motivation**: Wave-particle interactions are sensitive to collisional dynamics
  - Fundamental plasma physics problem
  - Fusion application: developing reduced models for energetic particle transport
  - Galactic application: understanding radial migration

- **Goal**: understand the influence of collisions on particle transport as a result of resonant interaction with *discrete* near-threshold instabilities

- **Main results**:
  - Near threshold, a quasilinear transport equation for δf naturally emerges from nonlinear theory
  - Collisional scattering broaden resonances while drag leads to their shifting and splitting

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Outline

- Establishing a particle transport framework for marginally unstable modes in a plasma

- Comparison of the developed theory with nonlinear simulations

- Applications:
  - energetic particle transport in a tokamak
  - dark matter dynamics in galaxies

- Summary and implications

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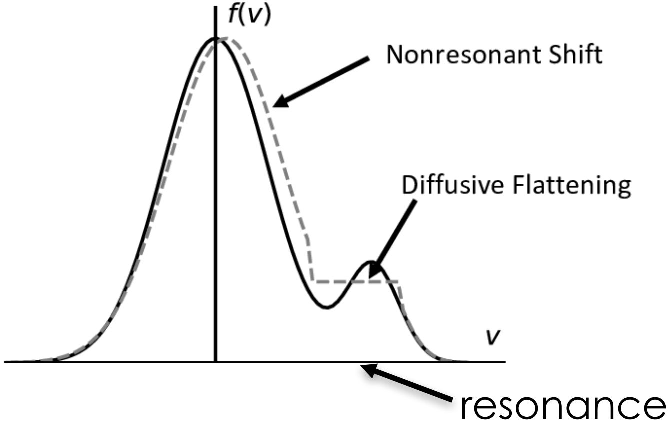
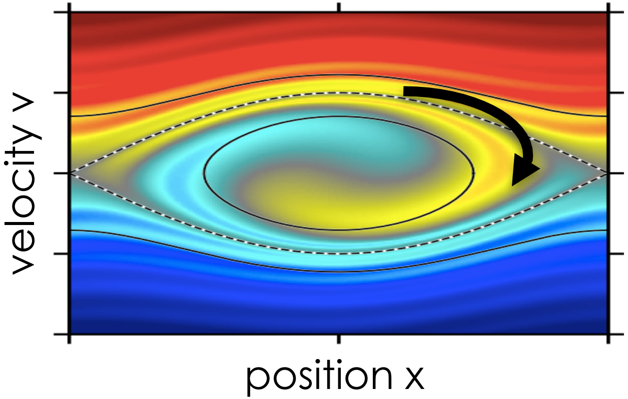
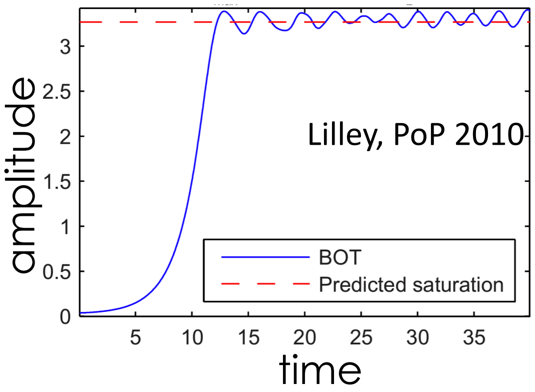
Basics of resonant dynamics







Three mechanisms at play:

- γ_L: wave growth rate due to a resonant minority species
- γ_d: wave background damping rate
- α and ν: effective collisional drag and scattering rates

For simplicity, position and velocity variables will be used in this talk but all results hold for actions and angles

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The effect of collisions is exacerbated within narrow layers

- Resonances are delicate interactions that occur within narrow layers

- Even if the non-resonant dynamics is hardly affected by collisions, the resonance can be highly affected by them
    - Su & Oberman, PRL 1968; Berk & Breizman, PoF 1990; Callen, PoP 2014; Catto, JPP 2021

- Heuristic arguments lead to $\nu \sim \left(\nu_\perp \omega^2\right)^{1/3}$

  effective collision rate within a resonance (ν)

  90-degree pitch angle scattering rate ($\nu_\perp$)

  mode frequency (ω)

- NOVA-K numerical evaluation indicates that ν is typically 2 orders of magnitude larger than $\nu_\perp$ for TAEs in present day tokamaks.

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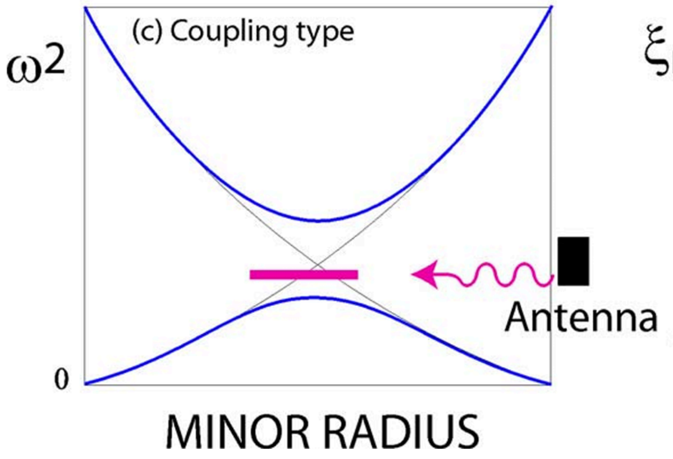
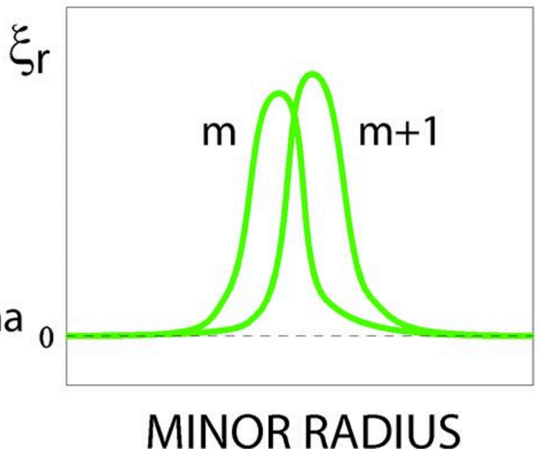
Discrete "gap" modes (e.g., Alfvénic eigenmodes) can be strongly excited by energetic particles





C.Z. Cheng, L. Chen, M.S. Chance, Annals Phys. 1985

Unlike in turbulence modeling, the modes are discrete in nature and their resonances can dynamically switch between isolated and overlapping scenarios →need for developing models for single modes

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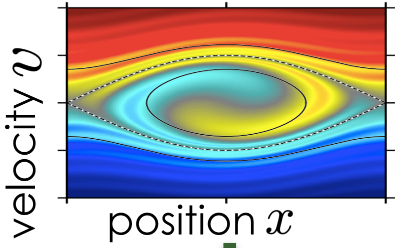
A new self-consistent formulation for the dynamics of a marginally unstable mode has been derived from first principles

Nonlinear kinetic theory in the vicinity of a resonance:

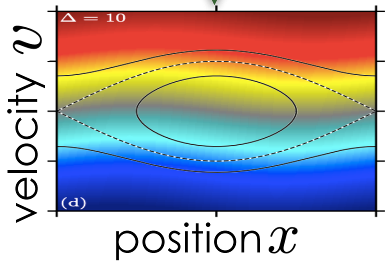
$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{1}{k}\mathrm{Re}\left(\underbrace{\omega_b^2}_{\text{bounce frequency}} e^{i(kx-\omega t)}\right)\frac{\partial f}{\partial v} = \underbrace{\frac{\nu^3}{k^2}\frac{\partial^2 (f-F_0)}{\partial v^2}}_{\text{scattering}} + \underbrace{\frac{\alpha^2}{k}\frac{\partial (f-F_0)}{\partial v}}_{\text{dynamical friction (drag)}}$$



Sufficiently near marginal instability, two small parameters emerge:

i) $|\omega_b^2|/\nu^2 \ll 1$: allows the integration of the kinetic equation to all orders via perturbation theory (already noted by Berk & Breizman, PRL 1996)

ii) $\nu \gg \gamma_{L,0} - \gamma_d$: leads to phase memory erasure → time delays become unimportant (new insight)



The distribution relaxation is then naturally cast as a resonance-broadened quasilinear diffusion equation:

$$\frac{\partial f(v,t)}{\partial t} - \frac{\partial}{\partial v}\left[\frac{\pi}{2k^3}\left|\omega_b^2\right|^2 \mathcal{R}(v)\frac{\partial f(v,t)}{\partial v}\right] = \frac{\nu^3}{k^2}\frac{\partial^2 (f-F_0)}{\partial v^2} + \frac{\alpha^2}{k}\frac{\partial (f-F_0)}{\partial v}$$

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The resulting set of transport equations

Distribution function evolution

$$\frac{\partial f(v,t)}{\partial t} - \frac{\partial}{\partial v}\left[\frac{\pi}{2k^3}|\omega_b^2|^2 \mathcal{R}(v)\frac{\partial f}{\partial v}\right] = C[f - F_0]$$

Resonance function
(emerges from the derivation
without further assumptions)

$$\mathcal{R}(v) = \frac{k}{\pi\nu}\int_0^\infty ds \cos\left(\frac{(kv-\omega)}{\nu}s + \frac{\alpha^2}{\nu^2}\frac{s^2}{2}\right)e^{-s^3/3}$$

Amplitude evolution

$$\frac{d|\omega_b^2(t)|^2}{dt} = 2[\gamma_L(t) - \gamma_d]|\omega_b^2(t)|^2$$

Growth rate

$$\gamma_L(t) = \frac{2\pi^2 q^2 \omega}{mk^2}\int_{-\infty}^{\infty} dv \mathcal{R}(v)\frac{\partial f(v,t)}{\partial v}$$

- The quasilinear approach involves a considerable dimensionality reduction that leads to numerical speed
- The above system is shown to lead to the same saturation level as nonlinear theory

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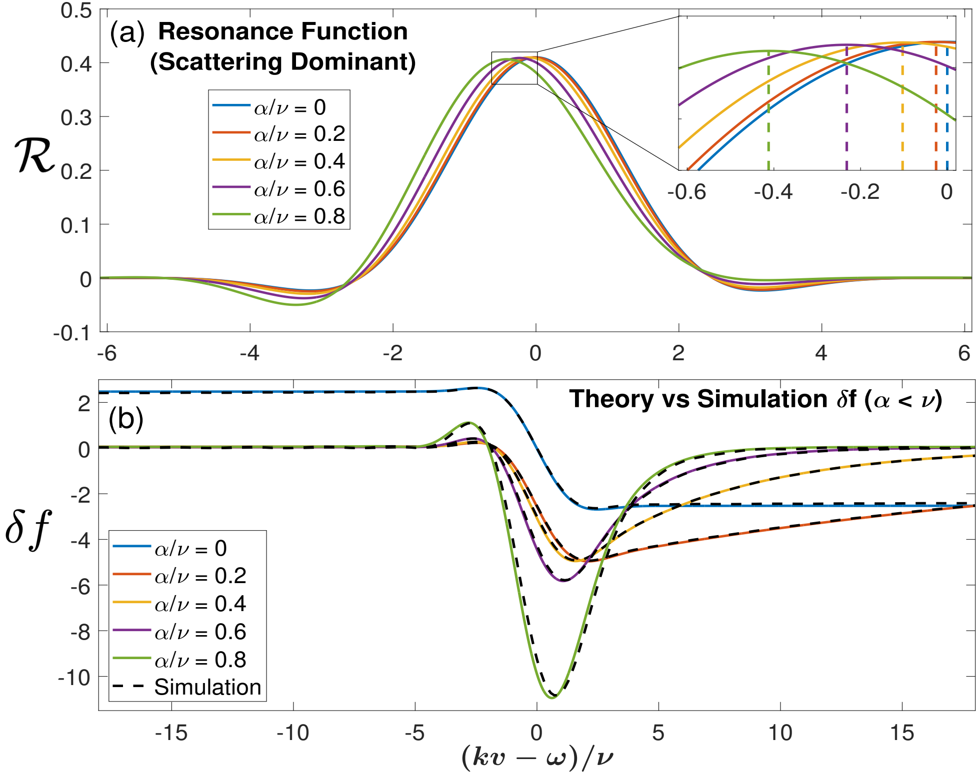
Scattering broadens the resonances while drag shifts them

verification against the Vlasov nonlinear code BOT



Any small amount of drag already leads to structural changes in the distribution function

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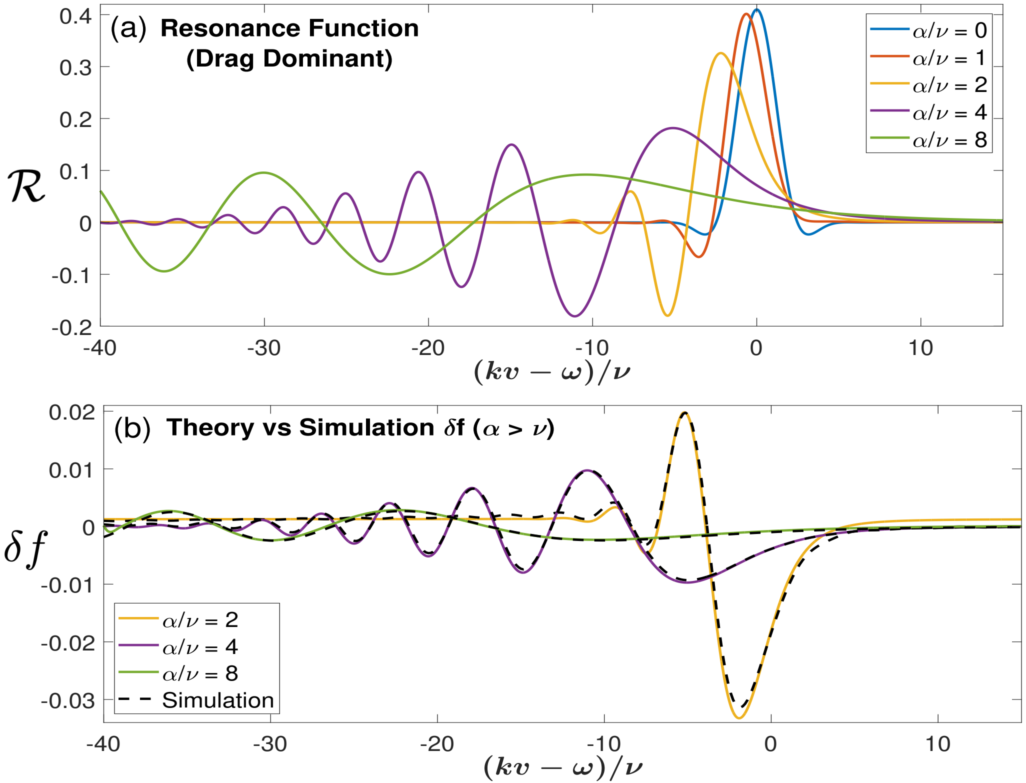
For large enough drag, resonances not only shift but also begin to split

different portions of phase space become resonantly activated



For reference, the analytic expression is

$$\delta f(v,t) = \frac{|\omega_b^2(t)|^2 F_0'}{2k\nu^3}\left\{ c\left(\frac{\alpha}{\nu}\right) - \int_0^\infty \frac{ds\, e^{-s^3/3}}{\alpha^4/\nu^4 + s^2}\left[\frac{\alpha^2}{\nu^2}\cos\left(\frac{(kv-\omega)s}{\nu} + \frac{\alpha^2}{\nu^2}\frac{s^2}{2}\right) + s\sin\left(\frac{(kv-\omega)s}{\nu} + \frac{\alpha^2}{\nu^2}\frac{s^2}{2}\right)\right]\right\}.$$

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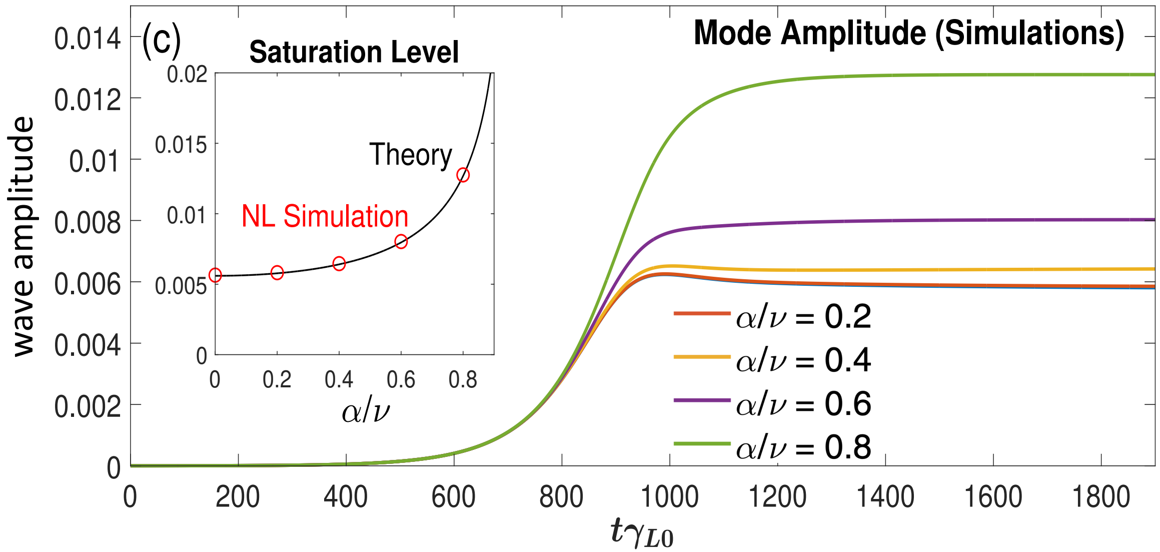
Drag increases the instability saturation level

Saturation levels match nonlinear simulations near threshold



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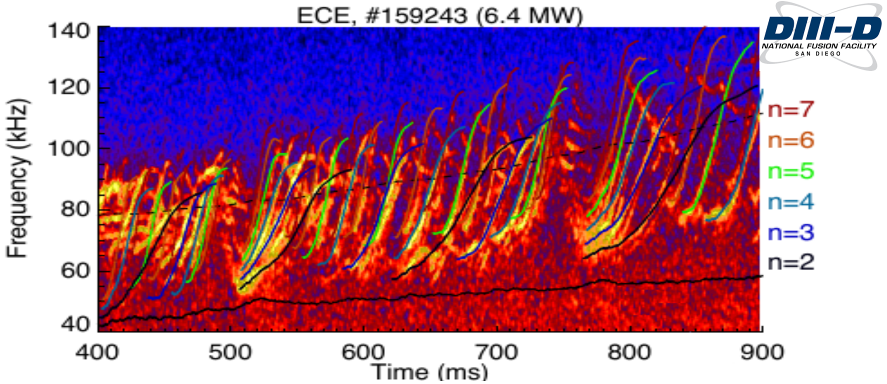
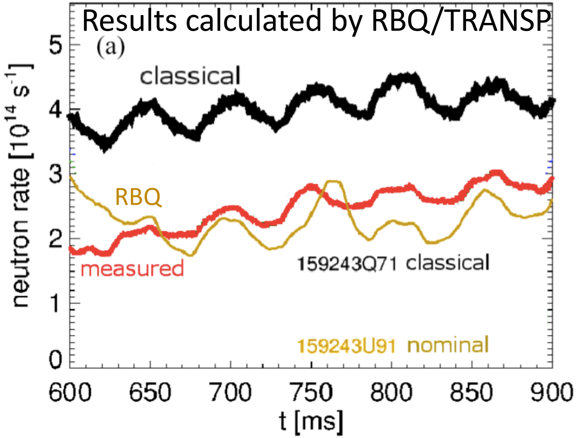
The new analytic development extends the physics of previous quasilinear simulations from interpretive to predictive

The Resonance-broadened quasilinear (RBQ) code is the numerical realization of the model:

- It uses eigenstructures calculated by an MHD solver
- It feeds NUBEAM/TRANSP with anomalous diffusivities due to wave-particle interaction



Previous analyses relied on mode amplitude measurements



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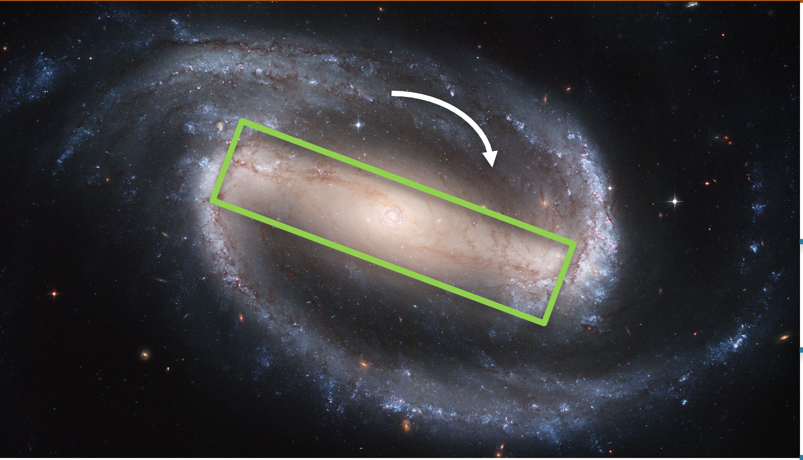
Application to a fundamental astrophysical problem: the rate of slowing down of galactic bar rotation



Chris Hamilton (IAS)

The galactic dynamics is isomorphic to the derived plasma quasilinear system

Galaxy	**Plasma**
Gravitational potential disturbance due to bar rotation	Discrete spectrum (e.g., Alfvén, Langmuir,…)
Stars and dark matter particles in the halo	Resonant sub-population (e.g., fast ions, electrons on the tail)
Diffusion due to non-resonant interaction	Pitch angle scattering

Solving the kinetic equation allows for **predicting the torque on the bar and its slowdown rate**

$$\mathcal{T}(t) = \int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{J}\, f(\boldsymbol{\theta},\boldsymbol{J},t)\frac{\partial\delta\Phi(\boldsymbol{\theta},\boldsymbol{J},t)}{\partial\theta_\varphi}$$

plasma kinetic theory enters here

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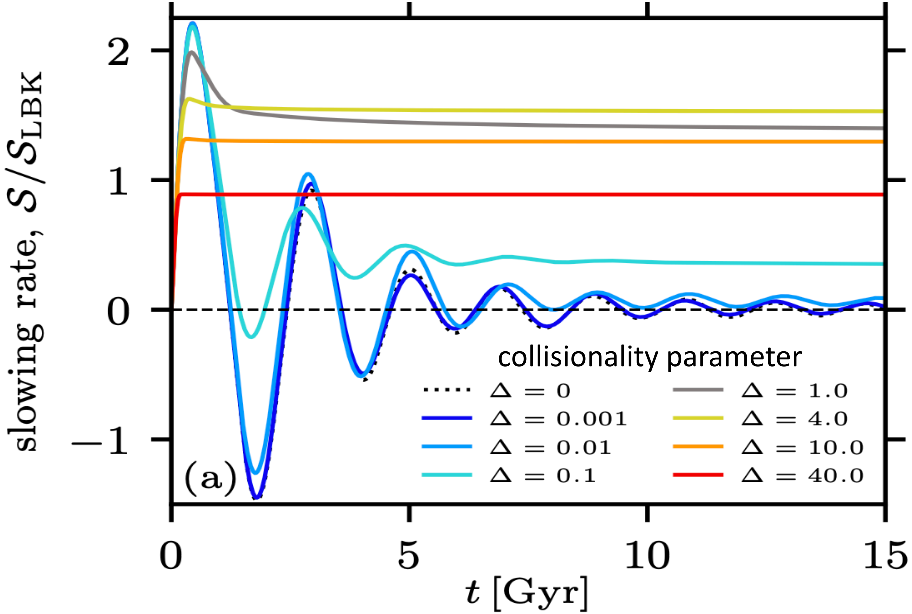
"Galactic bar resonances with diffusion: an analytic model with implications for bar-dark matter halo dynamical friction", C Hamilton, E A Tolman, L Arzamasskiy, V N Duarte, arXiv:2208.03855 (*Astrophys. J.*, 2023, accepted)

Collisions drastically alter the predicted galactic bar resonant dynamics



Collisionless prediction (Tremaine & Weinberg, 1984): friction on the bar asymptotically vanishes

Collisional prediction (Hamilton *et al*, 2023): <u>real galactic bars always decelerate</u>

Timely predictions in the Gaia observation era, with the distribution of over a billion stars mapped out→ quantitative comparison can constrain dark matter models

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Summary and implications of the work

Near an instability threshold, nonlinear kinetic theory naturally *becomes* resonance-broadened quasilinear theory, even for a single mode and in the absence of any resonance overlap

- fundamental features of nonlinear theory are automatically built into the quasilinear system, such as saturation levels and growth timescales
- the resonance function (shape of the resonance) emerges spontaneously in the derivation
  - scattering broadens the resonance
  - drag leads to resonance shifting and splitting
- physically transparent →useful in guiding and interpreting simulations
- the numerical realization of the model is the Resonance Broadening Quasilinear (RBQ) code
  - fully predictive and self-consistent, and interfaced with TRANSP;
  - fast enough for analysis between shots and for pilot plant design optimization

Duarte *et al*, "*Collisional resonance function in discrete-resonance quasilinear plasma systems*", *Phys. Plasmas* **26**, 120701 (2019)
Duarte *et al*, "*Shifting and splitting of resonance lines due to dynamical friction in plasma*", *Phys. Rev. Lett.* **130**, 105101 (2023)

Next steps

In fusion: extension and implementation for stellarators

—possible starting point is to use the orbits around rational surfaces obtained by Thomas Foster (yesterday's presentation)

In galaxies: treat radial migration due to the combined influence of bar modes and spiral (density) waves in the galactic disk

—being done by Nick Pham (2nd year grad student)

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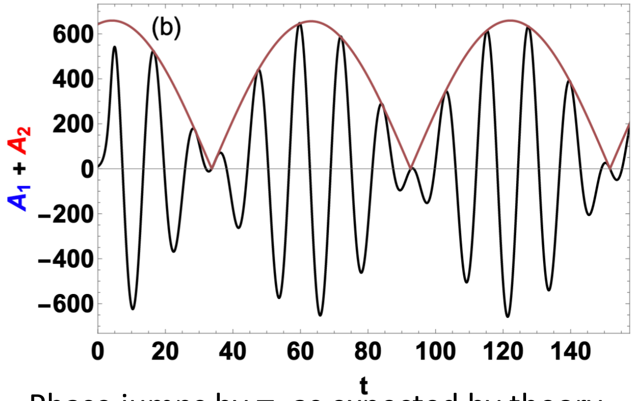
Thank you

E-mail: vduarte@pppl.gov

Extension to multiple instabilities

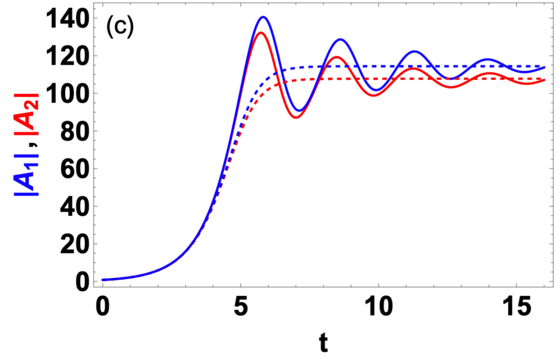
Nick Pham (2nd year grad student)

Relevant for the coupling between
- Overlapping Alfvénic eigenmodes (in fusion experiments)
- Spiral density waves + bar potential (in galaxies)



solid: numerical; dashed: reduced analytical approach

$\hat{\nu} = 6$



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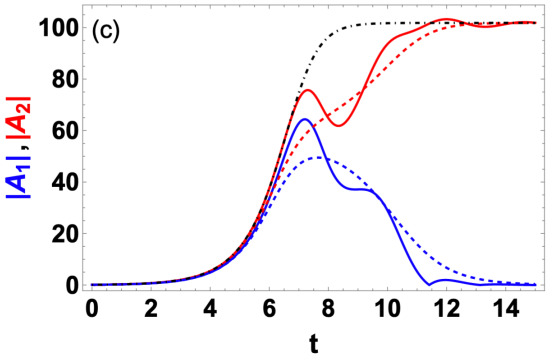
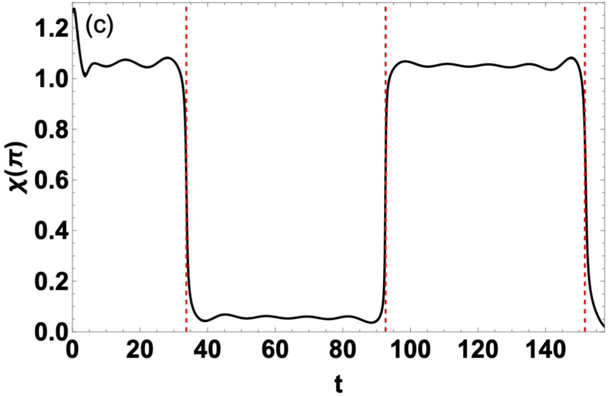


Phase jumps by π, as expected by theory



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N. M. Pham and VND, “*Evolution of coupled weakly-driven waves in a dissipative plasma*” (arXiv:2305.10322, submitted)

Quasilinear theory is a reduced approach to kinetic instabilities

In a regime where there is no effective particle trapping in resonances, the kinetic (Vlasov) description of phase mixing can be approximated by an irreversible, diffusive process

$$f\left(\varphi,\Omega,t\right)$$

$$\frac{\partial f}{\partial t}+\Omega\frac{\partial f}{\partial\varphi}+Re\left(\omega_b^2 e^{i\varphi}\right)\frac{\partial f}{\partial\Omega}=C\left[f,F_0\right]$$

→

$$f\left(\Omega,t\right)=\left\langle f\left(\varphi,\Omega,t\right)\right\rangle_\varphi$$

$$\frac{\partial f}{\partial t}-\frac{\partial}{\partial\Omega}D\frac{\partial f}{\partial\Omega}=C\left[f,F_0\right]$$

$$\dot{\varphi}=\partial H_0\left(J\right)/\partial J\equiv\Omega\left(J\right)$$

J represents $\left(\mathcal{E},P_\varphi,\mu\right)$

For quasilinear theory to be valid, the linear mode properties (e.g., eigenstructure and resonance condition) should not change in time

Quasilinear diffusion theory was independently proposed by
A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Sov. Phys. Usp. 4, 332 (1961).
W. Drummond and D. Pines, Nucl. Fusion Suppl. 2(Pt. 3), 1049 (1962).
Later generalized to action-angle variables:
A. N. Kaufman, Phys. Fluids 15, 1063 (1972).

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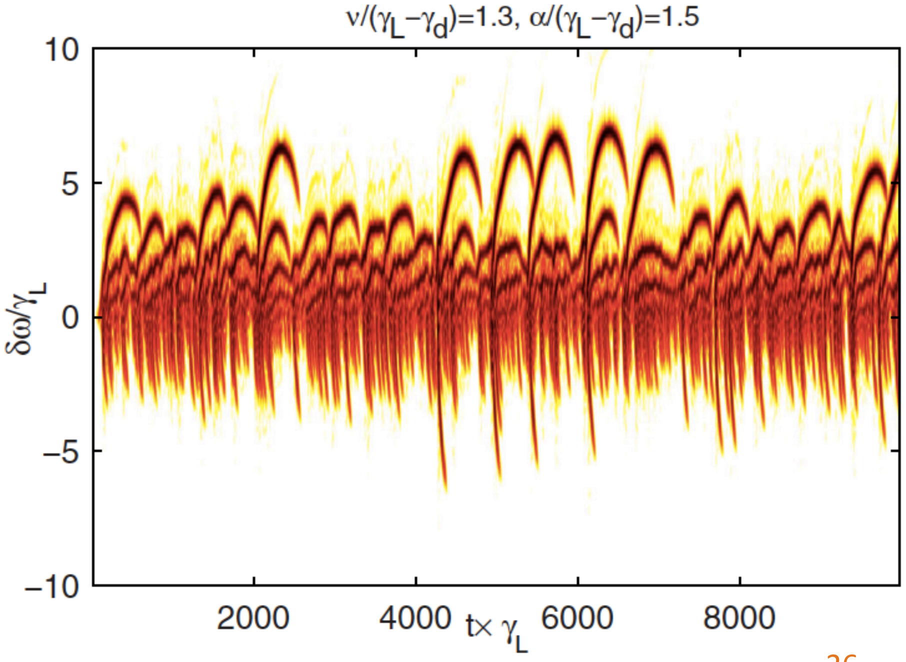
Drag features not covered in this presentation

Lilley, Breizman & Sharapov, PRL 2009, PoP 2010

- High drag/scattering ratio leads to less likelihood of steady saturation (more likelihood of chirping)
- Drag leads to asymmetric chirping



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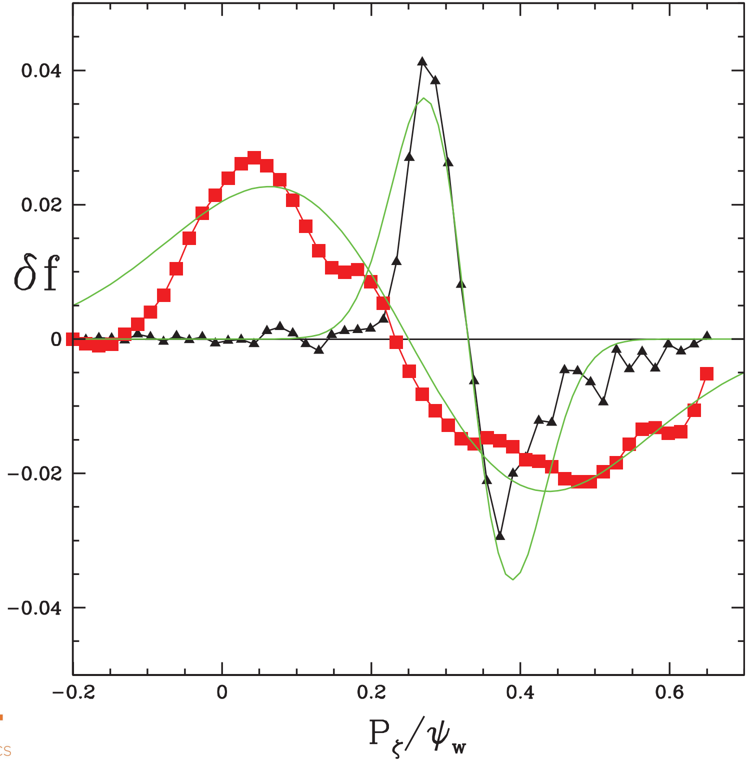
Verification of the analytical predictions against ORBIT simulations of Alfvénic resonances

Modification of the distribution function vs canonical toroidal momentum



Red and **black**: guiding-center ORBIT simulation results for two different levels of collisionality

Green: analytic fit

White, Duarte *et al*, *Phys. Plasmas* **26**, 032508 (2019)

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Energetic particles can be a show-stopper for burning plasmas

- Energetic particles can drive instabilities (e.g., Alfvénic) and lead to particle losses
    - >30% losses will likely prevent ignition
    - >5% losses will already be damaging for the first wall

    *(ITPA Physics Basis on Physics of energetic ions, Nucl. Fusion **47**, S264 (2007))*

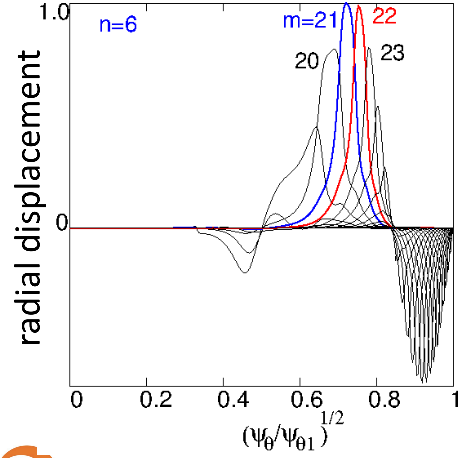
- EP-driven instabilities and their associated transport have been identified as one of the main physics gaps that need to be closed to confidently design a low-capital-cost tokamak fusion pilot plant

    (FESAC Long Range Plan, 2020)

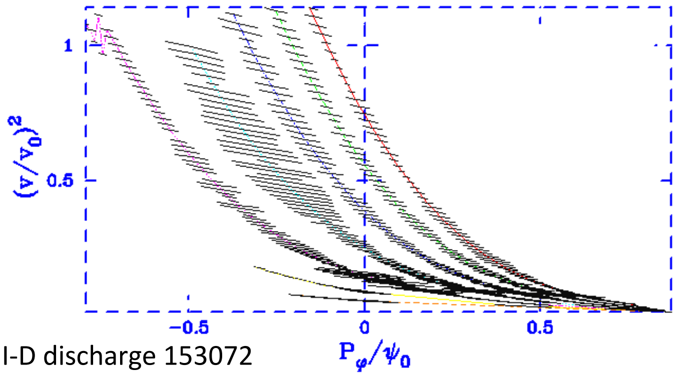
The Resonance-broadened quasilinear (RBQ) code is the numerical realization of the model: an efficient and realistic approach to fast ion transport

Workflow

1) Background plasma profiles read from the TRANSP code. Eigenstructure calculated by the linear MHD code NOVA



2) Damping rates and multi-dimensional resonance structure calculated by the NOVA-K code



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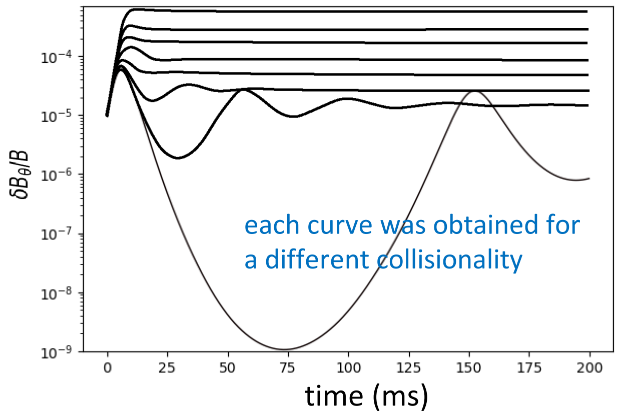
3) RBQ evolves the quasilinear distribution function together with the amplitudes of the modes and provides transport coefficients to TRANSP



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Early development of broadened quasilinear theory

- The broadening of resonances is a ubiquitous phenomenon in physics (e.g., in atomic spectra)
- In plasma physics, broadened strong turbulence theories for dense spectra have been developed (e.g., Dupree, Phys. Fluids 1966);

The line broadening model ($\delta(\Omega) \rightarrow \mathcal{R}(\Omega)$):

$$\frac{\partial f(\Omega,t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial\Omega}\left[\left|\omega_b^2\right|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial\Omega}\right] = C[f,F_0]$$

$$d\left|\omega_b^2\right|^2/dt = 2\left(\gamma_L(t)-\gamma_d\right)\left|\omega_b^2\right|^2$$

$$\gamma_L(t) = \tfrac{\pi}{4}\int_{-\infty}^{\infty} d\Omega\,\mathcal{R}\tfrac{\partial f(\Omega,t)}{\partial\Omega}$$

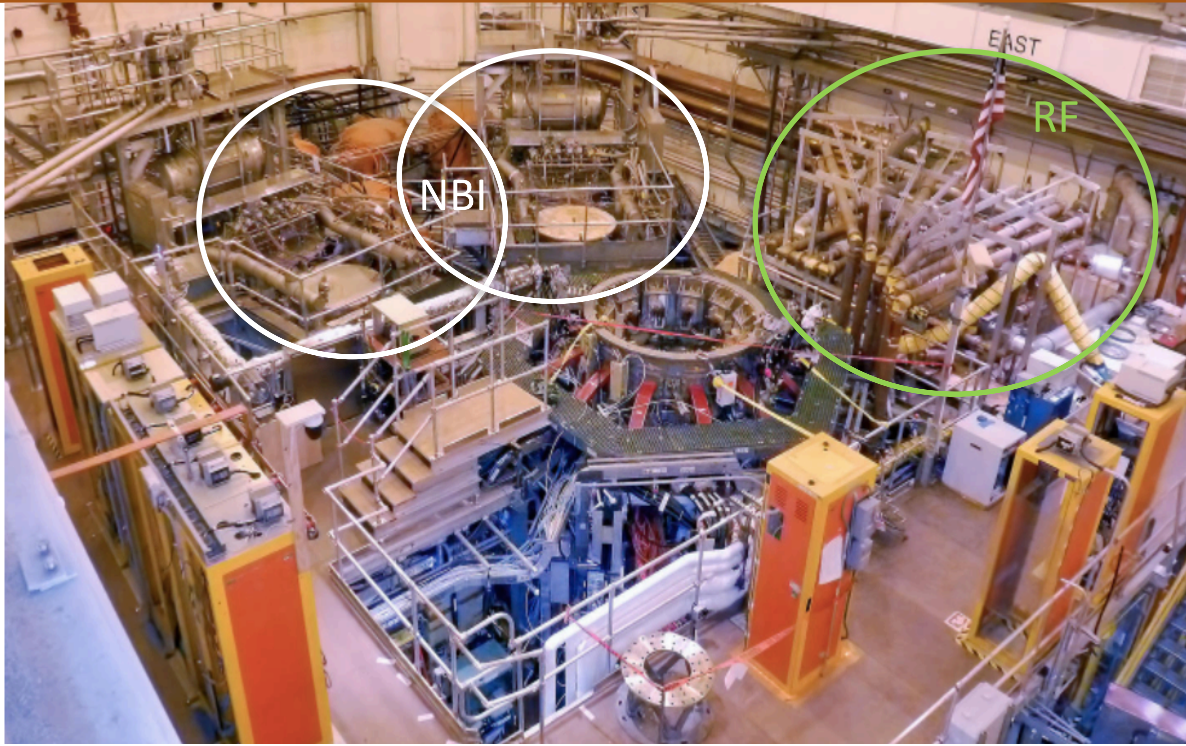
- $\mathcal{R}$ is an arbitrary resonance function (usually taken as in flat-top form) with $\int_{-\infty}^{\infty}\mathcal{R}(\Omega)d\Omega = 1$
- ω_b is the trapping (bounce) frequency at the elliptic point (proportional to square root of mode amplitude)

H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

A sub-population of energetic particles is ubiquitous in fusion plasmas

Sources of energetic particles:

- radiofrequency heating
- neutral beam injection
- alpha particles
- runaway electrons

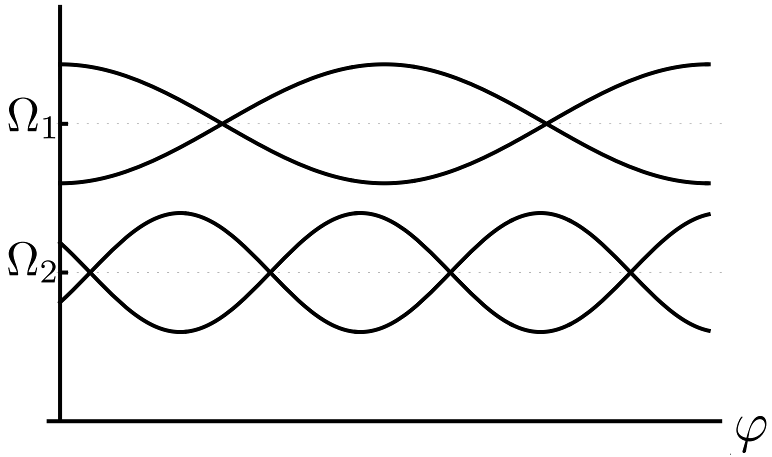


NSTX-U, Princeton Plasma Physics Laboratory

Historically, resonance overlap (Chirikov criterion) has been invoked to justify the applicability of QL theory

$$\omega_{b,1} + \omega_{b,2} \gtrsim |\Omega_1 - \Omega_2|$$

ω_b is the bounce (trapping) frequency



In this case, most trapped particles will not "belong" to a particular wave anymore but will be "shared" by the two waves.

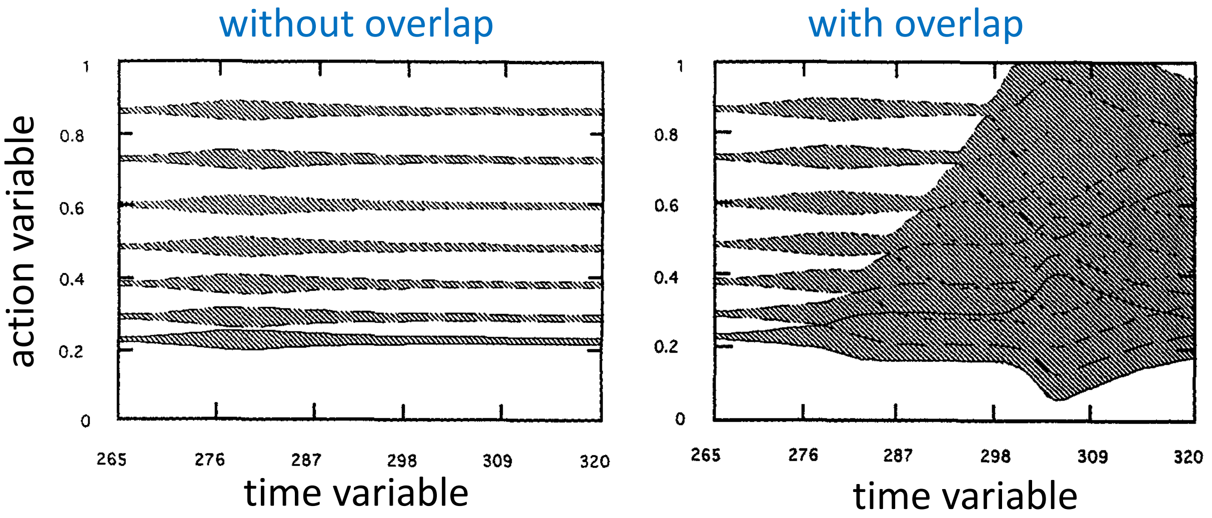
- Intrinsic stochastic diffusion: due to interaction with broad spectrum
- Extrinsic stochasticity: by collisions inducing randomization of phase

The end goal of this talk is to show that in the presence of collisions, a QL theory can be formulated from first principles near marginal stability, even for a single resonance. Interesting properties emerge:

(i) it recovers the saturation level predicted by nonlinear theory
(ii) the resonance function can be analytically calculated

The overlapping of resonances lead to losses due to global diffusion

- The resonance broadened quasilinear model is designed to address both regimes of isolated and overlapping resonances
  - the fast ion distribution function relaxes while self-consistently evolving the amplitude of modes



H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

First-principle analytical determination of the collisional resonance broadening – part I

Start with the kinetic equation: $$\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \varphi} + Re\left(\omega_b^2 e^{i\varphi}\right)\frac{\partial f}{\partial \Omega} = C\left[f, F_0\right]$$

$$C\left[f, F_0\right] \begin{cases} \nu_K\left(F_0 - f\right) \\ \nu_{scatt}^3 \partial^2\left(f - F_0\right)/\partial\Omega^2 \end{cases}$$

(from collisions, turbulence,…)

Periodicity over the canonical angle allows the distribution to be written as a Fourier series:

$$f\left(\varphi, \Omega, t\right) = F_0\left(\Omega\right) + \sum_{n=0}^{\infty}\left(f_n\left(\Omega, t\right)e^{in\varphi} + c.c.\right)$$

Near marginal stability, a perturbation theory can be developed in orders of $\omega_b^2/\nu_{K,scatt}^2$ which leads to the ordering $|F_0'| \gg \left|f_1'^{(1)}\right| \gg \left|f_0'^{(2)}\right|, \left|f_2'^{(2)}\right|$. When memory effects are weak, i.e., $\nu_{K,scatt}/\left(\gamma_{L,0} - \gamma_d\right) \gg 1$,

$$f_1 = \frac{\omega_b^2 F_0'}{2\left(i\Omega + \nu_K\right)} \qquad \frac{\partial f_0}{\partial t} + \frac{1}{2}\left(\omega_b^2\left[f_1'\right]^* + \omega_b^{2*} f_1'\right) = -\nu_K f_0$$

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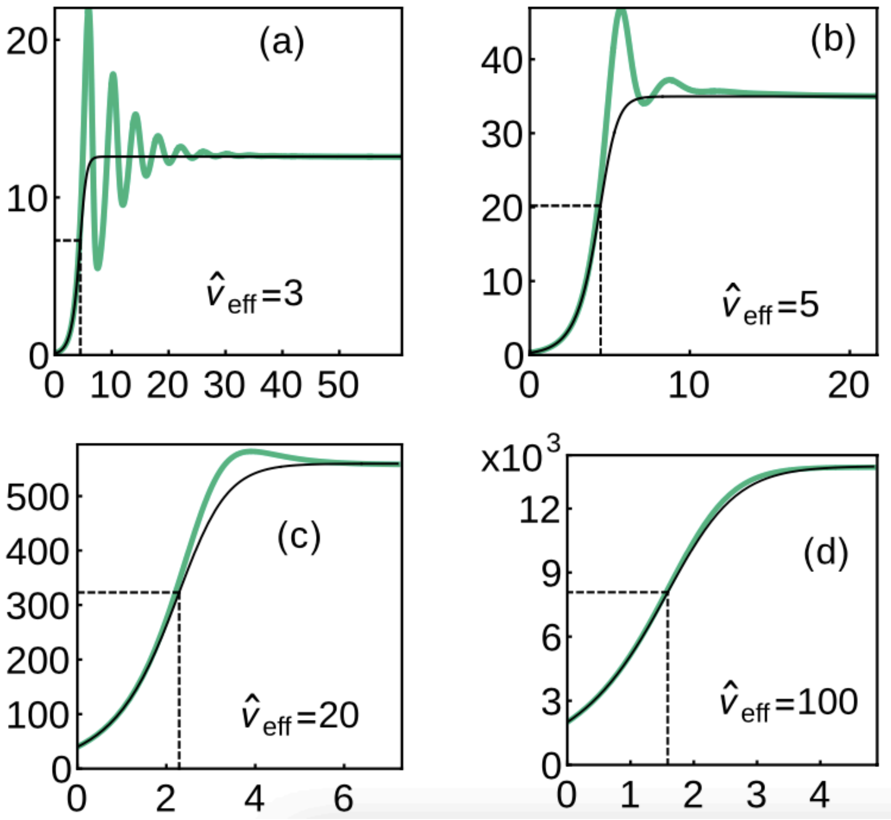
Simple analytical formula replicates essential features of nonlinear theory near threshold

Amplitude vs time
Green: nonlinear simulation
Black: analytical formula

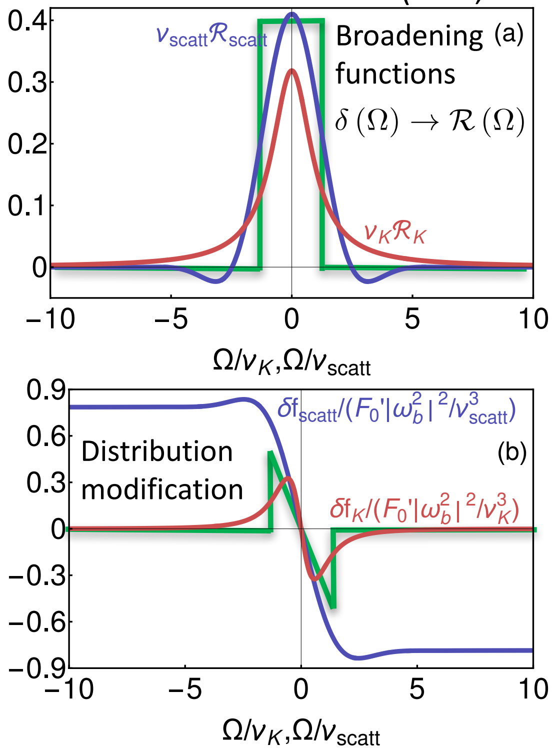
$$E(t) = \frac{E(0)e^{(\gamma_{L,0}-\gamma_d)t}}{\sqrt{1 - \frac{e^2k^2E^2(0)}{8m^2\nu_K^4}\frac{\gamma_{L,0}}{\gamma_{L,0}-\gamma_d}\left(1-e^{2(\gamma_{L,0}-\gamma_d)t}\right)}}$$



VND, *Nucl. Fusion* 2019

First-principles analytical determination of the collisional resonance broadening – part II

Blue curve: pitch-angle scattering
Red curve: Krook collisions
Green curve: previous heuristic broadening (Berk, NF '95)



When decoherence is strong, the distribution function has no angle dependence:

$$f(\Omega,t) \equiv F_0(\Omega) + f_0(\Omega,t)$$

In the limit $\nu_{K,scatt}/(\gamma_{L,0}-\gamma_d) \gg 1$, the distribution relaxation is naturally cast as a diffusion equation:

$$\frac{\partial f(\Omega,t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial\Omega}\left[\left|\omega_b^2\right|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial\Omega}\right] = C[f,F_0]$$

With the spontaneously emerged collisional resonance functions (both satisfy $\int_{-\infty}^{\infty}\mathcal{R}(\Omega)d\Omega = 1$):

$$\mathcal{R}_K(\Omega) = \frac{1}{\pi\nu_K\left(1+\Omega^2/\nu_K^2\right)} \qquad \mathcal{R}_{scatt}(\Omega) = \frac{1}{\pi\nu_{scatt}}\int_0^\infty ds\,\cos\left(\frac{\Omega s}{\nu_{scatt}}\right)e^{-s^3/3}$$

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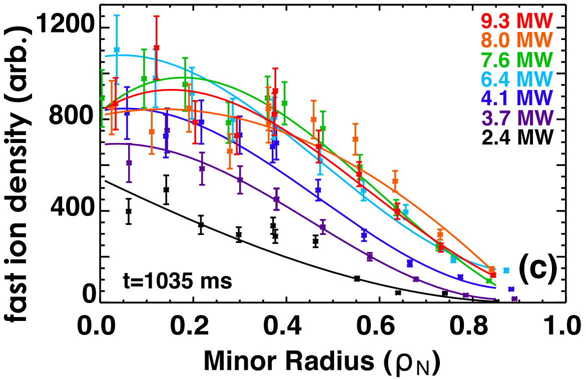
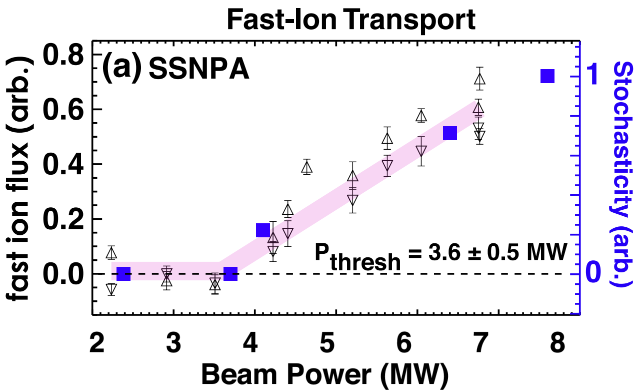
Many experiments are consistent with a quasilinear type of fast ion relaxation

The stiff transport suggests a stochastic fast ion transport mediated by overlapping resonances





C. Collins et al, PRL 2016

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Self-consistent formulation of collisional quasilinear transport near threshold replicates essential features of nonlinear theory

$$\frac{\partial f(\Omega,t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial \Omega}\left[\left|\omega_b^2\right|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial \Omega}\right] = C[f, F_0]$$

$$\gamma_L(t) = \tfrac{\pi}{4}\int_{-\infty}^{\infty} d\Omega \mathcal{R}\tfrac{\partial f(\Omega,t)}{\partial \Omega} \qquad d\left|\omega_b^2\right|^2/dt = 2\left(\gamma_L(t) - \gamma_d\right)\left|\omega_b^2\right|^2$$

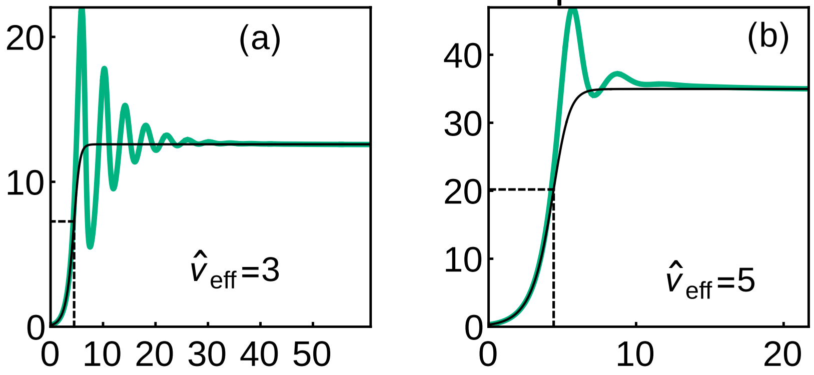
The QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels $|\omega_{b,sat}| = 8^{1/4}\left(1-\gamma_d/\gamma_{L,0}\right)^{1/4}\nu_K$ calculated from fully kinetic theory near marginality,

$$\frac{d}{dt}\,\omega_B^2 = (\gamma_L - \gamma_d)\omega_B^2(t) - \frac{\gamma_L}{2}\int_{t/2}^{t} dt'\,(t-t')^2\omega_B^2(t')\times$$
$$\times\int_{t-t'}^{t'} dt_1\,\exp[-\nu(2t-t'-t_1)]\,\omega_B^2(t_1)\omega_B^2(t'+t_1-t)$$

(Berk, Breizman and Pekker, *Phys. Rev. Lett.* 1996)

Amplitude vs time

Green: nonlinear **Black**: quasilinear



(Duarte & Gorelenkov, *Nucl. Fusion* 2019)

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