Moment methods for magnetized Vlasov equations

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Our model problem

It is **3D** magnetized transport (think of Tokamaks)

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \right) \cdot \nabla_v f = 0, \\ t > 0, \ \mathbf{x} \in \Omega \subset \mathbb{R}^3. \end{cases}$$

$\mathbf{E}(t, \mathbf{x})$ is a self consistent electric field evaluated with a Poisson or a Maxwell solver.
$\mathbf{B}_0(\mathbf{x})$ is a given magnetic field (application in Tokamaks, fusion plasmas).
The regime $\varepsilon \to 0^+$ corresponds to strong magnetic fields.

Objective of the MUFFIN project: explore and optimize original computational and numerical scenarios for multiscale and high dimensional transport codes.

References on moments

- Moment problem: first occurence Stieltjes 1894/1895.
See Akhiezer "the classical moment problem" 1965
- Moments and modeling : kinetic equations $\int f(v)v^i dv$ with $i = 1, 2, 3$,
Muller-Rugierri (extended thermodynamics 1993), Levermore 1996, ...

Physics+Num:
- Holloway: Spectral velocity discretizations for the Vlasov-Maxwell equations, 1996.
- Mandell/Dorland/Landreman: Laguerre-Hermite Pseudo-Spectral Velocity Formulation of **Gyrokinetics**, 2018.
- Grandgirard/.../Zarzoso: A 5D **gyrokinetic** full-f global semi-lagrangian code for flux-driven ion turb. sim., 2016.
- Phase mixing versus nonlinear advection in drift-kinetic plasma turbulence, 2016, Schekochihin→Hammett

- Math+Num Pham/Helluy/Crestetto 2012, Delzanno 2015,
Manzini/.../Markidis 2016
- Filbet-Xiong, Conservative Discontinuous Galerkin/Hermite Spectral Method for the Vlasov-Poisson System, 2020 (Filbet FKTW05 2022.)
- Filbet+Bessemoulin-Chatard, 2022.
- Charles+Dai+D.+Hirstoaga, Discrete moments models for Vlasov equations with non constant strong magnetic limit, HAL 2023 to appear in CRAS 2023.

Hermite polynomials (convenient for plasma physics)

Take the Maxwellian weight $G(v) = e^{-v^2/2}$.

- Hilbert basis of the space $\int_{\mathbb{R}} f^2(v)G(v)dv < \infty$.
  - Rodrigue's representation $H_n(v) = (-1)^n G(v)^{-1} \frac{d^n}{dv^n} G(v)$.
  - The degree of H_n is n. The parity of H_n is the parity of n.
  - $\int H_n(v)H_m(v)G(v)dv = (2\pi)^{\frac{1}{2}} n!\ \delta_{nm}, \quad n, m \in \mathbb{N}$.

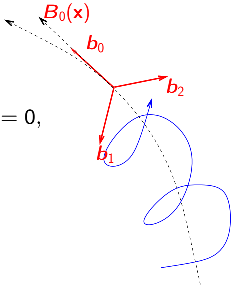
- Symmetric Hermite functions $\phi_n(v) = (2\pi)^{-\frac{1}{4}} n!^{-\frac{1}{2}} H_n(v)M(v)$

$$\varphi_0(v) = \frac{M(v)}{(2\pi)^{\frac{1}{4}}},\ \varphi_1(v) = \frac{vM(v)}{(2\pi)^{\frac{1}{4}}},\ \varphi_2(v) = \frac{(v^2-1)M(v)}{(8\pi)^{\frac{1}{4}}},\ \dots,$$

with recursion formulas

$$\begin{cases} v\varphi_n(v) = \sqrt{n+1}\varphi_{n+1}(v) + \sqrt{n}\varphi_{n-1}(v), & n \in \mathbb{N}, \\ \varphi_n'(v) = -\sqrt{\frac{n+1}{2}}\varphi_{n+1}(v) + \sqrt{\frac{n}{2}}\varphi_{n-1}(v), & n \in \mathbb{N}. \end{cases}$$

Vlasov versus Gyromodels



$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x})\right) \cdot \nabla_v f = 0, \\ t > 0, \ \mathbf{x} \in \Omega \subset \mathbb{R}^3. \end{cases}$$

- Gyrokinetic models (this version from Grandgirard et al)

$$B^*_{\|s} \frac{\partial \bar{F}_s}{\partial t} + \nabla_x \cdot (B^*_{\|s} \frac{d\mathbf{x}_G}{dt} \bar{F}_s) + \frac{\partial}{\partial v_{G\|}} (B^*_{\|s} \frac{dv_G}{dt} \bar{F}_s) = RHS.$$

- A rigorous limit (Filbet-Rodriguez 2021) is as follows. Set $\mathbf{b}_0(t, \mathbf{x}) = \frac{\mathbf{B}(t,\mathbf{x})}{|\mathbf{B}(t,\mathbf{x})|}$, $v = \langle \mathbf{v}, \mathbf{b}_0(t, \mathbf{x}) \rangle$, $w = \frac{|\mathbf{v}_\perp|^2}{2}$ and $\mathcal{V}_0(t, \mathbf{x}) = \left(v\mathbf{b}_0, E_\| + w\nabla_x \cdot \mathbf{b}_0, -vw\nabla_x \cdot \mathbf{b}_0\right)$. In the limit $\varepsilon \to 0$, one has

$$\partial_t G + \nabla_{\mathbf{x},v,w}(\mathcal{V}_0 G) = 0$$

Objective of this talk: alternative moment methods which capture the limit $\varepsilon \to 0$ without any modeling assumptions.

Main ingredient: anisotropic moment methods

Parallel dir. $\mathbf{b}_0(\mathbf{x}) = \frac{\mathbf{B}_0(\mathbf{x})}{|\mathbf{B}_0(\mathbf{x})|}$.

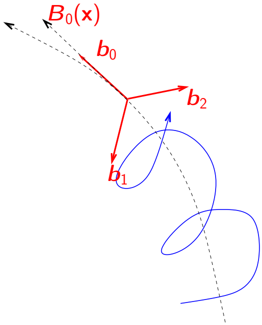
Complete $\mathbf{b}_0(\mathbf{x})$ as a local direct orthonormal basis $\mathbf{b}_i(\mathbf{x}) \cdot \mathbf{b}_j(\mathbf{x}) = \delta_{ij}$ and rescale the orthonormal directions as

$$\mathbf{d}_i(\mathbf{x}) = \frac{\mathbf{b}_i(\mathbf{x})}{\sqrt{T}}, \qquad i = 1, 2.$$

Rescale the parallel direction

$$\mathbf{d}_0(\mathbf{x}) = \frac{\mathbf{b}_0(\mathbf{x})}{\sqrt{T}}$$

$T > 0$ is the constant temperature.



Change-of-basis matrix is $M(\mathbf{x}) = \left(m_{ij}(\mathbf{x})\right)_{1 \le i,j \le 3} = (\mathbf{b}_0(\mathbf{x}) \mid \mathbf{b}_1(\mathbf{x}) \mid \mathbf{b}_2(\mathbf{x}))$.
The local change of variable is
$\mathbf{v} \mapsto \mathbf{w} = (\mathbf{v} \cdot \mathbf{d}_0(\mathbf{x}), \mathbf{v} \cdot \mathbf{d}_1(\mathbf{x}), \mathbf{v} \cdot \mathbf{d}_2(\mathbf{x}))^t = \frac{1}{\sqrt{T}} M^t(\mathbf{x})\mathbf{v}$ with $dv = T^{\frac{3}{2}} dw$.

Second ingredient: asymmetric shape functions

- A generic notation for a multi-index with three components is

$$\mathbf{n} = (n_0, n_1, n_2) \in \mathbb{N}^3 \text{ with } |n| = n_0 + n_1 + n_2.$$

- The shape functions are functions of $(\mathbf{x}, \mathbf{v})$

$$\boldsymbol{\varphi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v}) = \varphi_{n_0}(\mathbf{v} \cdot \mathbf{d}_0(\mathbf{x}))\, \varphi_{n_1}(\mathbf{v} \cdot \mathbf{d}_1(\mathbf{x}))\, \varphi_{n_2}(\mathbf{v} \cdot \mathbf{d}_2(\mathbf{x}))\,.$$

$$\text{Then } \boldsymbol{\psi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v}) = e^{-\frac{|\mathbf{v}|^2}{2T}} \boldsymbol{\varphi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v})$$

$$\text{and } \boldsymbol{\psi}^{\mathbf{n}}(\mathbf{x}, \mathbf{v}) = e^{\frac{|\mathbf{v}|^2}{T}} \boldsymbol{\varphi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v}) \quad (\text{ polynomials}).$$

Remark

Number of velocity basis-functions equal to

$$\text{Card}\,\{0 \le n_0 + n_1 + n_2 \le N\} = \frac{(N+1)(N+2)(N+3)}{6}.$$

Main model ($\mathbf{E} = 0$ for simplicity)

- The (Petrov-Galerkin) moment model writes

$f^N(\mathbf{x}, \mathbf{v}, t) = \sum_{|\mathbf{m}| \leq N} u_{\mathbf{m}}(\mathbf{x}, t)\psi_{\mathbf{m}}(\mathbf{x}, \mathbf{v}),$

$\left[\int_{\mathbf{v}} \left(\partial_t f^N + \mathbf{v} \cdot \nabla_x f^N + \frac{1}{\varepsilon}\mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \cdot \nabla_v f^N\right) \psi^{\mathbf{n}} dv\right](\mathbf{x}, \mathbf{v}) = 0, \ \forall(\mathbf{x}, t)$ and $|\mathbf{n}| \leq N.$

- The unknown is the vector

$$U(\mathbf{x}, t) = \left(u_{\mathbf{m}}(\mathbf{x}, t)\right)_{|\mathbf{m}| \leq N} .$$

Lemma

By construction the mass and the total energy are preserved since

$$1, |\mathbf{v}|^2 \in \underset{|\mathbf{n}| \leq N}{\mathrm{Span}} \left\{\psi^{\mathbf{n}}\right\}, \qquad \text{for } 2 \leq N.$$

Moment models
Stability issue
Miscellaneous numerics

- Notational simplicity $\mathbf{E} := 0$: one gets linear **Friedrichs system** with non constant matrices

$$\partial_t U(\mathbf{x},t) + \left[\sum_{i=1}^{3} \partial_{x_i}\left(A_i(\mathbf{x})U(\mathbf{x},t)\right) - B(\mathbf{x})U(\mathbf{x},t)\right] = \frac{1}{\varepsilon} C(\mathbf{x})U(\mathbf{x},t)$$

where for example

$$\begin{aligned} a^1_{\mathbf{nm}}(\mathbf{x}) &= \frac{T^2}{\sqrt{2}} m_{11}(\mathbf{x})\left(\sqrt{m_0+1}\delta_{m_0+1,n_0} + \sqrt{m_0}\delta_{m_0-1,n_0}\right)\delta_{m_1,n_1}\delta_{m_2,n_2} \\ &+ \frac{T^2}{\sqrt{2}} m_{12}(\mathbf{x})\delta_{m_0,n_0}\left(\sqrt{m_1+1}\delta_{m_1+1,n_1} + \sqrt{m_1}\delta_{m_1-1,n_1}\right)\delta_{m_2,n_2} \\ &+ \frac{T^2}{\sqrt{2}} m_{13}(\mathbf{x})\delta_{m_0,n_0}\delta_{m_1,n_1}\left(\sqrt{m_2+1}\delta_{m_2+1,n_2} + \sqrt{m_2}\delta_{m_2-1,n_2}\right). \end{aligned}$$

and

$$\begin{aligned} c_{\mathbf{nm}}(\mathbf{x}) = T^{\frac{3}{2}}|\mathbf{B}_0(\mathbf{x})|\delta_{m_0,n_0}\Big(&-\sqrt{(n_1+1)n_2}\,\delta_{m_1,n_1+1}\delta_{m_2,n_2-1} \\ &+\sqrt{n_1(n_2+1)}\,\delta_{m_1,n_1-1}\delta_{m_2,n_2+1}\Big). \end{aligned}$$

Lemma

The matrices A_i and B are symmetric, C is antisymmetric and one has $\sum_{i=1}^{3}\partial_{x_i}A^i(\mathbf{x}) = B(\mathbf{x}) + B^t(\mathbf{x})$.
The stability property holds $\partial_t|U|^2(\mathbf{x},t) + \sum_{i=1}^{3}\partial_{x_i}\left(A_iU\cdot U\right)(\mathbf{x},t) = 0$.

Formal limit $\varepsilon \to 0$

- Consider $U_\varepsilon = U_0 + \varepsilon U_1 + O(\varepsilon^2)$ solution of

$$\partial_t U_\varepsilon + \sum_{i=1}^{3} \partial_{x_i} (A_i U_\varepsilon) - BU_\varepsilon = \frac{1}{\varepsilon} CU_\varepsilon.$$

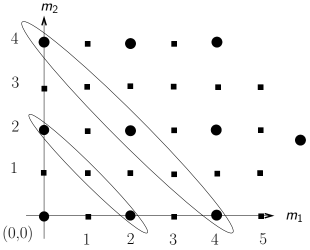
The hierarchy of equations starts with

$$\begin{cases} 0 = CU_0 \\ \partial_t U_0 + \sum_{i=1}^{3} \partial_{x_i} (A_i U_0) - BU_0 = CU_1, \\ \dots \end{cases}$$

The **gyro-kernel** does not depend of the space variable

$$\mathcal{K} = \{U \mid C(\mathbf{x})U = 0\} = \{U \mid C(\mathbf{y})U = 0\}$$

$$\mathcal{K} = \Big\{ U = (u_{\mathbf{n}})_{|\mathbf{n}| \le N} \ \Big| \quad -\sqrt{m_1(m_2+1)}\, u_{m_0, m_1-1, m_2+1} + \sqrt{(m_1+1)m_2}\, u_{m_0, m_1+1, m_2-1} = 0 \text{ for all } |\mathbf{m}| \le N \Big\}.$$



At the end of the analysis, one ellipse corresponds to a basis function which is a Laguerre polynomial (known in numerical plasma physics).

Remark

The number of basis functions is

$$\mathrm{Card}\, \{(m_0, s) \mid m_0 + 2s \le k\} = \frac{(N+1)(N+3)}{4} \text{ or } \frac{(N+1)(N+3)+1}{4}.$$

The Gyro-moment model

Let matrix P be the rectangular matrix from the projected space in the total space

$$P \in \mathcal{M}_{a,b}(\mathbb{R}) \text{ where } a \approx \frac{(N+1)(N+3)}{4} \text{ and } b = \frac{(N+1)(N+2)(N+3)}{6}.$$

- One has $CP = PC = 0$ and $P^t P = \widetilde{I}$.

Lemma

The gyro-moment model writes

$$\partial_t \widetilde{U} + \sum_{i=1}^{3} \partial_{x_i} \left(\widetilde{A}_i(\mathbf{x}, t) \widetilde{U}(\mathbf{x}, t) \right) = \widetilde{B}(\mathbf{x}) \widetilde{U}(\mathbf{x}, t) \tag{1}$$

where

$$\widetilde{A}_i(\mathbf{x}) = P^t A_i(\mathbf{x}) P, \quad \widetilde{B}(\mathbf{x}) = P^t B(\mathbf{x}) P.$$

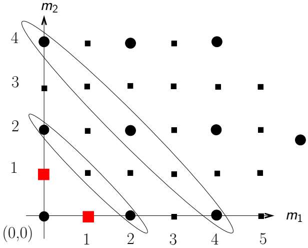
The magnetic force disappeared, as in gyrokinetics models.

- One can say it is an Asymptotic-Preserving model.

- This is an alternative to classical gyrokinetic models (Gysela, Gene, ...).

An interesting possibility: enriched gyro-moment model

Assume $0 < \varepsilon$ is small, and consider the graphics



The simplest non trivial idea is to add degrees of freedom for all

$$(m_0, m_1, m_2) \text{ where } m_0 + 1 \leq N \text{ and } m_1 + m_2 = 1.$$

Remark

The number of basis functions is

$$\frac{(N+1)(N+3)}{4} + 2N \text{ or } \frac{(N+1)(N+3)+1}{4} + 2N.$$

Structure of enriched gyro-moment model

- One gets naturally

$$U = (P, Q)\mathcal{U}, \qquad \mathcal{U} = \begin{pmatrix} \widetilde{U} \\ \widehat{U} \in \mathbb{R}^{2N} \end{pmatrix}$$

where P and $\widetilde{U} \in \mathcal{K}$ are unchanged, and we consider new notations

$$\widehat{\mathbf{n}} = (n_0, n_1, n_2) \text{ where } n_0 \in \mathbb{N},\ n_1 + n_2 = 1 \text{ and } |\widehat{\mathbf{n}}| = n_0 + 1.$$

The matrix is $Q = \left(q_{\mathbf{m},\widehat{\mathbf{n}}}\right)_{|\mathbf{m}|,|\widehat{\mathbf{n}}| \le N}$ with $q_{\mathbf{m},\widehat{\mathbf{n}}} = \delta_{m_0,n_0}\delta_{m_1,n_1}\delta_{m_2,n_2}$.

- This is a enriched gyro-kinetic model

$$\partial_t \mathcal{U} + \sum_{i=1}^{3} \partial_{x_i} \left(\mathcal{A}_i(\mathbf{x})\mathcal{U}\right) = \mathcal{B}(\mathbf{x})\mathcal{U} + \frac{1}{\varepsilon}\mathcal{C}(\mathbf{x})\mathcal{U}$$

where

$$\mathcal{C} = \begin{pmatrix} 0 & 0 \\ 0 & Q^t C Q \end{pmatrix},$$

and $\varepsilon > 0$ is presumably small, but non zero.

Numerical solver and numerical instability

1. First step

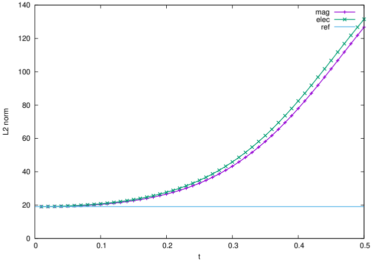
   $T^{\frac{3}{2}}\partial_t U(\mathbf{x},t) + \sum_{i=1}^{3} \partial_{x_i} \left(A_i(\mathbf{x})U(\mathbf{x},t)\right) - B(\mathbf{x})U(\mathbf{x},t) - \frac{1}{\varepsilon} C(\mathbf{x})U(\mathbf{x},t) = 0.$

   Cranck-Nicolson discretization in time + FE method in space.

2. Second step

   $T^{\frac{3}{2}}\partial_t U(\mathbf{x},t) + \sum_{i=1}^{3} E_i(\mathbf{x},t)D_i(\mathbf{x})U(\mathbf{x},t) = 0.$

   We solve the Poisson equation, with a Finite Element (FE) Poisson solver, to get the potential Φ. Then $\mathbf{E} = -\nabla\Phi$.



$t \mapsto \|U\|_{L^2}$: a numerical instability shows up in the electric step

Why this numerical instability

Consider the 1D equation $\partial_t f + e\partial_v f = 0$ ($e \in \mathbb{R}$ is constant) discretized with the moment method with asymmetric Hermite functions

$$\partial_t \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \dots \end{pmatrix} = eT^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \sqrt{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{4} & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{6} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \dots \end{pmatrix}.$$

The matrix D is triangular, which is the reason of the numerical instability. Indeed

$$\exp\left(teT^{-\frac{1}{2}}D\right) = \sum_{n\geq 0} \frac{(teT^{-\frac{1}{2}})^n}{n!} D^n.$$

This triangular structure is common to all recent papers on moment methods with asymmetric Hermite functions:

- G. Manzini, G. L. Delzanno, J. Vencels and S. Markidis, A Legendre-Fourier spectral method with exact conservation laws for the Vlasov-Poisson system, J. Comput. Phys. 317, 82-107, 2016.
- All recent papers by Delzanno
- F. Filbet and M. Bessemoulin-Chatard, On the stability of conservative discontinuous Galerkin/Hermite spectral methods for the Vlasov-Poisson system, J. Comput. Phys. 451 (2022).
- Charles+Dai+D.+Hirstoaga, Discrete moments models for Vlasov equations with non constant strong magnetic limit, HAL 2023 to appear in CRAS 2023.

A cure: introduce time in asymmetric functions

Redefine $\mathbf{d}_i(\mathbf{x}, t) = \frac{\mathbf{b}_i(\mathbf{x})}{\sqrt{T(t)}}$ and

$$\boldsymbol{\varphi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v}, t) = \varphi_{n_0}\left(\mathbf{v} \cdot \mathbf{d}_0(\mathbf{x}, t)\right) \varphi_{n_1}\left(\mathbf{v} \cdot \mathbf{d}_1(\mathbf{x}, t)\right) \varphi_{n_2}\left(\mathbf{v} \cdot \mathbf{d}_2(\mathbf{x}, t)\right)$$

$$\boldsymbol{\psi}_{\mathbf{n}}(\mathbf{x}, \mathbf{v}, t) = T(t)^{-\frac{3}{2}} \psi_{n_0}\left(\mathbf{v} \cdot \mathbf{d}_0(\mathbf{x}, t)\right) \psi_{n_1}\left(\mathbf{v} \cdot \mathbf{d}_1(\mathbf{x}, t)\right) \psi_{n_2}\left(\mathbf{v} \cdot \mathbf{d}_2(\mathbf{x}, t)\right)$$

and

$$\boldsymbol{\psi}^{\mathbf{n}}(\mathbf{x}, \mathbf{v}, t) = \psi^{n_0}\left(\mathbf{v} \cdot \mathbf{d}_0(\mathbf{x}, t)\right) \psi^{n_1}\left(\mathbf{v} \cdot \mathbf{d}_1(\mathbf{x}, t)\right) \psi^{n_2}\left(\mathbf{v} \cdot \mathbf{d}_2(\mathbf{x}, t)\right)$$

The new abstract form of the moment model is

$$\begin{aligned} & f^N(\mathbf{x}, \mathbf{v}, t) = \textstyle\sum_{|\mathbf{m}| \leq N} u_{\mathbf{m}}(\mathbf{x}, t) \boldsymbol{\psi}_{\mathbf{m}}(\mathbf{x}, \mathbf{v}, t), \\ & \int g^N(\mathbf{x}, \mathbf{v}, t) \boldsymbol{\psi}^{\mathbf{n}}(\mathbf{x}, \mathbf{v}, t) dv = 0 \quad \text{for } |\mathbf{n}| \leq N, \end{aligned}$$

where $g^N(\mathbf{x}, \mathbf{v}, t) = \partial_t f^N(\mathbf{x}, \mathbf{v}, t) + \mathbf{v} \cdot \nabla_x f^N(\mathbf{x}, \mathbf{v}, t) + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \cdot \nabla_v f^N(\mathbf{x}, \mathbf{v}, t)$.

The moment model with electric field writes

$$\partial_t U(\mathbf{x},t) + \sum_{i=1}^{3} \partial_{x_i} \left(A_i(\mathbf{x})U(\mathbf{x},t)\right) - B(\mathbf{x})U(\mathbf{x},t) \tag{2}$$

$$+\sum_{i=1}^{3} E_i(\mathbf{x},t)D_i(\mathbf{x})U = \frac{1}{\varepsilon}C(\mathbf{x})U(\mathbf{x},t) - T'(t)R(t)U.$$

To be coupled with the Poisson equation and $\mathbf{E} = -\nabla\varphi$.

Lemma

The model is stable in quadratic norm under the condition $T' = \gamma||\mathbf{E}||_\infty^2$ where $\gamma > 0$ is a coefficient. More precisely one obtains

$$\|U(t)\|^2 \le e^t |U(0)\|^2.$$

Idea adapted from Filbet/Bessemoulin-Chatard 2022.

Idea of proof: use weighted norm

One notices that

$$\int_{v\in\mathbb{R}} e^{\frac{v^2}{T(t)}} |f(x,v,t)|^2 dv = \int \left| \sum_m u_m(x,t)\varphi_n\left(\frac{v}{\sqrt{T(t)}}\right)\right|^2 = \sum_{m\geq 0} |u_m(x,t)|^2.$$

Then

$$\begin{array}{rcll}
\frac{d}{dt}\sum |u_m|^2 & = & 2\int e^{\frac{v^2}{T(t)}} f\partial_t f & -\frac{T'(t)}{T(t)^2}\int e^{\frac{v^2}{T(t)}}|f|^2 v^2 \\
& = & 2\int e^{\frac{v^2}{T(t)}} fe\partial_v f & -\frac{T'(t)}{T(t)^2}\int e^{\frac{v^2}{T(t)}}|f|^2 v^2 \\
& = & -2\int e^{\frac{v^2}{T(t)}} |f|^2\frac{e}{T}vdv & -\frac{T'(t)}{T(t)^2}\int e^{\frac{v^2}{T(t)}}|f|^2 v^2 \\
& \leq & 2\int e^{\frac{v^2}{T(t)}} |f|^2\left(\frac{1}{2}+\frac{1}{2}\left(\frac{e}{T}v\right)^2\right)dv & -\frac{T'(t)}{T(t)^2}\int e^{\frac{v^2}{T(t)}}|f|^2 v^2 \\
& \leq & \int e^{\frac{v^2}{T(t)}} |f|^2 dv & +\left(-\frac{T'}{T^2}+\frac{\gamma e^2}{T^2}\right)\int e^{\frac{v^2}{T(t)}}|f|^2 v^2 \\
& \leq & \int e^{\frac{v^2}{T(t)}} |f|^2 & \\
& \leq & \sum |u_m(x,t)|^2. &
\end{array}$$

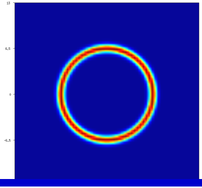
Diocotron without stabilization (first seconds, $N = 70$)

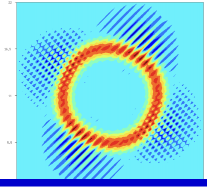
$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \right) \cdot \nabla_v f = 0, \\ \nabla \cdot \mathbf{E} = \int f dv - \rho_e. \end{cases}$$

$$f_0(\mathbf{x}, \mathbf{v}) = \begin{cases} \frac{n_0}{(\sqrt{2\pi})^3} (1 + \eta \cos(k\theta)) \exp^{-4(r-6.5)^2} \exp^{-|\mathbf{v}|^2/2}, & r^- \le r \le r^+, \\ 0, & \text{otherwise}, \end{cases}$$

Non-homogenous magnetic field

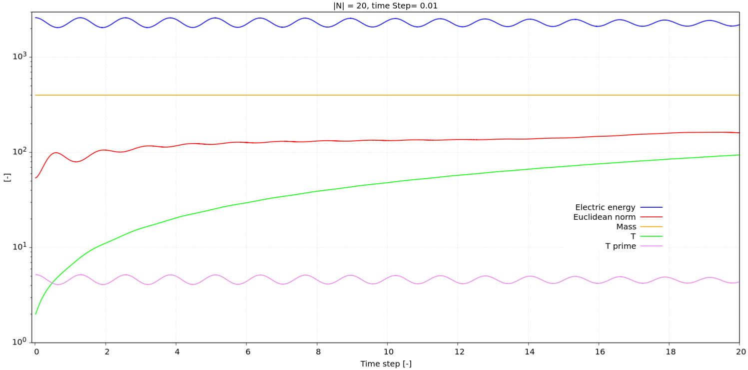
$\mathbf{B}_0(\mathbf{x}) = \omega_c(\mathbf{x}) \frac{1}{\sqrt{1 + \alpha^2 x_3^2 + \alpha^2 x_2^2}} (1, \alpha x_2, -\alpha x_3)^\top$ and $\varepsilon > 0$ (=1 or 10^{-2}).





With stabilization

Data similar to Muralikrishnan-Cerfon-et al JCP 2021.



This numerical stabilizer works fine, but it is physically disgusting.

The number of moments for the reduced model

Finite Element in space
Separation of variables = tensorialization $x - v$

N	Moments ($|\mathbf{m}| \le N$)	Moments (in cyclotron kernel)
5	56	12
10	286	36
15	816	72
20	1771	121
200	1373701	10201

Table: The number of moments.

Complexity

1. For finite element matrices, one needs $\mathcal{O}(ws^3)$ double floats;
2. The number of moments for the complete-moment model
$$r = (N+1)(N+2)(N+3)/6;$$
3. The number of moments for the gyro-moment model
$$r = \begin{cases} (N+1)(N+3)/4, & N \text{ odd}, \\ ((N+1)(N+3)+1)/4, & N \text{ even}. \end{cases}$$

Matrix storage	Storage for the Krylov method	Computational cost
$\mathcal{O}(ws^3)$	$\mathcal{O}(krs^3)$	$\mathcal{O}(kr^2s^3)$

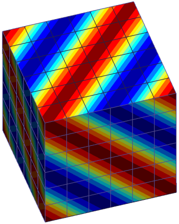
Table: Complexity of storage and computational cost.

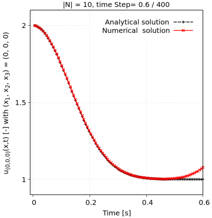
Validation: the transport equation

Vlasov equation: $\partial_t f_\epsilon + \mathbf{v} \cdot \nabla_x f_\epsilon + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \cdot \nabla_v f_\epsilon = 0,$

with $\mathbf{B}_0(\mathbf{x}) = \mathbf{b}_0(\mathbf{x}) = \{1, 0, 0\}^\top$, $\varepsilon = 10^{15}$.

Exact solution: $u_{0,0,0}(t = 0, \mathbf{x}) = 1 + \cos\left(2\pi(x_1 + x_2 + x_3)\right) e^{-\pi^2 t^2}$.





Finite Element Mesh.

- Similar as in: Pham/Helluy/Crestetto, A. Space-only hyperbolic approximation of the Vlasov equation, 2012.

- Recurrence described in: M. Mehrenberger L Navoret, N. Pham, Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh, Commun. in Comput. Phys., 2020.

Linear Landau damping

We consider the non-linear Vlasov-Poisson system:

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \left(\mathbf{E} + \dfrac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}_0(\mathbf{x}) \right) \cdot \nabla_v f = 0, \\ \nabla \cdot \mathbf{E} = \displaystyle\int f dv - \rho_e. \end{cases}$$

We keep periodic solutions in x_1-direction, and consider the initial density function

$$f(0, \mathbf{x}, \mathbf{v}) = \left(\frac{1}{\sqrt{2\pi}} \right)^3 (1 + \alpha \cos(k x_1)) \exp^{-|\mathbf{v}|^2/2}.$$

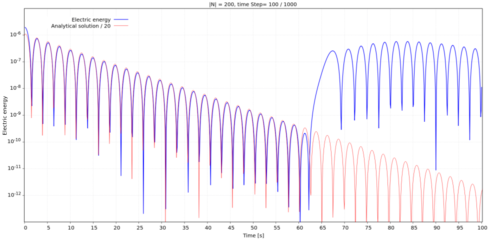


Figure: Landau damping. Damped electric field with $k = 0.4$, and $\epsilon = 10^{15}$.

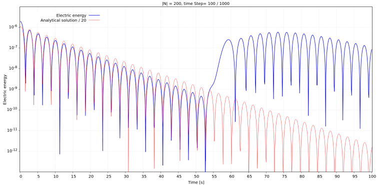
The Bernstein-Landau paradox

We consider the non-linear Vlasov-Poisson system:

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}_0(\mathbf{x})) \cdot \nabla_v f = 0, \\ \nabla \cdot \mathbf{E} = \int f dv - \rho_e. \end{cases}$$

We keep periodic solutions in x_1-direction, and consider the initial density function

$$f(0, \mathbf{x}, \mathbf{v}) = \left(\frac{1}{\sqrt{2\pi}} \right)^3 (1 + \alpha \cos(k x_1)) \exp^{-|\mathbf{v}|^2/2}, \quad k = 0.4.$$



- Rege-Charles-Weder-D. 2020.

Conclusions

- Extension of the validity of moments models/Friedrichs systems to 3D anisotropic magnetized Vlasov equations:
  main tool is anisotropic basis function $\varphi_{\mathbf{n}}(\mathbf{x}, \mathbf{v}, t)$.

  It provides moment methods which capture the strong magnetic field limit without any modeling assumptions.

- Our *MUF* implicit code uses also standard space FEM+parallelism+preconditionner
  Ask Ruiyang Dai for all implementation techniques.

- Another stabilization procedure is under development, not disgusting at all.