Generalized entropy in collisionless plasmas: *navigating an uncertain landscape*

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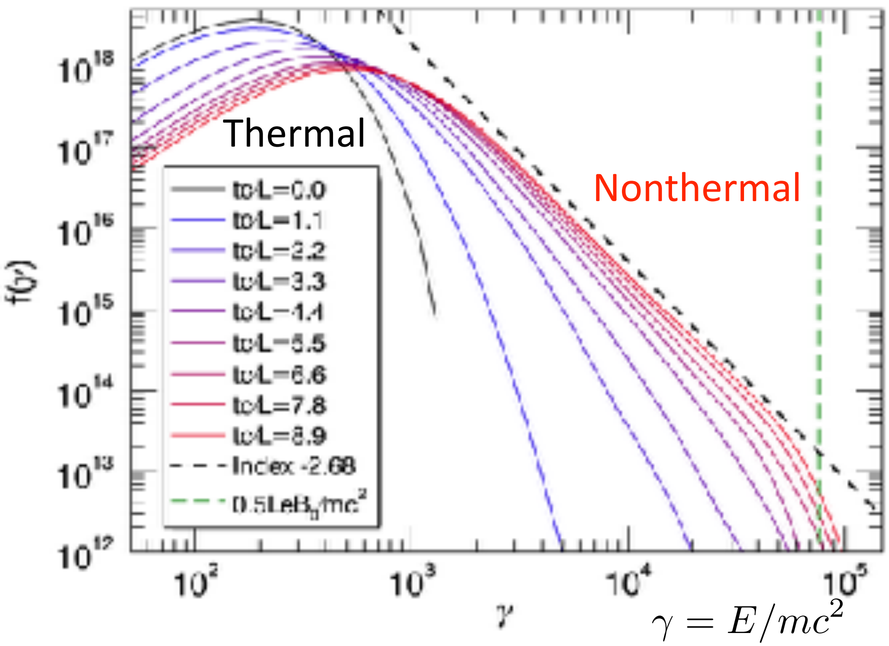
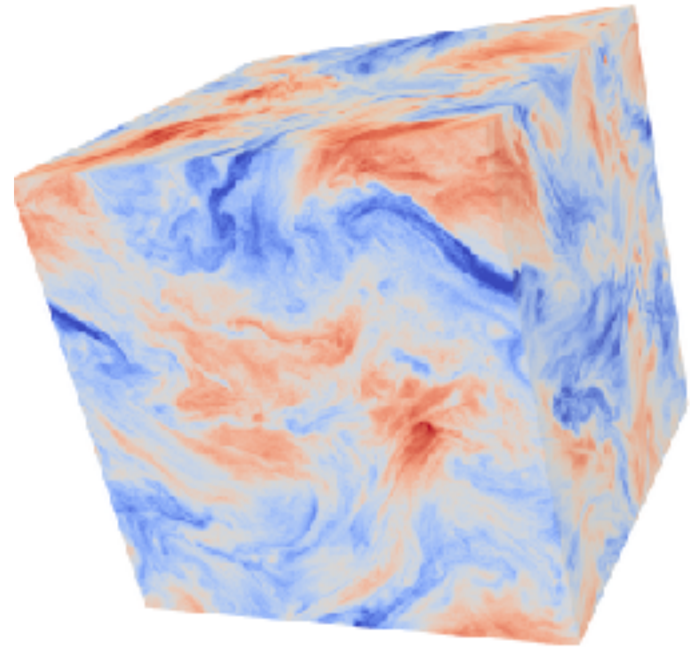
The Ninth Wave, Ivan Aivazovsky

“Seed motivation”: relativistic turbulence





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Kinetic Turbulence in Relativistic Plasma:
From Thermal Bath to Nonthermal Continuum

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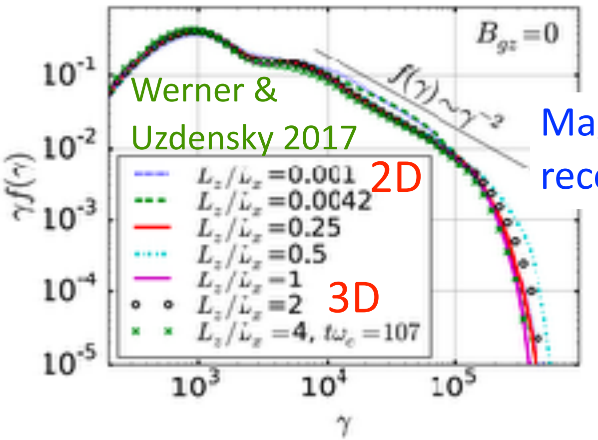
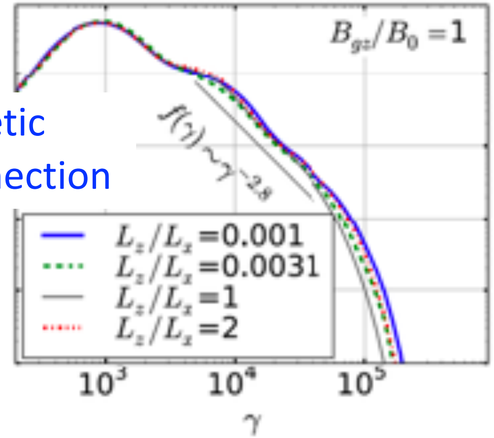
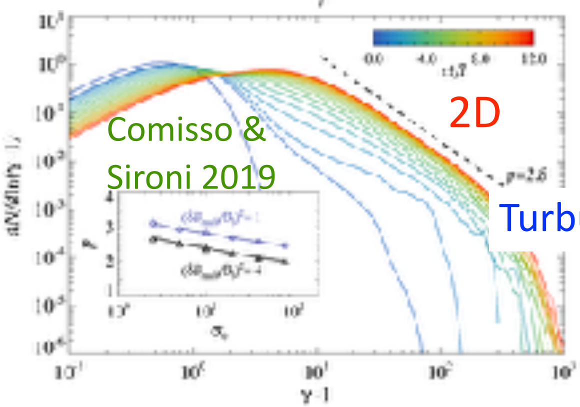
Why are nonthermal, power-law distributions so common?

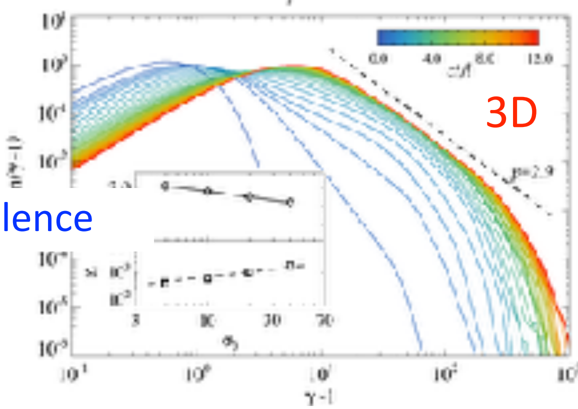
$$f(E) \propto E^{-\alpha}$$

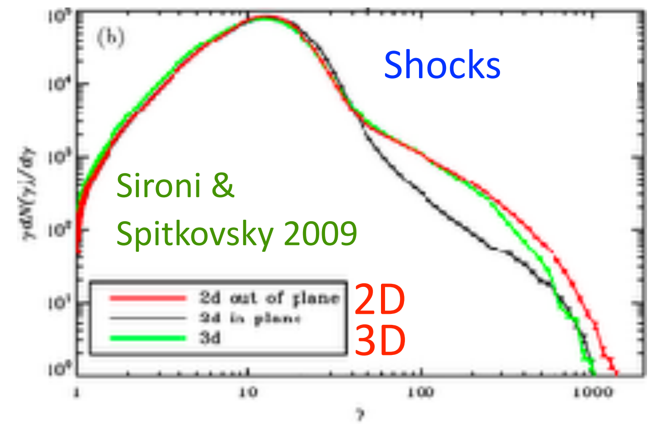
all around!

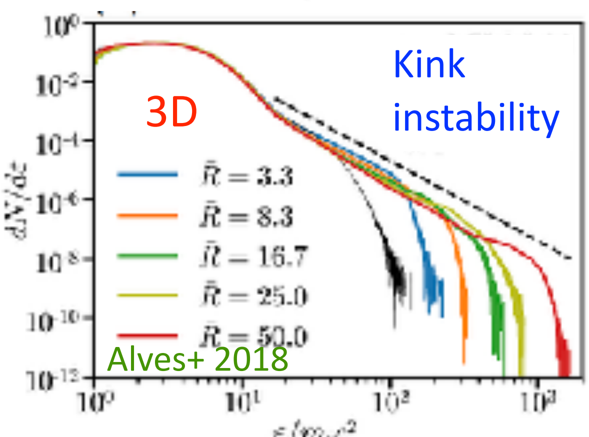


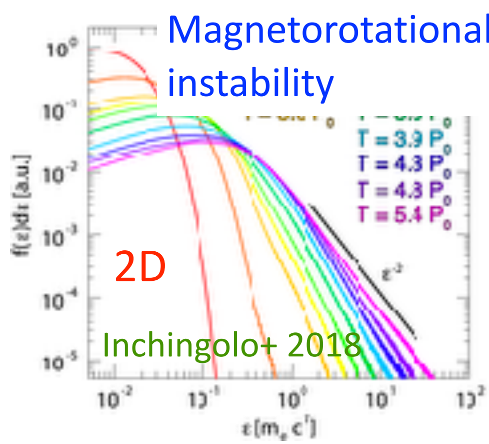


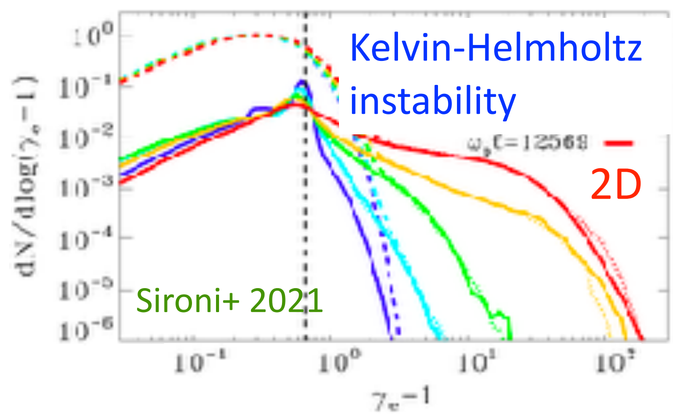












How does entropy fit in?

- Collisionless plasma processes exhibit irreversibility, but characterizing it is nontrivial due to nonthermal nature
- Prevalence of power-law particle distributions in systems with varying acceleration/trapping/escape mechanisms suggests universal underlying principles
- Why don't collective effects cause collisionless plasmas to relax to thermodynamic state of maximum entropy (the thermal distribution)?
- What is role of entropy in all of this? Can it be used as a constraint or as a guiding principle? Many current theories of energization are agnostic to entropy…
- There is a gap in our understanding…

"If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations — then so much the worse for Maxwell's equations. If it is found to be contradicted by observation — well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in the deepest humiliation." — Arthur Eddington

What this talk is about

- New/speculative ideas for understanding entropy production (irreversibility) in collisionless plasmas and its role in shaping nonthermal distributions

- How should we characterize entropy in a collisionless plasma?
  - Dimensional representation of generalized entropy: “Casimir momenta”
- What happens to entropy during dissipative processes in collisionless plasmas?
  - Case study: particle-in-cell simulations of relativistic turbulence
- What is generalized entropy “useful” for?
  - Modeling power-law energy distributions arising from dissipative processes

Part I: Characterizing generalized entropy

What happens to entropy in a collisionless plasma?

1. **Entropy production via violations of Vlasov equation?**
   - Nonlinear entropy cascades (Schekochihin+ 2009, Eyink 2018; see Nastac+)
   - Other routes to singularities (e.g., phase mixing)
2. **Coarse-grained entropy production, fine-grained entropy conservation?**
   - Vlasov valid microscopically, but system irreversible macroscopically (see Ewart+)
   - Scrambling of information at small (kinetic) scales where nobody can see/care
3. **Entropy conservation at both coarse-grained and fine-grained scales?**
   - Would explain prevalence of nonthermal distributions
   - Consistent with low entropy production rates in PIC simulations (Liang+ 2019)
   - But how does one understand irreversibility?

“Competition” between entropy conservation and entropy production?
(combo of irreversible “thermal heating” and reversible “nonthermal acceleration”?)

Roll over Boltzmann, and tell Gibbs the news…

- A note before proceeding… Boltzmann-Gibbs entropy *S* is not the only game in town!
- Infinite number of "generalized entropies" exist from information theory

Renyi (1961): $$H_\alpha = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{n} p_i^\alpha \right)$$

Tsallis (1988): $$S_q = \frac{k}{q-1} \left(1 - \sum_{i=1}^{n} p_i^q \right)$$

reduce to Shannon (Boltzmann-Gibbs) entropy when $\alpha \to 1$ or $q \to 1$

"Superstatistics" (Beck & Cohen 2003), etc.

- Generalized entropies are nonextensive/nonadditive, useful for systems with long-range correlations where "information" not expected to be additive
- Applications to finances, cold atoms, solar wind, dusty plasmas, spin glass relaxation, turbulent flow, galactic dynamics, …
- Notably, Tsallis statistics have been suggested as a framework for explaining kappa distributions of nonthermal populations in solar wind (e.g., Milovanov & Zelenyi 2000, Leubner 2002, Livadiotis & McComas 2009)

Vlasov framework

- Vlasov equation for collisionless plasma [feel free to add collisions]:

$$\partial_t f + \boldsymbol{v} \cdot \nabla f + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \qquad \frac{\partial}{\partial \boldsymbol{p}} \cdot \boldsymbol{F} = 0$$

where $f(\boldsymbol{x}, \boldsymbol{p}, t)$ is "fined-grained" plasma distribution for a given species

$N = \int d^3x d^3p f$ is number of particles (assumed asymptotically large)

$\boldsymbol{v} = \frac{\boldsymbol{p}c}{\sqrt{m^2c^2 + p^2}}$ is (relativistic) velocity

$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}, t)$ is force field (Lorentz force + external force + …)

- "First principles"… but possibly incomplete (singularity formation?)
  - Collisions ultimately needed?
  - Finite *N* effects?
- Note: in practice, must consider "coarse-grained" particle distribution, which may deviate from Vlasov equation

Casimir invariants

- Vlasov: $$\partial_t f + \boldsymbol{v} \cdot \nabla f + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \qquad \frac{\partial}{\partial \boldsymbol{p}} \cdot \boldsymbol{F} = 0$$

- Conserves phase-space volume: parcels of f are pushed around reversibly
- Equivalently, for closed/periodic boundary conditions, conserves the infinite set of "Casimir invariants": $$\mathfrak{C}_g(f) \equiv \frac{1}{N} \int d^3x d^3p g(f)$$

  where $g(f)$ is *any* differentiable function (subject to convergence)

$$\begin{aligned}
\frac{d\mathfrak{C}_g}{dt} &= \frac{1}{N} \int d^3x d^3p \frac{dg}{df} \partial_t f \\
&= -\frac{1}{N} \int d^3x d^3p \left[\nabla \cdot (\boldsymbol{v} g) + \frac{\partial}{\partial \boldsymbol{p}} \cdot (\boldsymbol{F} g) \right] \\
&= -\frac{1}{N} \int d^3p d\boldsymbol{S}_x \cdot \boldsymbol{v} g - \frac{1}{N} \int d^3x d\boldsymbol{S}_p \cdot \boldsymbol{F} g = 0
\end{aligned}$$

- Casimir invariants include Boltzmann-Gibbs entropy *S* (for $g = -f \log f$) and infinite number of other quantities (e.g., $g = f^\chi$)... **these are generalized entropies**!

How do we make sense of Casimir invariants?

- For simplicity, consider power-law functions:

$$\mathcal{C}_\chi(f) \equiv \frac{1}{N}\int d^3x d^3p f^\chi$$

where $\chi > 0$ is a "weight" parameter

- **Issue:** these Casimir invariants do not have physically meaningful dimensions, since distribution has units of inverse phase volume; $[f] = L^{-3} \times p^{-3}$

$$[\mathcal{C}_\chi] = L^{3(1-\chi)} \times p^{3(1-\chi)}$$

- Get physical dimensions (of angular momentum) by raising to another power:

$$\mathcal{C}_\chi^{1/3(1-\chi)} \qquad [\mathcal{C}_\chi^{1/3(1-\chi)}] = L \times p$$

- **Interpretation:** Length scale related to a typical number density n_0, momentum scale to a typical momentum/energy $\langle p \rangle$
- For applications with fixed mean density, but injected energy, can factor out $n_0^{-1/3}$…

Casimir momenta

- A dimensional representation of generalized entropy, with units of momentum:

$$p_{c,\chi}(f) = n_0^{1/3}\left(\frac{1}{N}\int d^3x d^3p f^{\chi}\right)^{-1/3(\chi-1)}$$

"Casimir momenta"

(VZ, PRX 2022)

- Represents a characteristic "spread" of distribution in momentum space
- Ideally conserved by Vlasov equation!
- Evolution indicates violation of Vlasov (irreversibility!) at corresponding energy: large weight $\chi \gg 1$ is low energy, small weight $\chi \lesssim 1$ is high energy
- Integral resembles generalized (*non-extensive*) entropies of Renyi (1961) and Tsallis (1988), with overall form similar to "exponential entropy" of Campbell (1966)
- Upon energy injection, measures nonthermality of dissipation:
    - For thermal dissipation, $p_{c,\chi}(t) \propto \langle p \rangle (t)$ for all χ
    - For nonthermal dissipation, $p_{c,\chi}(t)$ will vary with χ

Interpreting the index χ

Casimir momenta: $p_{c,\chi} = n_0^{1/3}\left(\frac{1}{N}\int d^3x d^3p f^\chi\right)^{-1/3(\chi-1)}$

- $\chi \to 1$ recovers dimensionalized Boltzmann-Gibbs entropy *S*: $p_{c,\chi\to 1} = n_0^{1/3} e^{S/3N}$

- For uniform isotropic distribution, χ maps to different values of momentum/energy

- Example: thermal (Maxwell-Juttner) distribution $f = \frac{n_0}{4\pi m^2 c T K_2(mc^2/T)} \exp\left(-\frac{\sqrt{m^2c^4+p^2c^2}}{T}\right)$

Ultra-relativistic limit: $p_{c,\chi} = \frac{(8\pi)^{1/3}}{3}\chi^{1/(\chi-1)}\langle p\rangle$
$(T/mc^2 \gg 1)$

$\chi^{1/(\chi-1)} \to \infty$ as $\chi \to 0$

$\chi^{1/(\chi-1)} \to 1$ as $\chi \to \infty$

Non-relativistic limit: $p_{c,\chi} = \frac{\pi}{2}\chi^{1/2(\chi-1)}\langle p\rangle$
$(T/mc^2 \ll 1)$

Reduction of entropy by anisotropy and inhomogeneity

- Inhomogeneities and anisotropies will decrease the Casimir momenta relative to the uniform/isotropic case:

$$f(\boldsymbol{x},\boldsymbol{p}) = \overline{f}(\boldsymbol{p}) + \delta f(\boldsymbol{x},\boldsymbol{p}) \quad \Longrightarrow \quad p_{c,\chi}(f) \le p_{c,\chi}(\overline{f})$$

$$f(\boldsymbol{x},\boldsymbol{p}) = f_{\rm iso}(p) + \delta f(p,\theta,\phi) \Longrightarrow \quad p_{c,\chi}(f) \le p_{c,\chi}(f_{\rm iso})$$

This follows from Holder's inequality: $\left(\int d^3x f/V\right)^\chi \ge \int d^3x f^\chi/V$ if $\chi < 1$

$$\left(\int d^3x f/V\right)^\chi \le \int d^3x f^\chi/V \quad \text{if} \quad \chi > 1$$

- Generalized maximum entropy state will be isotropic, uniform!
- Interpretation: *any* nontrivial structure will lower entropy

Growth of Casimir momenta for global distribution

- Casimir invariants of global (system-averaged) distribution evolve via:

$$\overline{f}(\boldsymbol{p},t) \equiv \int d^3x f(\boldsymbol{x},\boldsymbol{p},t)/V$$

(1) $$\frac{d\mathfrak{C}_g(\overline{f})}{dt} = \int d^3p g'(\overline{f})\partial_t \overline{f} = \int d^3p g''(\overline{f}) \frac{\partial \overline{f}}{\partial \boldsymbol{p}} \cdot \boldsymbol{\mathcal{F}}$$

$$\boldsymbol{\mathcal{F}}(\boldsymbol{p},t) \equiv \int d^3x \boldsymbol{F} f/V$$

- Compare to "heating" rate (increase in average $E = (m^2c^4 + p^2c^2)^{1/2}$) given by:

(2) $$\frac{Q}{V} = \frac{d}{dt}\int d^3p d^3x \frac{Ef}{V} = \int d^3p E \partial_t \overline{f} = \int d^3p \boldsymbol{v} \cdot \boldsymbol{\mathcal{F}}$$

- If energy injected to system, then $Q > 0$ and $\boldsymbol{v} \cdot \boldsymbol{\mathcal{F}}$ must have net positive part…
- If $\overline{f}$ is monotonically decreasing with p, then $\partial \overline{f}/\partial \boldsymbol{p} \cdot \boldsymbol{\mathcal{F}}$ will tend negative
- (1) then implies $\mathfrak{C}_g(\overline{f})$ will typically grow if $g'' < 0$ and decline if $g'' > 0$
- Taking $g = f^\chi$, then $p_{c,\chi}(\overline{f})$ will tend to grow for all values of χ!

- While heuristic, this suggests: when energy is injected to a system, spatial structure will develop that lowers entropy, which (via Vlasov) must be compensated by increasing entropy of global distribution, as measured by Casimir momenta

Casimir momenta in PIC simulations

- Now that we've introduced the Casimir momenta,

$$p_{c,\chi} = n_0^{1/3} \left(\frac{1}{N} \int d^3x d^3p f^\chi \right)^{-1/3(\chi-1)}$$

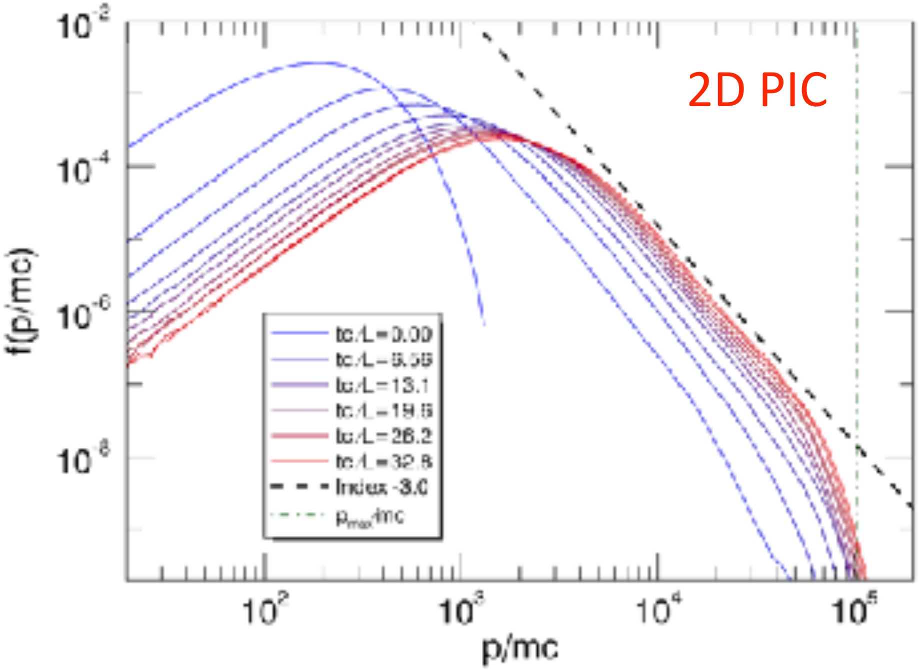
  let's see what happens to them in PIC simulations!

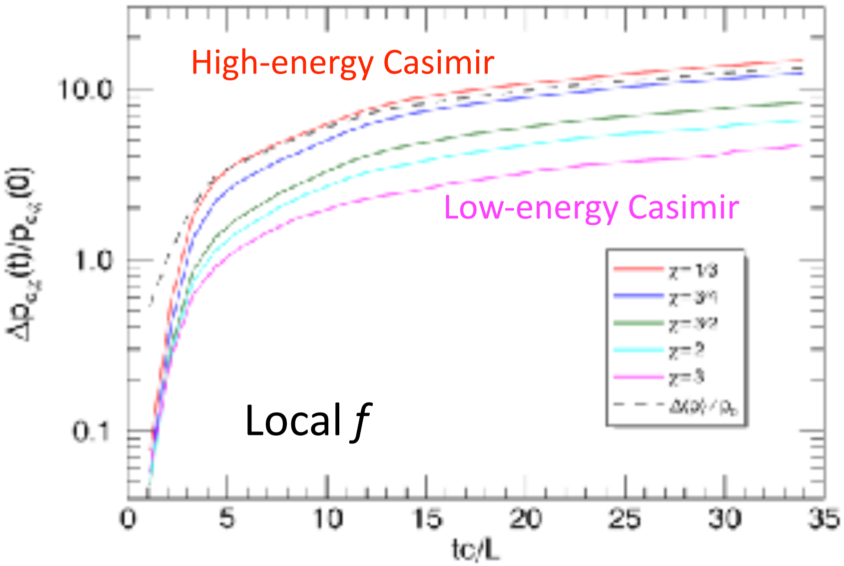
- Recall: upon energy injection,
  - For thermal dissipation, $p_{c,\chi}(t) \propto \langle p \rangle (t)$ for all χ
  - For nonthermal dissipation, $p_{c,\chi}(t)$ will vary with χ
- 2D PIC simulations (3D in momentum) using *Zeltron* (code: Cerutti+ 2013)
- Relativistically hot pair plasma (motivated by nonthermal particle acceleration)

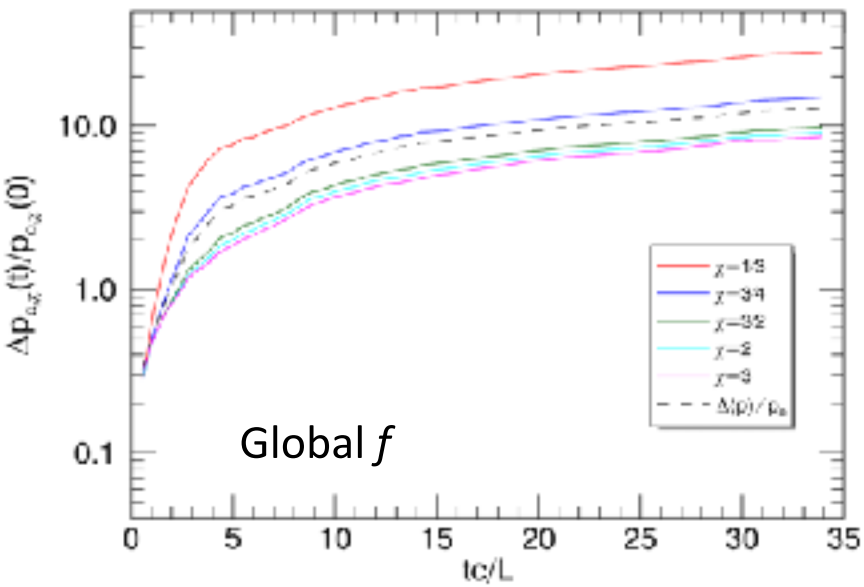
$T_0/m_e c^2 = 100$ $\beta_0 = 16\pi n_0 T_0/B_0^2 = 1/4$ $L/2\pi\rho_{e0} \approx 109$

- Casimir momenta calculated from distribution on "coarse-grained" grid
  - Up to 64^2 position-space bins (32^2 cells per bin), 256^3 momentum-space bins
  - Momentum space bin size adapts to local average: $\Delta p_{i,\mathrm{bin}} = p_{i,\mathrm{rms}}/4$

Casimir momenta in (2D) turbulent flow







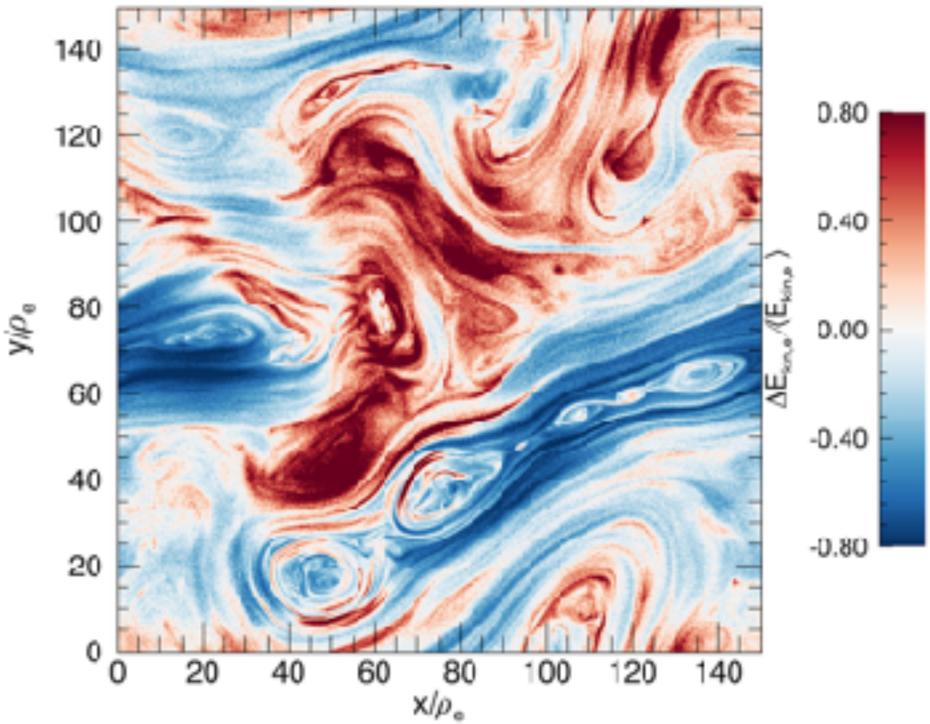
Verdict: Vlasov is violated
(especially at high energy)
Entropy is produced!

Probable cause: entropy cascade

Spatial structure of entropy

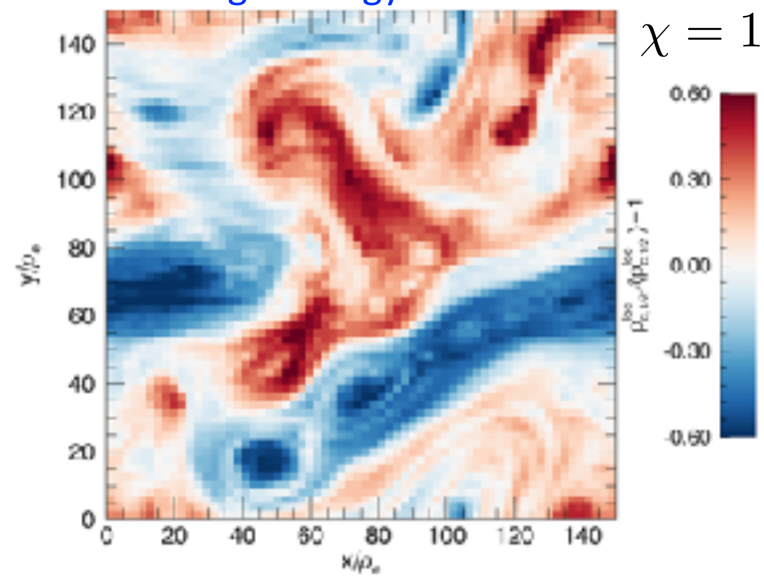
Local Casimir momenta are proxy for irreversible dissipation

Plasma energy



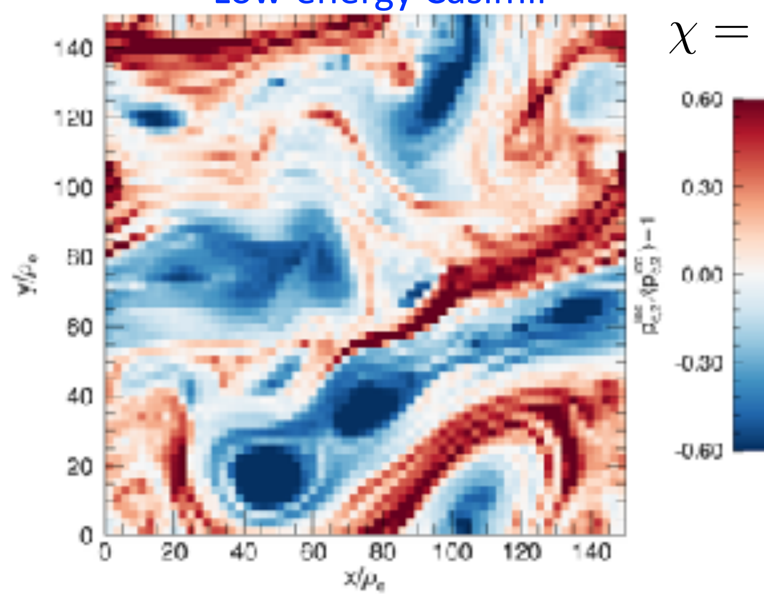
High-energy Casimir

$\chi = 1/2$



Low-energy Casimir

$\chi = 2$



Part I summary: characterizing generalized entropy

- Anomalous entropy production can be characterized by non-conservation of infinite set of Casimir momenta (representing generalized entropy):

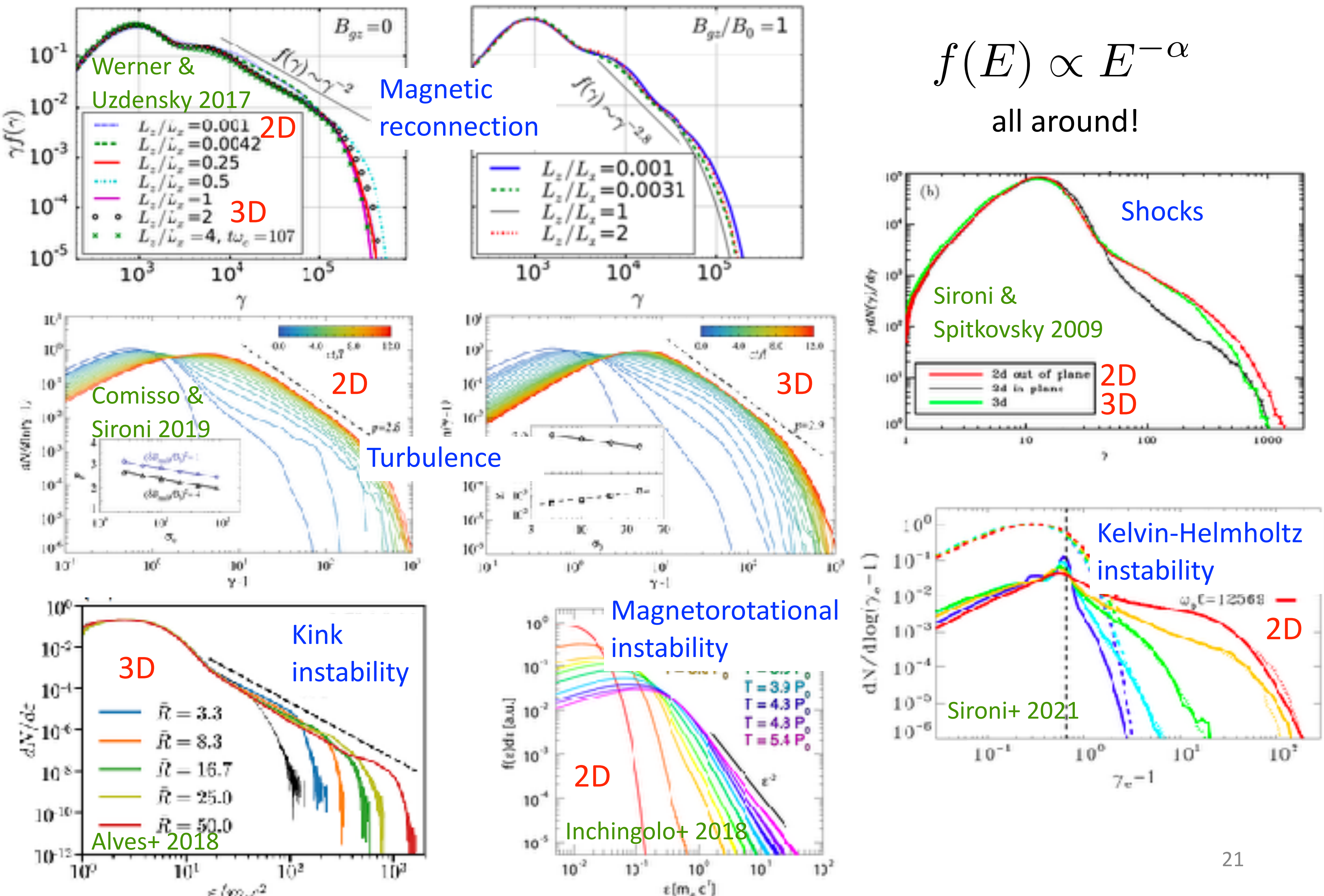
$$p_{c,\chi} = n_0^{1/3} \left(\frac{1}{N} \int d^3x d^3p f^\chi \right)^{-1/3(\chi-1)}$$

- Growth of Casimir momenta (following injection of energy) indicates violation of Vlasov equation, and thus irreversibility
- By this merit, PIC simulations indicate that (relativistic) turbulence leads to efficient entropy production in collisionless plasmas, mainly at high energies

- Future directions:
  - local Casimir momenta as a proxy for sites of energy dissipation (applications to solar wind and Earth's magnetosphere; see Pezzi+ 2021)
  - more analytical investigation on simplified problems (e.g., density fluctuation)
  - more numerical investigation on complex problems (e.g., 3D turbulence)
  - connections with other areas of statistical physics (e.g. gravitational dynamics)

Part II: Generalized maximum entropy and particle acceleration

Back to the motivation…



$$f(E) \propto E^{-\alpha}$$

all around!

Why do power law distributions even exist?

- Acceleration mechanisms are often Fermi-type processes described by quasilinear theory:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(D_{pp} \frac{\partial f}{\partial \boldsymbol{p}} \right) \qquad D_{pp} = \frac{1}{4} \frac{p^2}{\tau_{\rm acc}}$$

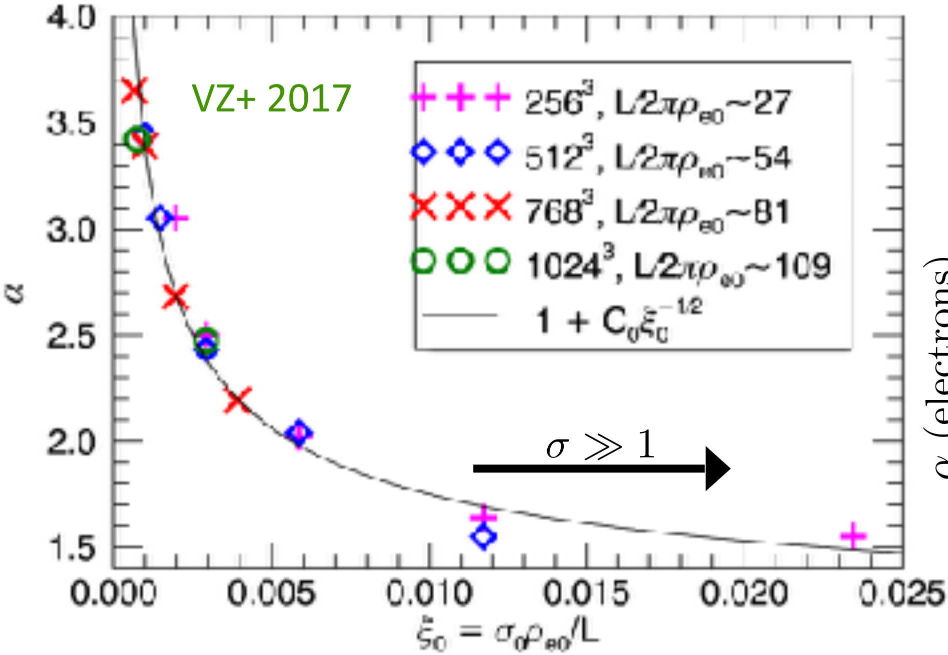
- However, knowledge of acceleration mechanism alone is insufficient to predict power law and its index α
- Classical picture: Fermi acceleration must be balanced by escape or trapping mechanism to get a power law
- PIC simulations: no escape (periodic box), unclear trapping, diverse mechanisms

- Not obvious how to model power-law distributions seen in PIC simulations

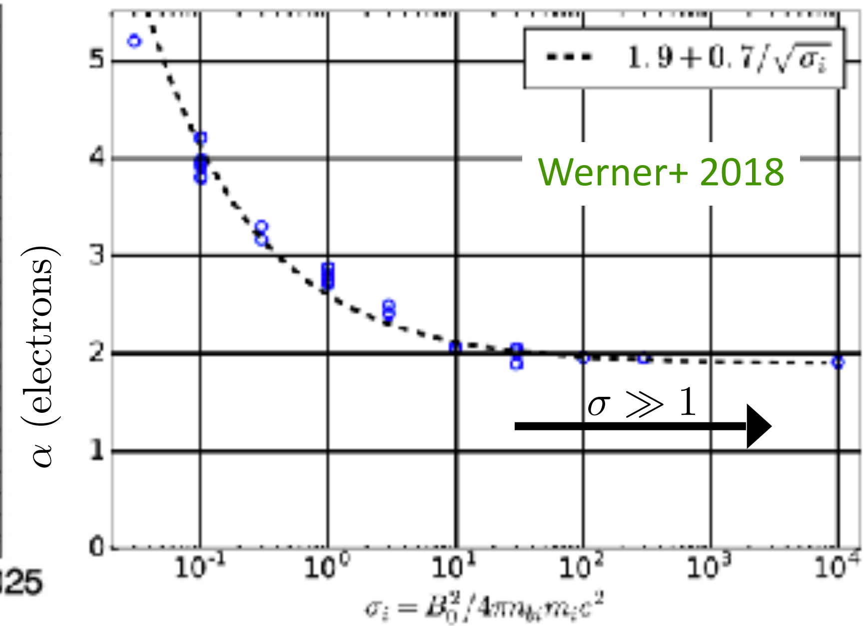
Mysteries of particle acceleration

- Relativistic turbulence and magnetic reconnection both exhibit similar scalings of power-law index α versus magnetization, or beta (e.g., Werner+ 2018, VZ+ 2017)
- Similar in 2D and 3D domains (e.g., Werner & Uzdensky 2017 for relativistic reconnection, Comisso & Sironi 2019 for relativistic turbulence)

Relativistic turbulence



Relativistic magnetic reconnection



Magnetization: $\sigma = B^2/4\pi h$ $\qquad$ $v_A/c = \sqrt{\sigma/(1+\sigma)}$

Modeling particle acceleration with Casimir momenta

- Suppose dynamics cause irreversible dissipation mainly at a super-thermal energy
- Model: maximize Casimir momentum at that scale! ("dissipation momentum" p_{c,χ_d})

$$\mathcal{L} = N^{1/3}\left(\int d^3p f^{\chi_d}/N\right)^{-1/3(\chi_d-1)} - \lambda_1\left(\int d^3p f - N\right) - \lambda_2\left[\int d^3p E(p) f - N\overline{E}\right]$$

Casimir momentum — Density constraint — Energy constraint

$\delta\mathcal{L} = 0$ upon variations δf

- Generalized maximum entropy distribution:

$$f(\boldsymbol{p}) \propto [E(p)/E_b + 1]^{-1/(1-\chi_d)}$$

where

$$E(p) = (m^2c^4 + p^2c^2)^{1/2} - mc^2$$

E_b is determined by $\overline{E}$

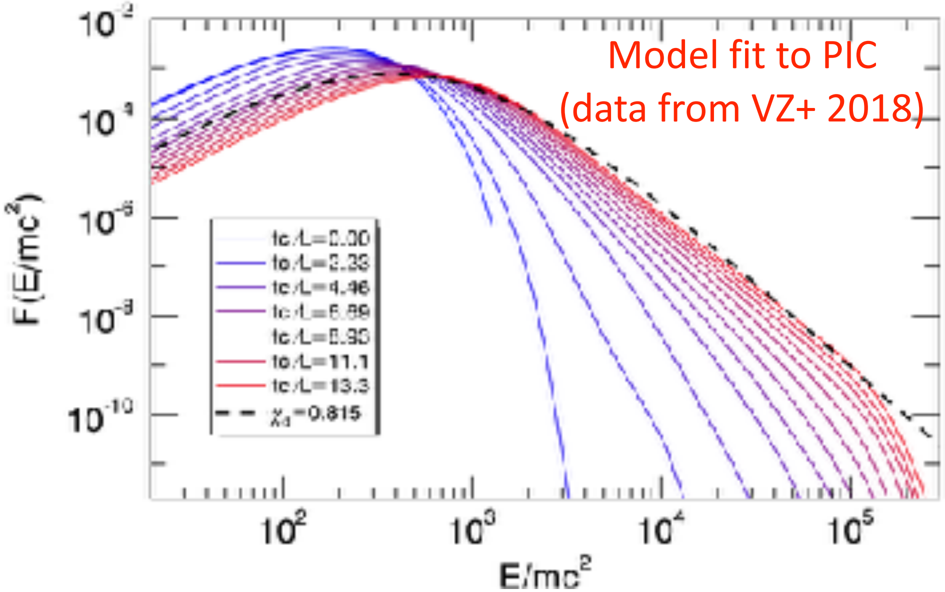
- One "free" parameter: index χ_d representing dissipation scale
- Power law if $\chi_d < 1$, thermal if $\chi_d = 1$, and flat-topped if $\chi_d > 1$

VZ, JPP 2020

Generalized maximum entropy distribution

Generalized maximum entropy distribution:

$$f(\boldsymbol{p}) \propto [E(p)/E_b + 1]^{-1/(1-\chi_d)}$$



- Fair fit to PIC simulations of relativistic turbulence
- Equivalent to "Tsallis distribution" obtained from maximizing Tsallis entropy
- Reduces to kappa distribution in non-relativistic limit, commonly used in space and astrophysical applications (e.g., Livadotis & McComas 2009)

Connecting power-law index to Casimir momenta

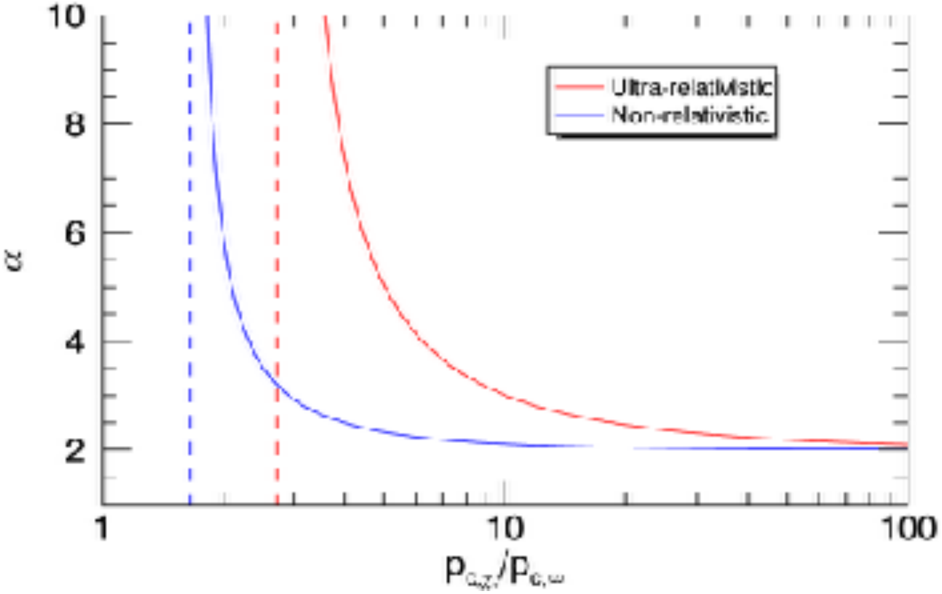
- Power-law index of energy distribution, α , can be related to ratio between "entropy-maximizing" momentum (p_{c,χ_d}) and "typical" momentum $(p_{c,\infty})$

Ultra-relativistic limit: $(E \gg mc^2)$

$$\frac{p_{c,\chi_d}}{p_{c,\infty}} \xrightarrow{\text{UR}} \left(\frac{\alpha+1}{\alpha-2}\right)^{(\alpha+2)/3}$$

Non-relativistic limit: $(E \ll mc^2)$

$$\frac{p_{c,\chi_d}}{p_{c,\infty}} \xrightarrow{\text{NR}} \left(\frac{\alpha-1/2}{\alpha-2}\right)^{(2\alpha+1)/6}$$



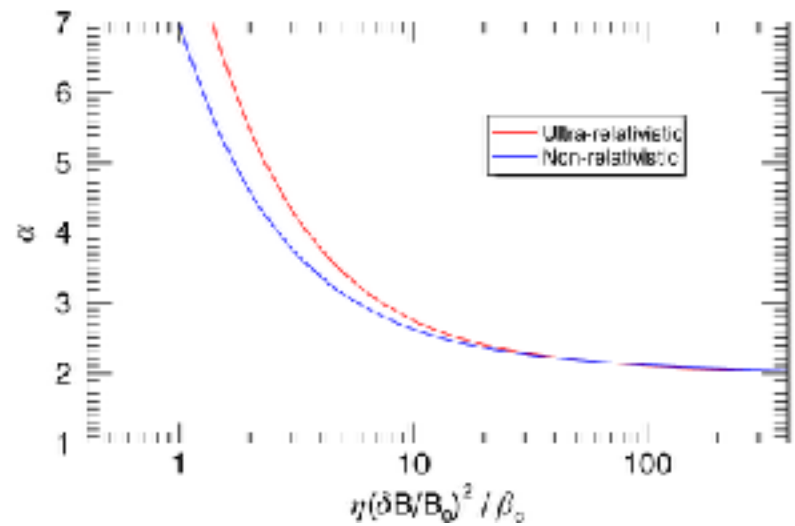
- Can we predict $p_{c,\chi_d}/p_{c,\infty}$ for given plasma parameters and energization mechanism?

Power-law index from "magnetic dissipation"

- Idealized model: suppose particles are energized by an amount comparable to the free magnetic energy before equilibration

$$E_{c,\chi_d} \sim eE_0 + \eta E_{\rm free} \qquad E_{\rm free} = \delta B^2/8\pi n_0 \qquad \eta \text{ is conversion efficiency}$$

$$\frac{p_{c,\chi_d}}{p_{c,\infty}} = \left[\frac{E_{c,\chi_d}(E_{c,\chi_d}+2mc^2)}{E_{c,\infty}(E_{c,\infty}+2mc^2)}\right]^{1/2}$$
$$\sim \left[\frac{(eE_0+\eta E_{\rm free})(eE_0+\eta E_{\rm free}+2mc^2)}{E_0(E_0+2mc^2)}\right]^{1/2}$$
$$\sim \left[\frac{[e+\eta(\delta B/B_0)^2/\beta_c][e+\eta(\delta B/B_0)^2/\beta_c+2/\theta_c]}{1+2/\theta_c}\right]^{1/2}$$



where

$\delta B/B_0$ fluctuation amplitude

$\theta_c = E_0/mc^2$ characteristic temperature

$\beta_c = 8\pi n_0 E_0/B_0^2$ characteristic plasma beta

Comparison of "magnetic dissipation" model to PIC

Idealized model: particles are energized by free magnetic energy before equilibrating

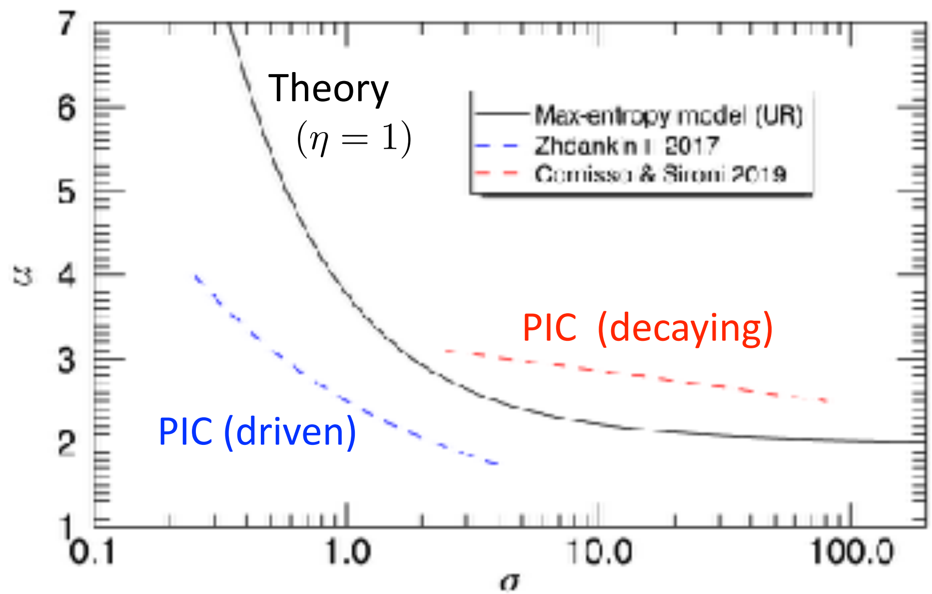
Model prediction (ultra-relativistic limit):

$$\eta \left(\frac{\delta B}{B_0}\right)^2 \frac{1}{\beta_c} = \left(\frac{\alpha+1}{\alpha-2}\right)^{(\alpha+2)/3} - e$$

In relativistic turbulence,

$$\beta_c = 1/4\sigma$$

$$\delta B/B_0 = 1$$



Theory close to relativistic turbulence simulations (VZ+ 2017, Comisso & Sironi 2019)

Similar to relativistic magnetic reconnection simulations (e.g. Werner+ 2019, Ball+ 2018)

Merits and limitations of generalized max entropy model

Merits:

- Explains ubiquitous appearance of power-law tails in particle distributions
- Predicts similar particle acceleration in 2D and 3D domains (*a priori*)
- May apply to turbulence and magnetic reconnection

Limitations:

- Assumes dynamics are sufficiently complex to enable generalized maximum entropy state, which may not always be the case
- Ignores dynamical constraints (such as anisotropy of global distribution)
- Assumes entropy maximization at a “single” energy scale, while mechanisms might compete over a range of energy scales in realistic cases
- Hysteresis (memory of initial distribution and time-dependent parameters) not accounted for

Part II summary: maximum entropy modeling

- Casimir momenta form a foundation for modeling particle acceleration from maximum entropy principles
- Generalized maximum-entropy distribution provides a fair fit to PIC simulations (which may be improved with more sophisticated modeling)
- Simple model for power-law index from "magnetic dissipation" is able to reproduce scaling of index versus magnetization observed in turbulence

- Future work:
  - more rigorous treatment of dissipation mechanisms
  - connect maximum-entropy modeling with Fokker-Planck equation, quasilinear theory, etc.
  - broader tests of model: numerical + experimental (e.g., solar wind distributions consisting of core and halo populations)
  - other processes: shocks, wave damping, etc.

Take-home messages

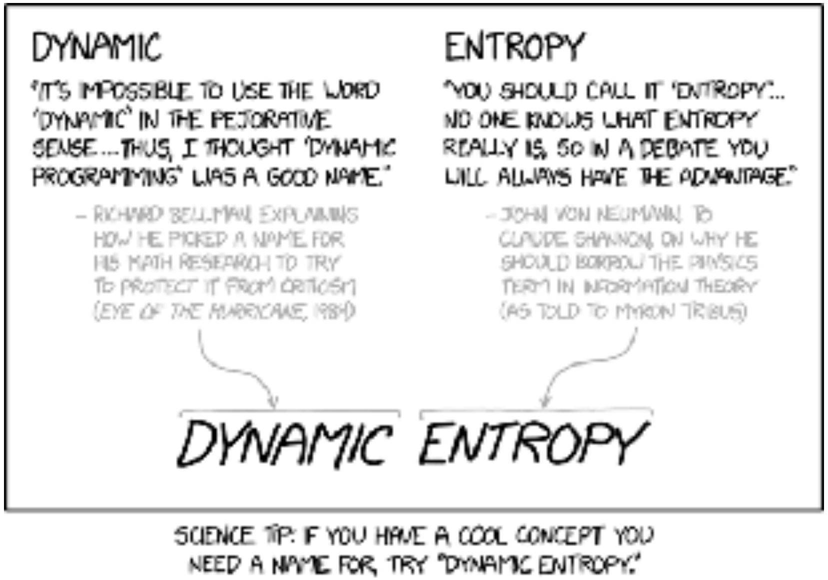
- Entropy is at the frontier of plasma physics
- New mathematical approaches such as the Casimir momenta, as well as increasing quality of kinetic simulations, may allow us to finally confront fundamental questions about entropy production in collisionless plasmas
- Incorporating entropy production into reduced modeling of nonthermal particle distributions is a promising avenue, and should be taken seriously

Thank you!

For more details:

1) V. Zhdankin PRX 2022, arXiv:2110.07025

2) V. Zhdankin JPP 2022, arXiv:2203.13054



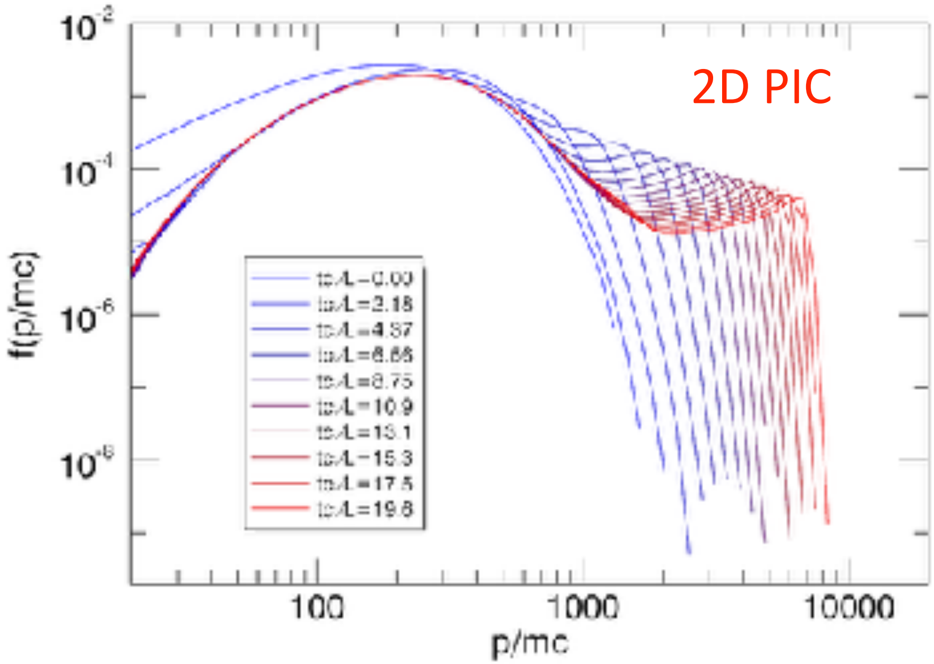
xkcd.com/2318

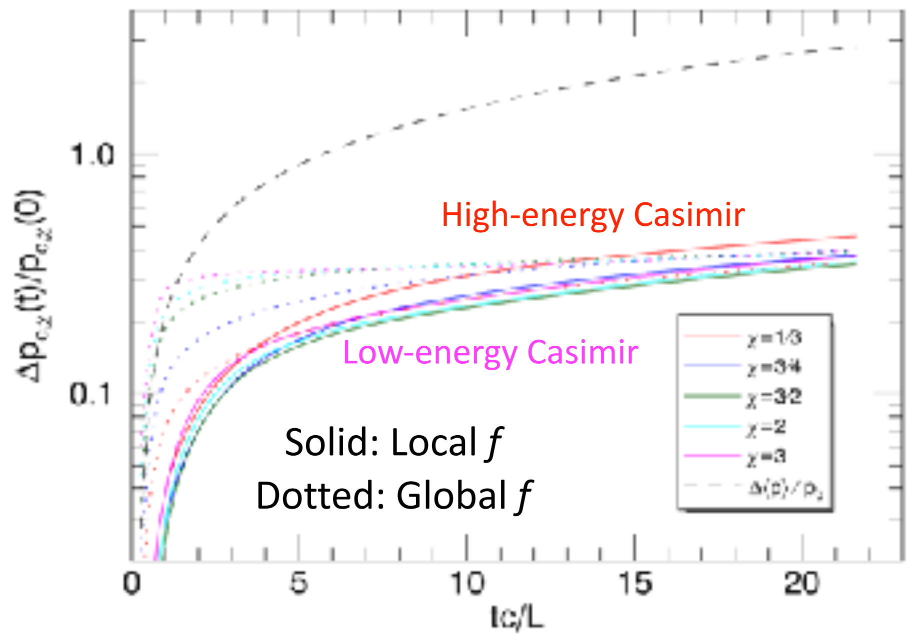
Open questions on generalized entropy

- Are Casimir momenta a sufficient basis, or does one need to expand to even more generalized entropies?
- Are Casimir momenta a useful measure of free energy?
- Can one build a generalized statistical mechanics? (see Schekochihin, Ewart, etc.)
- Are there other statistical applications? (note widespread use of Tsallis entropy in modeling nonlinear systems)

Example: Casimir momenta in a neutral shear flow

Uncharged particles, shear force: $\boldsymbol{F}_{\text{shear}}(x) = F_0 \sin(kx)\hat{\boldsymbol{y}}$





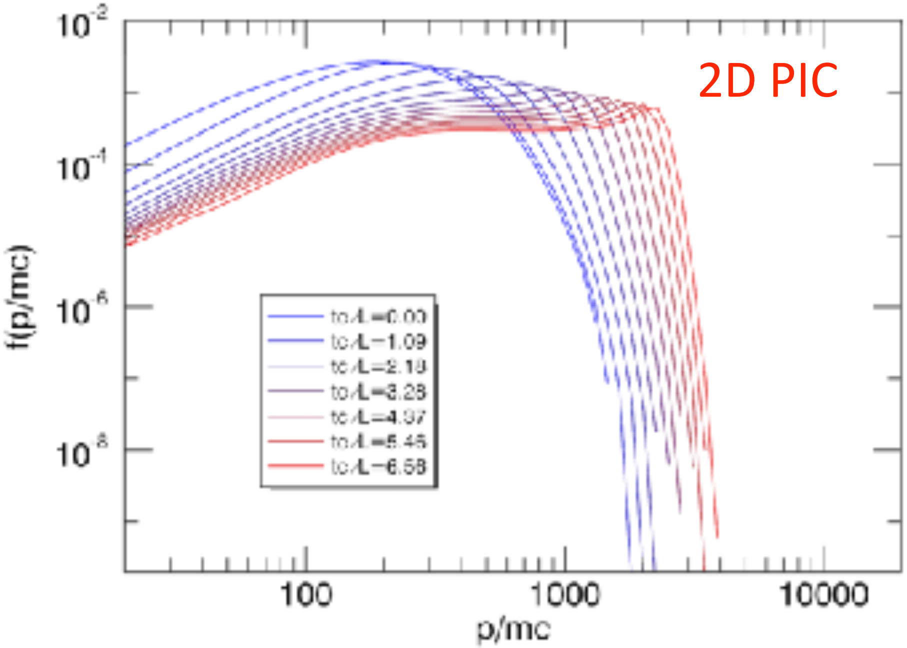
Increase in Casimir momenta is small relative to amount of energy injected

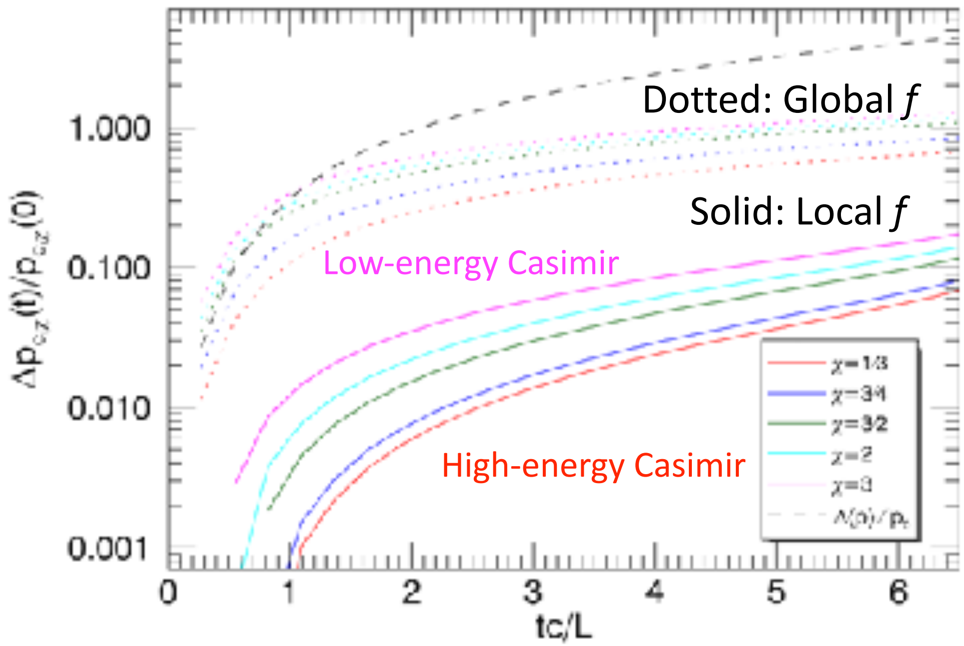
Verdict: Vlasov is satisfied (dynamics are reversible)

Implication: linear phase-mixing only leads to modest entropy production

Example: Casimir momenta in parallel shear flow

Pair plasma, parallel shear flow: $\boldsymbol{F}_{\text{shear}}(x) = F_0 \sin{(kx)}\hat{\boldsymbol{y}}$ $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{y}}$



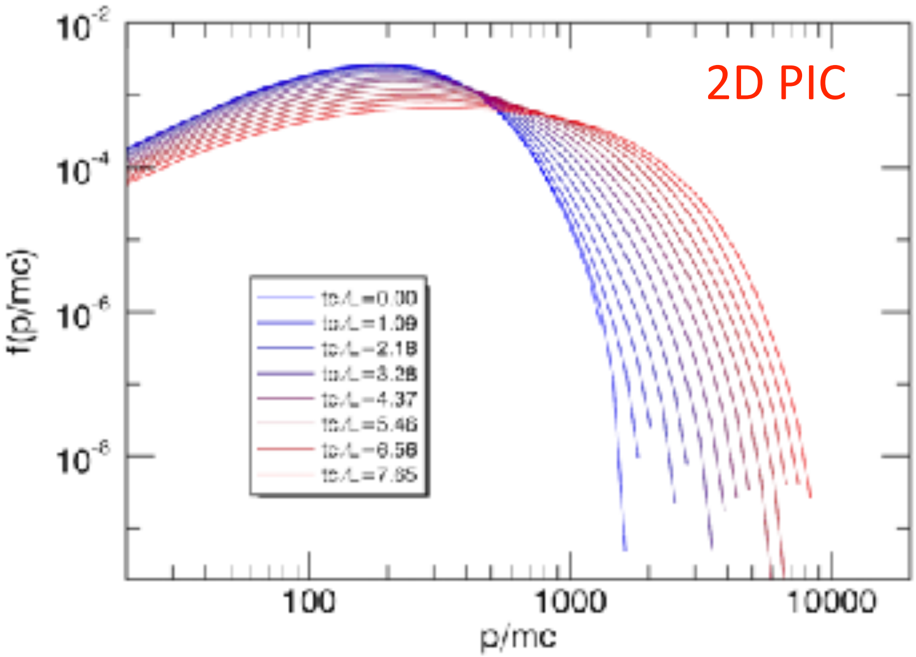


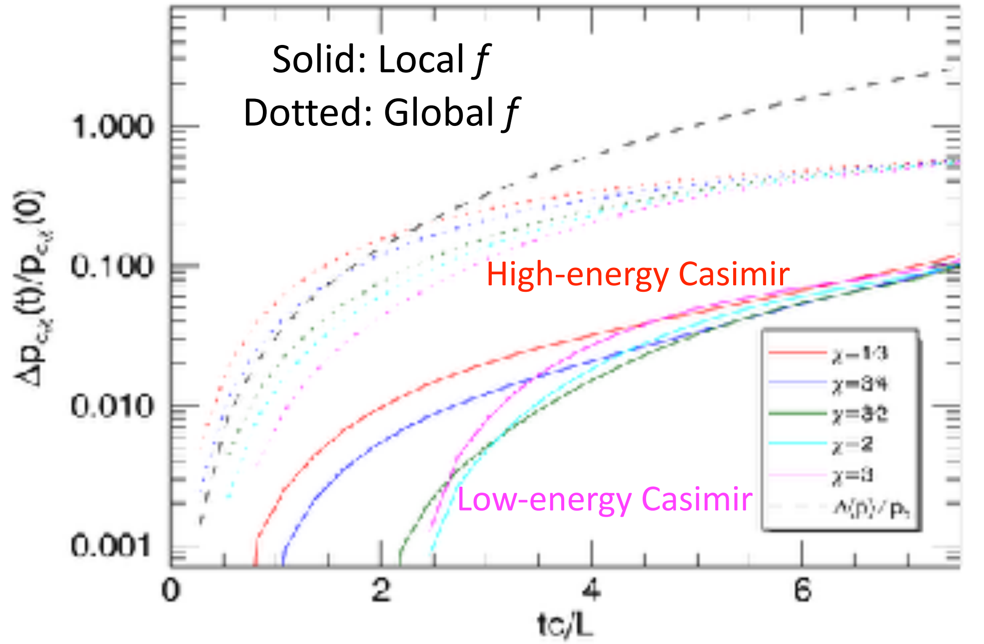
Increase in Casimir momenta is small relative to amount of energy injected

Verdict: Vlasov is satisfied at microscale (dynamics are reversible)

Example: Casimir momenta in perpendicular shear flow

Perpendicular shear force: $\boldsymbol{F}_{\text{shear}}(x) = F_0 \sin{(kx)}\hat{\boldsymbol{y}}$ $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$





Increase in Casimir momenta is small relative to amount of energy injected

Verdict: Vlasov is satisfied at microscale (dynamics are reversible)

Future: connecting entropy with Fokker-Planck equation

Quasilinear theory suggests particle acceleration is described by Fokker-Planck equation:

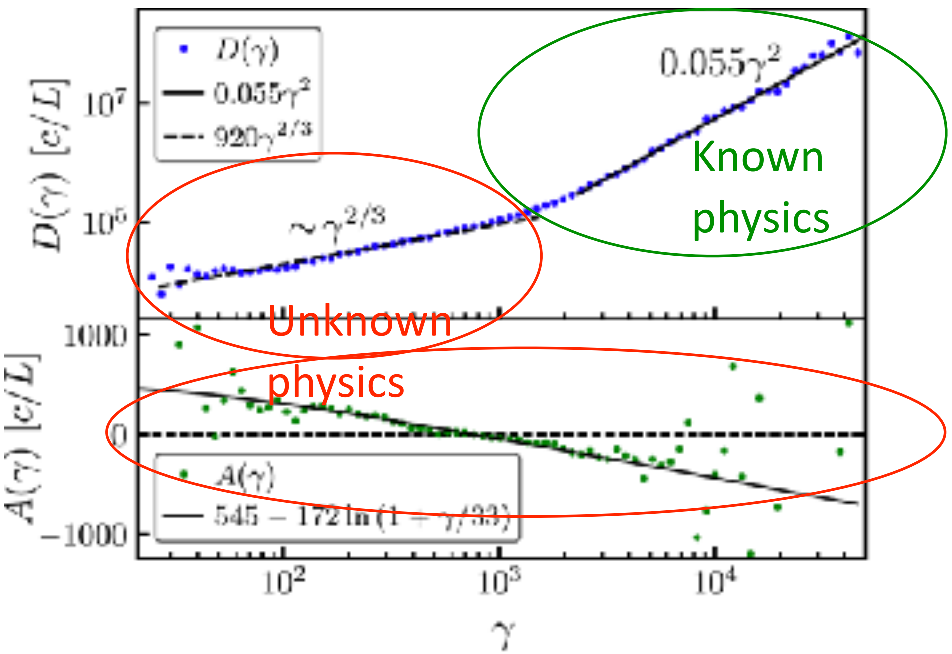
$$\partial_t f = \partial_\gamma (D \partial_\gamma f) - \partial_\gamma (A f)$$ (advection-diffusion in energy space)

2nd order Fermi acceleration / gyroresonance by Alfven waves: $D(\gamma) \sim \frac{u_A^2}{3cL}\gamma^2$ $(\gamma = E/m_e c^2)$

Confirmed by PIC simulations of relativistic turbulence!

Kai Wong, VZ+ 2020

Coefficients from tracked particles in PIC



Fokker-Planck solution