About solutions of the Vlasov equations.

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Based on ongoing work and discussion with N. Besse F. Golse, T. Nguyen and C. Mouhot.

The equations

$$\text{In} \quad \mathbb{D} = (\Omega \subset \mathbf{R}^d) \times \mathbf{R}^d_v \quad \partial_t F + v \cdot \nabla_x F + E(f) \cdot \nabla_v F = 0$$

$$-\Delta V = \rho = \int_{\mathbf{R}^d_v} F(x, v, t) dv - \int_{\mathbf{R}^d_v} F(x, v, t) dx dv \,, \hat{V}(k) = \frac{e}{|k|^2}$$

Λ Plasma Parameter Electrons $E = \Lambda \nabla V$.

• Exhibit several some refined properties present in physic and numerical simulations.

• The Vlasov flow preserves the positivity (density of particles) the total density and any $L^p(\mathbb{D})$ norm, and the total energy;

$$\frac{d}{dt}(\|F\|_{L^p} + \int_{\mathbb{D}} F(x,v,t)\frac{|v|^2}{2}dxdv + \frac{1}{2}\int_{\mathbf{R}^d_v} |\nabla V|^2 dx) = 0\,.$$

• A super play ground for mathematicians and what I intend to do is to draw a road map of the mathematical state of the art focusing on the global profile:

$$G(t) = \int_{\Omega} F(x,v,t)dv \quad \text{or} \quad E(t,x)\,. \tag{1}$$

and qualitatives related issues.

Elementary observations for the spectra of the linearized operator (as dubbed projective dynamic in other fields) in a bounded set Ω or in $\mathbf{T}^d$:

$$\begin{aligned} &f \to A(t)f = A_d f + K(t)(f) \\ &A_d f = v \cdot \nabla_x f \quad K(t)(f) = \nabla_v G(t,v) \cdot E(f) \end{aligned} \tag{2}$$

$\sigma(\pm A_d) = i\mathbf{R}$ and $\sigma(-A(t) \cap \Re\lambda \geq \delta > 0)$ is a finite sum of eigenvalues invariant under conjugaison:

$$\lambda \in \sigma(A(t) \cap \Re\lambda > 0) \Leftrightarrow \exists f \neq 0 \quad (I + (\lambda I + A(t))^{-1} K(t))f = 0 \tag{3}$$

Since the L^p norm and energy of $F(x,v,t)$ are bounded the $\Re\lambda_k(t)$ cannot grow for ever .

Hence for a solution $t \mapsto F(x, v, t)$ a formal description of the evolution in term of the spectra of the linearized operator

$$
\begin{aligned}
&\mathbf{R}_t^+ = \{0 < T_{ns} \leq T_{tur} \leq T_{LD} \leq \infty\} \\
&\sigma_{us} \subset \{0 < t < T_{ns} \leftrightarrow \Re\lambda > 0 \cap \sigma(-A(t) \neq \emptyset\,,\} \\
&\sigma_{st} \subset \{\,T_{LD} < t \leftrightarrow \Re\lambda > 0 \cap \sigma(-A(t) = \emptyset\,, \\
&\{\,T_{us} < t < T_{LD}\} = \text{Transitional or turbulent time.}
\end{aligned}
\tag{4}
$$

For the mathematical community it concerns solutions $t > 0$, $F(t) = G(t) + f(t)$ with no unstable eigenvalues for $-A(t)$.
Hence for $G(t) = G(0)$ given the solution of

$$\partial_t f + v \cdot \nabla_x f + \nabla_v (G(0) E(f) = 0 \qquad (5)$$

should go to 0 fast (exponentially for analytic data, $exp(-t^{\frac{1}{\beta}})$ for data in Gevrey space) which in turns gives, under small perturbations the same decay for

$$\partial_t f + v \cdot \nabla_x f + \nabla_v (G(t) E(f) = 0 \qquad (6)$$

and with

$$\partial_t G + \nabla_v \langle E(f), f)\rangle = 0 \qquad (7)$$

then with a boot strap argument the fast decay of the electric field $E(f)$ to zero.

This is already written in any physic text book, systematically studied in the torus $\mathbf{T}^d$ and it is by now well known that this is a much diverse and rich issue:

- Forerunner Cagliotti Maffei 1998, Hwang, Velazquez, 2009.
- Complete proof for analytic data a Mouhot Villani,
- For Gevrey data Mouhot Bedrossian Masmoudi, Grenier, Nguyen, and I. Rodnianski.

The state of the art by now is

• Convergence to 0 of the electric field $E(x,t)$ is proven only as "small perturbation" of a "stable" regular profile $G(0,v)$ with the Coulomb potential , with data and small perturbation in the space Gevrey < 3 .

• "Plateau" for $1d$ $G(0,v)$ (< 3) is compatible with the decay.

• Small perturbations in Gevrey 3 does not prevent non convergence to 0 of the electric field. Hence Gevrey 3 appears as a threshold.

• For $\Omega = \mathbf{R}^d$ one can adapt the point of view of classical scattering theory (cf for instance, for non linear Klein Gordon equation, Morawetz and Strauss (1973)) under a stability condition and for a screened potential:

$$\lim_{t\to 0} \| f(x+vt,v,t) - f_\infty(x,v) \| = 0 \,.$$

Masmoudi Mouhot 2016, Han-Kwan, Nguyen Rousset 2020.

Fourier spectra $\sigma(-A(t)) \cap \Re\lambda > 0$. in $\mathbf{T}^d$.

$$f(x,v) = \sum_k f_k e^{i\cdot x} \Rightarrow (\lambda + ik\cdot v) f_k - \frac{ik}{|k|^2} \cdot \partial_v G(v,t)\rho_k = g_k(v)$$

$$D(\lambda,k,t)\rho_k = (1 - \frac{ik}{|k|^2} \cdot \int \frac{\partial_v G(v,t)}{\lambda + ik\cdot v} dv)\rho_k = \int \frac{g_k(v)}{\lambda + ik\cdot v} dv\,.$$

- $D(\lambda,k,t)$ is called the " Penrose" dispersion relation.
- λ_k with $\Re\lambda_k > 0$ is an eigenvalue if and only if there exists k such $D(\lambda,k,t) = 0$
- $D(\lambda_k,k,t) = 0 \Leftrightarrow D(\overline{\lambda_k},-k,t) = 0$.
- For λ_k, t given the number of solutions of $D(\lambda_k,k,t) = 0$ is finite. That does not prevent, for given $\mathbf{k}$ the number of λ such that $D(\lambda,\mathbf{k},t) = 0$ to be infinite

Introduce the "modulated linear" operator

$$f \mapsto A(t)f = v \cdot \nabla_x f + \nabla_x (-\Delta)^{-1} f \nabla_v G(t,v)$$

Assume that $\sigma(-A(t)) \cap \{\Re\lambda > 0\}$ is the union simple smoothly time dependent eigenvalues $(\lambda_k(t), f_k(t))$ each of them contained in non intersecting closed time independent closed curves $\Gamma_k(t)$

$$f_k(t) = \frac{1}{2i\pi} \int_{\Gamma_k} (\tau I + A(t))^{-1} f d\tau \qquad (8)$$

$$f_k(t) = -\frac{E_k(t) \cdot \nabla_v G(v,t)}{\lambda + ik \cdot v} \qquad (9)$$

$$(\partial_t + A(t))(e^{\int_0^t \lambda_k(s)} f_k(t)) = e^{\int_0^t \lambda_k(s)ds} f_k(t)$$
$$\frac{1}{2i\pi}\int_{\Gamma_k} (\tau + A(t))^{-1}\partial_t(\nabla_v G(v,t))\cdot E((\tau + A(t))^{-1} f) d\tau \qquad (10)$$

For slowly varying $\nabla_v G(t) = O(\epsilon^2)$

$$f = \sum_k (e^{\int_0^t \lambda_k(s)} f_k(t)) \text{ or } \sum_k \Re(e^{\int_0^t \lambda_k(s)} f_k(t)) \qquad (11)$$

are approximate solutions of the equation

$$(\partial_t + A(t))(e^{\int_0^t \lambda_k(s)} f_k(t)) = O(\epsilon^2), .$$

In the equations:

$$
\begin{aligned}
&\partial_t F + v\partial_x F + E(f)\partial_v G(t,v) = 0\\
&F(t,v) = G(t,v) + f(x,v,t) \qquad \langle f\rangle = 0\\
&\partial_t G + \partial_v\langle E(f), f\rangle = 0\,,\\
&\partial_t f + v\cdot\partial_x f + E(f)\partial_v G(t) = -((E(f),f) - \langle E(f)f\rangle)
\end{aligned} \tag{12}
$$

Insert $\quad F(x,v,t) = G(t,v) + \epsilon(f \simeq \sum_{k>0} \Re(e^{\int_0^t \lambda_k(s)} f_k(t) e^{ikx}))$

in the equations

$$
\begin{aligned}
&\partial_t F + v\partial_x F + E(f)\partial_v G(t,v) = 0\\
&F(t,v) = G(t,v) + f(x,v,t)\\
&\partial_t G + \partial_v\langle E(f), f\rangle = 0\,,\\
&\partial_t f + v\cdot\partial_x f + E(f)\partial_v G(t) = -((E(f),f) - \langle E(f)f\rangle)
\end{aligned} \tag{13}
$$

This leads to the following formulas:

$$\nabla_v \langle E(\epsilon f, \epsilon f\rangle \simeq -\epsilon^2 \nabla_v (\mathbf{D} \nabla_v G)$$

$$\mathbf{D}(v, \nabla_v G(v,t)) = \sum_{D(k,\lambda)=0} \frac{|E(0,k)|^2 2\Re\lambda e^{2\Re\int_0^t ds\lambda(s)}}{(k\cdot v - \lambda)^2 + (\Re\lambda)^2}$$

$$\partial_t G - \epsilon^2 \nabla_v (\mathbf{D}(v, \nabla_v G(v,t) \nabla_v G) = o(\epsilon^2)$$

$$\partial_t f + v\cdot\partial_x f + E(f)\partial_v G(t) = ((E(f), f) - \langle E(f) f\rangle) + o(\epsilon^2)$$

The above estimates are given for uniformly smooth solutions and as such they lead to proof of validity in a very small time $t \in (0, T_\epsilon) with T_\epsilon = 0(\epsilon^2)$ Such type of observations are present in recent derivations (Boblylev , Pulvirenti , Saffirio) for the Landau equation, and Duerinck for Balescu Lenhard).
Observe that the error come from terms of the right hand $R(t)$ side of the equation and under the action of a propagator $||U(t,s)|| \leq e^{(t-s)L}$ according to a Duhamel formula: $\int_0^t U(t,s)R(s)ds$ that will generate an error of order $\epsilon^2 e^{Lt}$ and a time of validity

$$T = O(\log \frac{C}{\epsilon})$$

about the same time as the diffusion..

Time validity and accuracy of the QLI approximation for a basic growing mode:

$$\lambda(t)_\pm = \gamma(t) \pm i\omega(t)$$

$$f_s(t) = \Re|E(0)|e^{\int_0^t \gamma(s)ds + ikx} \cdot \frac{\nabla_v G(v,t)}{\lambda + ik\cdot v}$$

$$F(x,v,0) = G(v,0) + \epsilon f_s(x,v,0)\,; \quad h_\epsilon(x,v,0) = 0$$

$$F_\epsilon(x,v,t) = G_\epsilon(v,t) + \epsilon f_{s,\epsilon}(x,v,t) + \epsilon^2 h_\epsilon(x,v,t)$$

$$\mathbf{D}_\epsilon(v,t) = \frac{2|E(0)|^2\gamma_\epsilon(t)e^{2\int_0^t ds\gamma_\epsilon(s)}}{(k\cdot v - \omega_\epsilon(t))^2 + (\gamma_\epsilon(t))^2}$$

Then for $\log\frac{1}{\epsilon} \simeq \int_0^t \gamma_\epsilon(s)ds$ one has:

$$\partial_t G_\epsilon(x,v,t) - \epsilon^2\nabla_v(\mathbf{D}(v,\nabla_v G_\epsilon)\nabla_v G_\epsilon(v,t)) = 0(\epsilon^3)$$

$$F_\epsilon - (G_\epsilon + \epsilon f_{s,\epsilon}) = O(\epsilon^3)$$

In the torus $\mathbf{T}^d_x$ at the present date *"fully well established qualitative mathematical results"* concern

- Large time behavior in the absence of unstable mode (Mouhot, Villani, Bedrossian, Toan Grenier...)
- Short time behavior in the presence of strong growing modes (B. Besse....)

Lol ! This is not compared to the diversity of phenomena in the most interesting region

$$T_{tur} \leq t < T_{LD}$$

where following approaches may be more relevant, in conjunction with a larger time scale and reiteration up to higher order of the Duhamel formula:

$$\partial_t F_\epsilon + \frac{1}{\epsilon^2} v \cdot \nabla_x F_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot \left(E_\epsilon F_\epsilon\right) = 0$$
$$\partial_t \int_{\mathbf{R}^d_v} F_\epsilon dv + \nabla_v \cdot \Big(\int_{\mathbf{R}^d_v} \frac{E_\epsilon F_\epsilon}{\epsilon} dv \Big) = 0$$

• The validity of the above approximation has been proven for non consistant detailed random phase hypothesis (B.,Besse)

$$
\begin{aligned}
&E^\varepsilon(t,k)e^{-i\frac{\omega_k t}{\varepsilon^2}}\\
&\mathbb{E}[E^\varepsilon(t,k)\otimes E^\varepsilon(s,k')] = A_k\Big(\frac{t-s}{\varepsilon^2}\Big)\delta(k+k')\\
&|\sigma| > \tau \Longrightarrow A_k(\sigma) = 0\,,
\end{aligned}
$$

Following numerical experiments N. Besse, Y. Elskens, D. Escande, P. Bertrand, Validity of quasilinear theory: refutations and new numerical confirmation, Plasma Phys. Control. Fusion **53** 025012–48 (2011).

• Introduction of randomness rises the issue generation of transition from the unstable region to the "turbulent" region and the propagation of such randomness in the turbulent region.

• • For the transition on can use the Langmuir approximation of the dispersion relation near an oscillating mode: For $\sigma \to 0_+$ this gives for the eigenvalue $(0, \tilde\omega)$ the expression:

$$\text{Bump on tail} \quad \partial_v G(\omega = 0, t) \quad \text{and} PV \int \frac{\partial_v G(v,t)}{(v-\tilde\omega)} dv = k^3 \tag{14}$$

and the Langmuir approximation:

$$\tilde\sigma(t) \simeq \frac{\pi}{2}\tilde\omega(t)\partial_v G(\tilde\omega(t)) \quad \text{with} \quad \partial_v G(\tilde\omega(T^*)) = 0 \Rightarrow \tilde\sigma << \tilde\omega$$

• Using the Langmuir approximation and following Nicholson seection 7.16 one deduces in the transition region an approximation of the evolution of the density

$$\rho = \int f(x, v, t)dv$$

in term of a quasilinear Sine Gordon type system.

$$\partial_t \rho(x,t) - \partial_x^2 \rho(x,t) = \partial_x^2 |E(x,t)|^2 \quad \partial_x E(x,t) = 4\pi\rho(x,t)$$

• • And finally study the propagation of the randomness of $\rho(x,t)$ by weak trubulence analysis. P. Germain C. Cotta...And many more.