Further thoughts on the asymptotic behaviour of a Rapidly Rotating Magnetic Mirror

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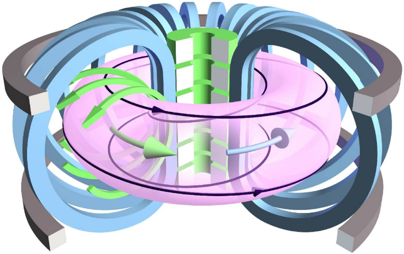
Outline

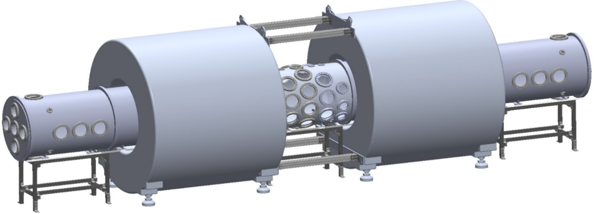
Tokamaks aren't that trivial!



Plasma Physics

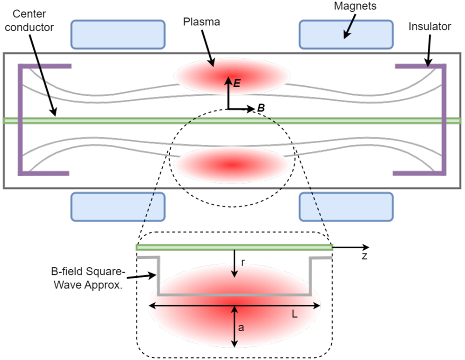
- Current drive and current-driven instabilities
- Geometrically complex (c.f. ITER / SPARC engineering costs) – coils link the plasma
- Turbulence!

Centrifugal Mirrors: What Are They



- Novel device (preceded by PSP-2 & MCX)
- Simple geometry, reducing build & maintainance costs
- Very long confinement times are possible

Centrifugal Mirrors: How do They Work?



- Ions are pushed away from the ends of the plasma, confining particles
- Elecrons follow (quasineutrality), confinining heat
- Flow shear stabilizes macro- and micro- instabilities

Equilibrium Modelling

- Self-consistent asymptotic equilibrium already derived in the limit of $M \gg 1$
- How robust is this ?

Theory and Modelling for CM

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Neoclassical Effects

- Neoclassical Fluxes?
- Neoclassical Toroidal Viscosity ?

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Macrostability

- The Infamous Interchange
- Kelvin-Helmholtz-Rayleigh

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Microstability

- ITG / ETG ?
- Interchange?

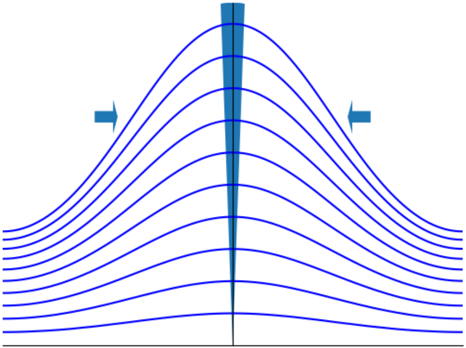
Ordering

Even though we have already assumed a rapidly-rotating plasma, we have not made any other assumptions.

- The plasma is rapidly rotating $M = u/c_s \gg 1$.
- The electron and ion temperatures are comparable $T_i/T_e \sim 1$.
- The Alfvén Mach number (defined with the line-average density) is of order unity

We also assume reflection symmetry in the vertical plane $z = 0$.

Disc-like structure



Solution structure

- Narrow Plasma layer near $z = 0$, with width scaling as $\delta z \sim R_{\text{mid}} M^{-2}$
- Inside the layer centrifugal forces balance curvature.
- Vacuum solution outside the layer

Layer Solution: 1

Dropping terms small in β, we have to solve

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R\boldsymbol{\nabla} R = -\boldsymbol{\nabla}\frac{B^2}{2\mu_0} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B}, \qquad (1)$$

we expect the rapid rotation to localise the density into a disc-like layer near the midplane (i.e at $z = 0$). Making the assumption that the density localises and that gradients in z dominate over gradients in R, we have to solve for a field that balances centrifugal forces and magnetic tension in the radial direction:

$$m_i N_i \exp\left[\frac{m_s\omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0}B_z\frac{\partial B_R}{\partial z} \qquad (2)$$

Layer Solution: 2

Introducing the field line shape as $R = R(\psi, z)$ we can write this as an equation for R:

$$m_i N_i \exp\left[\frac{m_s\omega^2}{4T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0} B_z \frac{\partial}{\partial z}\left(B_z \left.\frac{\partial}{\partial z}\right|_{\psi} R\right). \tag{3}$$

To reduce the complexity of the system, we note that

$$\nabla \cdot \boldsymbol{B} = \frac{\partial B_z}{\partial z} + \frac{\partial B_R}{\partial R} \approx \frac{\partial B_z}{\partial z} = 0, \tag{4}$$

and so B_z is constant (with respect to z) inside the layer. Then, we observe that

$$\begin{aligned} B_z \left.\frac{\partial}{\partial z}\right|_R &= B_z \left.\frac{\partial}{\partial z}\right|_{\psi} - B_z \left.\frac{\partial R}{\partial z}\right|_{\psi} \left.\frac{\partial}{\partial R}\right|_z \\ &= B_z \left.\frac{\partial}{\partial z}\right|_{\psi} - B_R \left.\frac{\partial}{\partial R}\right|_z \approx B_z \left.\frac{\partial}{\partial z}\right|_{\psi} \end{aligned} \tag{5}$$

Layer Solution: 3

Hence we have an equation purely along the field line:

$$m_i N_i \exp\left[\frac{m_s \omega^2}{2T_s}\left(R^2 - R_{\max}^2\right)\right]\omega^2 R = \frac{1}{\mu_0}\left(B_z^2 \left.\frac{\partial^2}{\partial z^2}\right|_{\psi} R\right). \tag{6}$$

Simplifying by assuming that R changes only by a small amount inside the layer, we write

$$R \approx R_{\max}(\psi) - \delta R \tag{7}$$

we can solve to find that

$$\delta R = \frac{4}{M^2} R_{\max} \ln\left[\cosh\left(\frac{M^2}{4}\lambda\frac{z}{R_{\max}}\right)\right], \tag{8}$$

with $M = \omega R_{\max}/c_s$ and

$$\lambda = \left(\frac{4}{M^2}\frac{N_i m_i \omega^2 R_{\max}^2}{B_z^2/\mu_0}\right)^{1/2} \tag{9}$$

where $\lambda \sim 1$ as $M^2 \to \infty$.

Layer Solution: 4

From this solution we can now compute the density profile:

$$n_i = N_i \operatorname{sech}^2\left(\frac{M^2}{4}\lambda\frac{z}{R_{\rm max}}\right) \qquad (10)$$

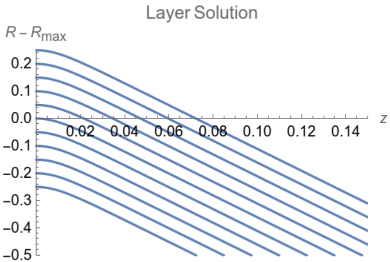
and calculate the field-line-averaged density $\bar{n}_i$ in terms of N_i to finally eliminate N_i:

$$N_i = \frac{1}{32}\frac{\bar{n}_i}{R_{\rm max}}M^2\overline{M}_A^2, \qquad (11)$$

where the average Alfven Mach number is

$$\overline{M}_A^2 = \frac{\bar{n}_i m_i \omega^2 R_{\rm max}}{B_z^2\,/2\mu_0} = 4\lambda, \qquad (12)$$

which allows us to eliminate λ.



Exterior Solution

Now we need to solve outside the layer. Thankfully, our solution for n_i is such that it becomes a delta function (consistent with our assumptions). The current layer due to the plasma is

$$J_\phi = [B_R]_{0-}^{0^+} = 2\lambda B_z = \frac{1}{2}\overline{M}_A^2 B_z \quad (13)$$

Greens Function

For a current layer at $z = 0$:

$$G(R,z,R') = \frac{1}{2\pi}\sqrt{(R+R')^2+z^2}\left[\left(1-k^2\right)K(k) - E(k)\right] J_\phi \quad (14)$$

Giving

$$\psi(R,z) = \psi^{\mathrm{coil}} + \int_a^b G(R,z,R')J_\phi(R')dR' \quad (15)$$

Nonlinear Integral Equation for ψ. We solve it numerically.

Exterior Solution: 2

Simple Two-Coil Setup

- Exterior solution formed from the plasma plus two coils of unit strength at $R = 0.5$ and $z = \pm 1.0$
- The plasma pressure is uniform and we assume an ω profile of

$$\omega = \omega_0(\psi - \psi_{\min}) \times (\psi_{\max} - \psi)/ (\psi_{\max} - \psi_{\min})^2 \quad (16)$$

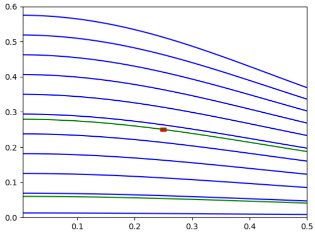


Figure: Vacuum Solution

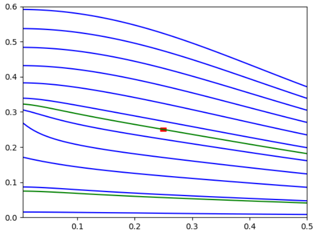


Figure: Plasma Solution

Exterior Solution: 3

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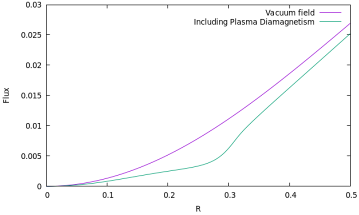


Figure: Midplane Psi

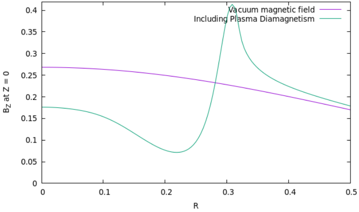


Figure: Midplane Magnetic Field

Non-Axisymmetric Perturbations

Perturb with a non-axisymmetric $\tilde{\boldsymbol{B}}$

- Caused by error fields in the coils – circles are easier than tori but nothing is perfect.
- Pick the simplest possible error, only modify the field strength $\tilde{\boldsymbol{B}} = \cos(N\phi)\,\boldsymbol{B}$.
- Define $\delta = \tilde{B}/B$.

Naïve estiamtes

If we assume that the torque τ_{NTV} is due to Braginskii η_0 then

$$\tau_{NTV} \sim \delta^2 \frac{n_i T_i}{\nu_{ii}} \frac{\omega}{a} \tag{18}$$

Compared to the classical perpendicular torque

$$\tau_\perp \sim \nu_{ii} \left(\rho_i / a \right)^2 m_i n_i R^2 \omega \tag{19}$$

Then $\tau_{NTV} \ll \tau_\perp$ requires

$$\delta \ll \nu_{ii} / \Omega_i \; . \tag{20}$$

Non-Axisymmettric Perturbations II: Drift Kinetics

To leading order, the toroidally-varying part of the drift kinetic equation is

$$\left(v_{\|}\boldsymbol{b} + \boldsymbol{u}\right) \cdot \nabla \tilde{F}_s + \tilde{\boldsymbol{V}}_{\mathrm{D}s} \cdot \nabla \psi \frac{\partial F_{0s}}{\partial \psi} = C_L\left[\tilde{F}_s\right] \tag{21}$$

If we assume that $\boldsymbol{u} \gg v_{\mathrm{th}_s}$ and go to the collisionless limit, then

$$\tilde{F}_s \sim \int^{\phi} \frac{\boldsymbol{V}_{Ds} \cdot \nabla \psi}{\omega} d\phi' \tag{22}$$

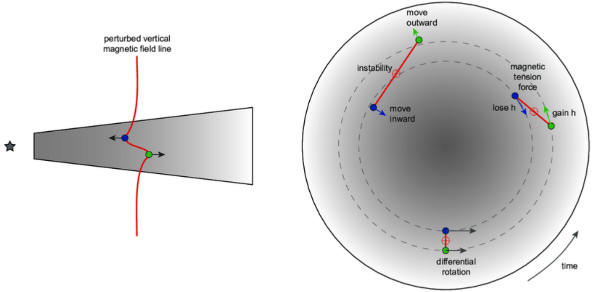
leading to

$$\tau_{NTV} \sim \frac{\delta^2 N}{M} \left(\frac{\rho_i}{a}\right)^2 n_i m_i v_{\mathrm{th}_i}^2 \tag{23}$$

and a limit of

$$\delta^2 \ll \frac{\nu_{ii} M^2}{\left(v_{\mathrm{th}_i}/R\right) N} \tag{24}$$

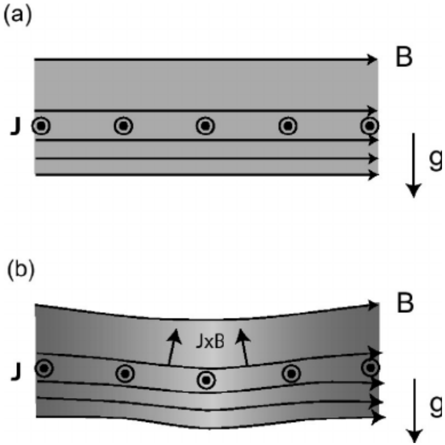
Axisymmettric Stability I: MRI



Magnetorotational Instabiltiy for $B_\phi = 0$

- Driven by differential rotation ($\frac{d\omega}{d \ln R} < 0$) coupled to magnetic tension ($k_{||} \neq 0$)
- Stabilised by $k_{||} v_A$ if it is large eough ($k^2 v_A^2 \gtrsim -\frac{d\omega^2}{d \ln R}$).

Axisymmettric Stability II: Parker



The Parker Instability

- Driven by magnetic buoyancy (effective gravity $\omega R^2 > 0$) and parallel compressibility (i.e. $k_{||} \neq 0$)
- Stabilised by resistance to compression, i.e. the sound wave, when $k_{||}^2 c_s^2 \gtrsim \omega^4 R^2 / v_{\mathrm{A}}^2$

Axisymmettric Stability III: What is $k_{\| \, \mathrm{min}}$

Stabilisation due to $k_\|$

- Both instabilities require $k_\|$ but are also stabilised by it.
- Thus, the most unstable mode is at the smallest non-zero $k_\|$
- Given our plasma is **narrow**, with width δ, we estimate

$$k_{\| \, \mathrm{min}} \approx \delta^{-1} = \left(\frac{M^2 \bar{M}_A^2}{16 R_{\mathrm{max}}} \right) \tag{25}$$

- This is asymptotically larger (as $M \to \infty$) than the drive terms, and so we expect to stabilise these modes completely

Axisymmettric Stability & Waves: A sketch of a calculation

Eigenfunction Expansion

- Following Ogilvie (1998) and Papaloizou & Szuszkiewicz (1992) we solve

$$B_z^2 \frac{\partial}{\partial z} u_n + \rho \hat{\omega}_n^2 u_n = 0, \qquad \lim_{z \to \infty} \frac{\partial u_n}{\partial z} = 0, \tag{26}$$

  and solve the linear theory as an expansion in the eigenmodes u_n.
- For our problem B_z is constant and $\rho = \rho_0 \operatorname{sech}^2 \left(\frac{16 R_{\max}}{M^2 \bar{M}_A^2} z \right)$
- The u_n are then given by

$$u_n = \mathrm{P}_l \left(\tanh \left(\frac{16 R_{\max}}{M^2 \bar{M}_A^2} z \right) \right), \quad l \in \mathbb{N} \tag{27}$$

  with eigenvalues

$$\hat{\omega}_n^2 = \left(\frac{M^4 \bar{M}_A^4 \hat{v}_A^2}{256 R_{\max}^2} \right) \left[(2l+1)^2 - 1 \right] \tag{28}$$

Axisymmettric Stability & Waves: Torsional Alfven Time

Torsional Alfven Waves

- Taking the lowest nonzero eigenvalue

$$\tau_A \sim \left(\frac{4}{M}\right)^4 \frac{\pi}{4} \frac{1}{\bar{M}_A^3} \frac{1}{\omega} \tag{29}$$

- The time to radiate an Alfven wave to infinity is shorter than the cyclic time – very limited winding up of the field.
- **TODO: Apply torsional breaking theory from star formation (Gillis, Mestel & Paris 1973) to check if any radiated momentum is important**

Non-axisymmettric Theory: Kelvin-Helmholtz

Rayleigh's Theorem

- Flow profile is ideally stable if

$$\frac{d}{dR} R^{-1} \frac{d}{dR} R^2 \omega \neq 0 \tag{30}$$

- Trivially satisfied for constant viscosity solution
- Preliminary evidence of transport profiles suggests that realistic profiles also satisfy this theorem.

Viscosity / FLR Effects

- Plane Pouseille Flow is known to be stable at infinite Reynolds number, but unstable at large finite Reynolds number. This is due to viscosity exciting "negative energy waves".
- FLR Effects play the same role as viscosity for a rotating plasma. For $\rho^* \sim 0.05 - 0.1$, perhaps the effective Reynolds number is not enough to excite these modes?

Conclusions

- We have found a new magnetic equilibrium for rapidly-rotating mirrors from an asymptotic expansion in $M^2 \gg 1$
- This equilibrium consists of a disc-like plasma and an exterior vacuum solution.
- There is evidence of an equilibrium limit where this asymptotic equilibrium breaks down.
- The equilibrium is robust to ripple.

Current work

- Detailed calculations being prepared for publication.
- Application of this model to CMFX - integrate it into the 0D model (currently uses a square well))
- Integration of this model with transport models, much faster than a Grad-Shafranov solve