AlbaNova–NORDITA Colloquium 2 June 2022

IN SEARCH OF UNIVERSALITY:
Towards a Statistical Mechanics of Collisionless Plasma

Alex Schekochihin (Oxford)

with Toby Adkins, Robbie Ewart & Michael Nastac

[JPP 84, 905840107 (2018)]

[arXiv: 2201.03376]

[in preparation]

KINETIC DESCRIPTION OF COLLISIONLESS PLASMA

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot \underbrace{\dot{\vec{r}}}_{\vec{v}} f + \frac{\partial}{\partial \vec{v}} \cdot \underbrace{\dot{\vec{v}}}_{-\frac{e}{m}\vec{E}} f = \cancel{C[f]}$$

← collisions ("particle noise") neglect if rare

$\vec{E} = -\nabla\varphi$ for simplicity, electrostatic, $\vec{B}=0$

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$\dot{\vec{r}} \to \vec{v}$

$\dot{\vec{v}} \to -\frac{e}{m}\vec{E}$, $\quad \vec{E} \to -\nabla\varphi$ for simplicity, electrostatic, $\vec{B}=0$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \dfrac{e}{m}(\nabla\varphi)\cdot\dfrac{\partial f}{\partial \vec{v}} = 0 \\ \nabla^2\varphi = 4\pi e\left[\int d^3\vec{v}\, f - \bar{n}\right] \end{cases}$$

Vlasov's Equation (Liouville's, really)

Poisson

$(\vec{r},\vec{v})$

$$\frac{d}{dt}\int d^6Q\; G(f) = 0$$

$\forall$ function $G(f)$ (including, but not only, $f\log f$)

"Casimir invariants"

a.k.a. phase-volume conservation

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$G(f) = \delta(f(Q) - \eta)$

$\int d^6Q\, \delta(f(Q)-\eta) = \rho(\eta) = \text{const}$

"waterbag content" of the distribution

constrained dynamics

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Vlasov's Equation (Liouville's, really)

Poisson

If $\varphi = 0$ (no fluctuations),

$\forall f(\vec v)$ is an equilibrium

If equilibrium is unstable, $\varphi(\vec r) \neq 0$ appear, f evolves towards stability.

If after that fluctuations are still there, or are injected externally, even stable f continues evolving.

What does it evolve to? ← constrained dynamics

$$\frac{d}{dt}\int d^6\overset{(\vec r, \vec v)}{Q}\, G(f) = 0$$

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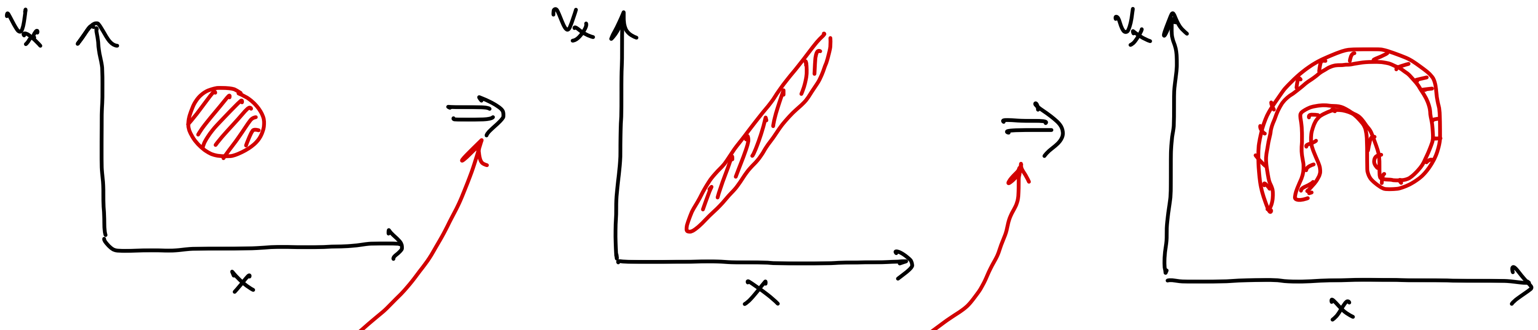

Phase mixing in $\vec{v}$ and $\vec{r}$ of the perturbed distribution $\delta f(\vec{r},\vec{v}) = f - \bar{f}(\vec{v})$

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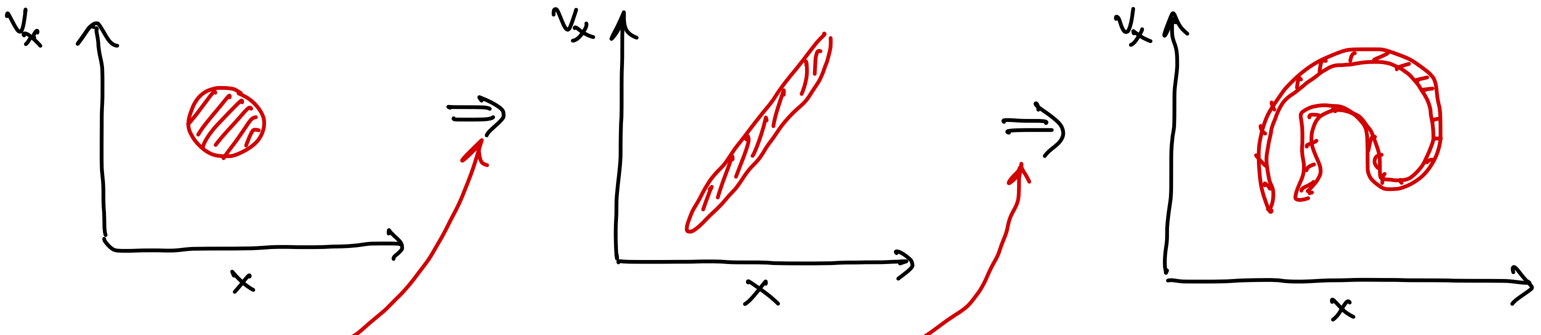

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Vlasov's Equation (Liouville's, really)

Poisson

nonlinearly? see below...

fluctuations are (linearly) damped as δf is mixed in $\vec{v}$ " $\int d^3\vec{v}\,\delta f \to 0$
("Landau damping")

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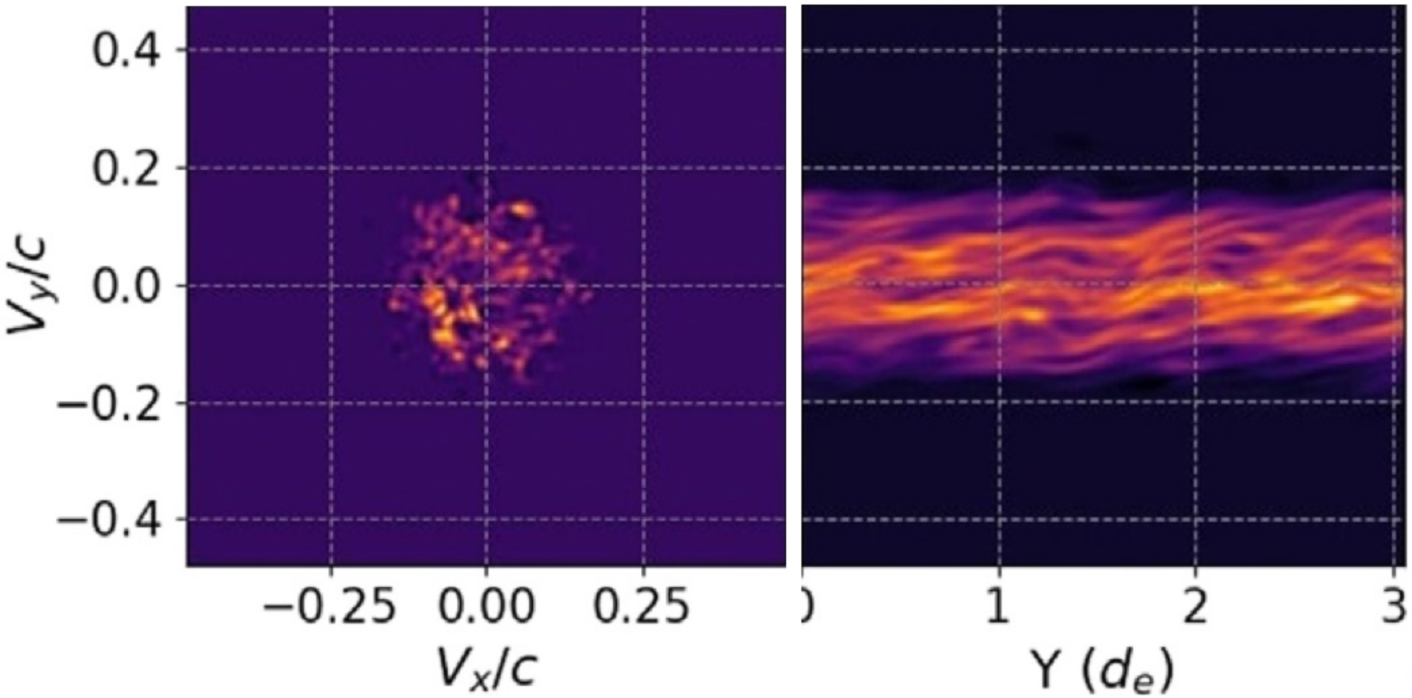
UNIVERSAL EQUILIBRIA & PHASE-SPACE TURBULENCE

Here is a recent example of a relaxed distribution from a "truly collisionless" simulation (using Gkeyll code; see Skoutnev+ ApJ 872, L28, 2019 – a distribution resulting from relaxation of two beams)



$$f = \bar{f}(\vec{v}) + \delta f(\vec{r}, \vec{v})$$

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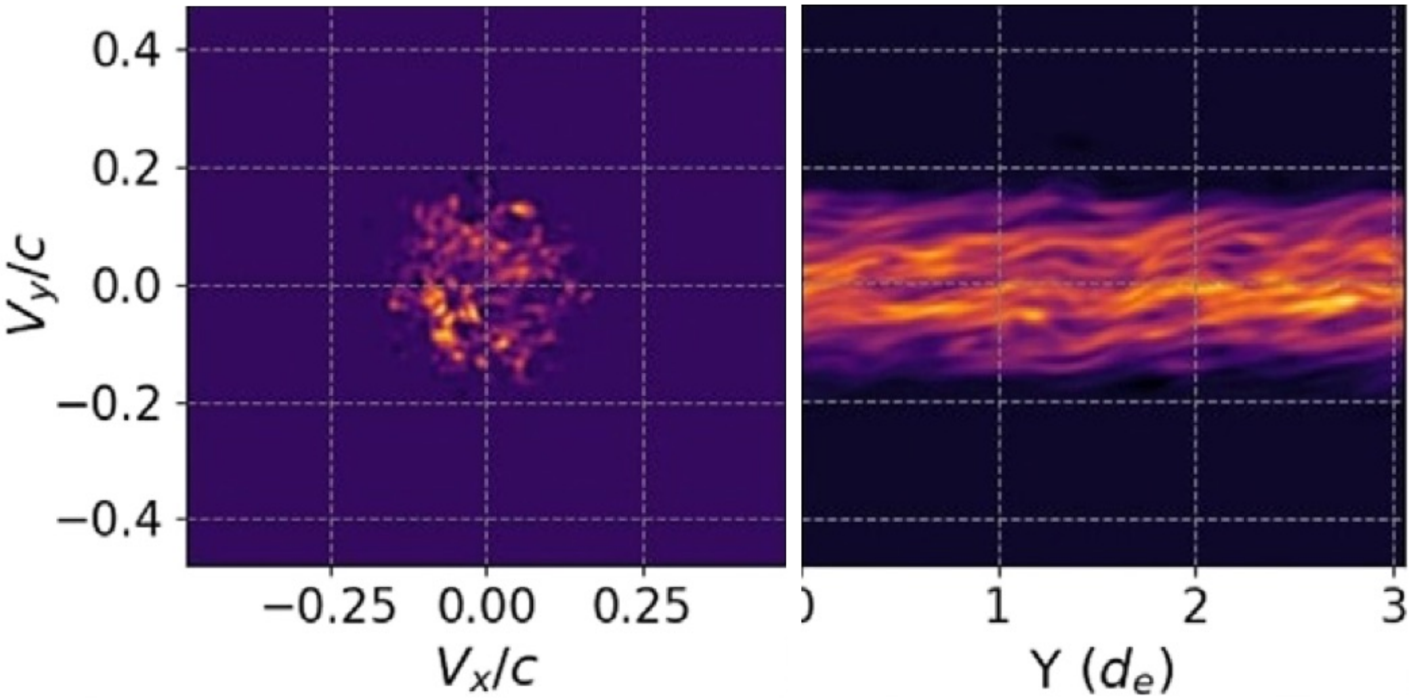

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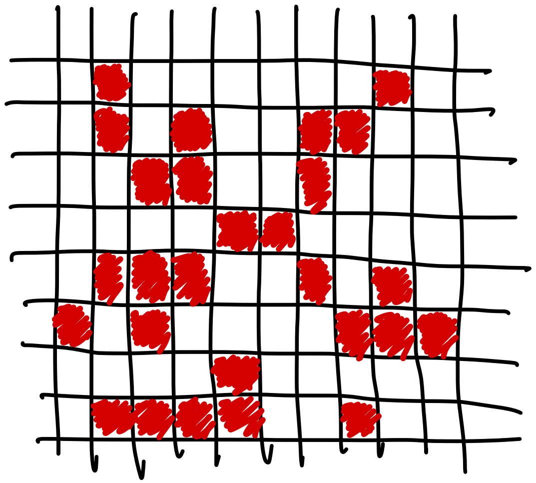
Is there a universal collisionless equilibrium? (or classes of them)

What is the structure of phase-space turbulence?

STATISTICAL MECHANICS OF COLLISIONLESS PLASMA

(Lynden Bell 1967 – for steller kinetics
Kadomtsev & Pogutse 1970 realized relevance
to plasmas)



- Break phase space into microcells $\delta\Gamma$

$$f(Q) = \eta \text{ or } 0 \quad (\text{"waterbag"})$$

Q: $(\vec{r}, \vec{v})$, or whatever

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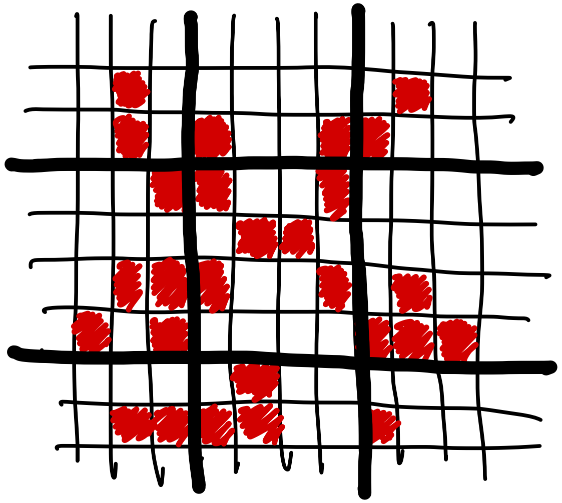

- Break phase space into microcells $\delta\Gamma$ ($\mathcal{N}$ occupied)

$$f(Q) = \eta \text{ or } 0 \quad (\text{"waterbag"})$$

Q: $(\vec{r}, \vec{v})$, or whatever

- Coarse-grain into macrocells, with M microcells in each

$$\bar{f}_i = \eta \frac{N_i}{M}$$

N_i: occupation # of i-th macrocell

$$W = \frac{\mathcal{N}!}{\prod_i N_i} \prod_i \frac{M!}{(M-N_i)!}$$

distinguishable "particles" with exclusion principle (because phase volume is conserved)

↳ Entropy $S = \ln W = \text{const} - \frac{1}{\delta\Gamma}\int dQ \left[\frac{\bar{f}}{\eta}\ln\frac{\bar{f}}{\eta} + \left(1-\frac{\bar{f}}{\eta}\right)\ln\left(1-\frac{\bar{f}}{\eta}\right)\right]$

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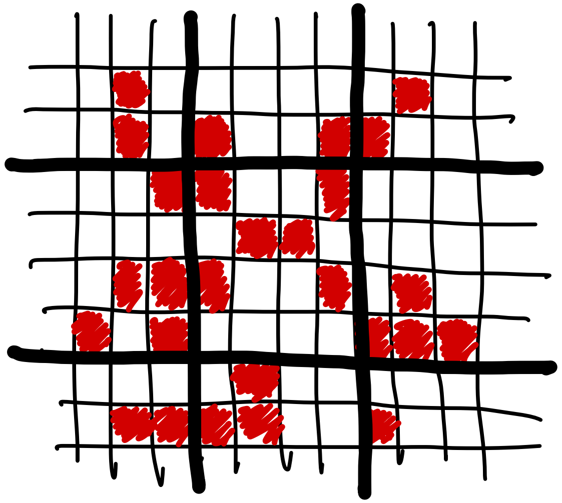

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$$\Downarrow$$

$$\int dQ\,\bar{f} = N$$

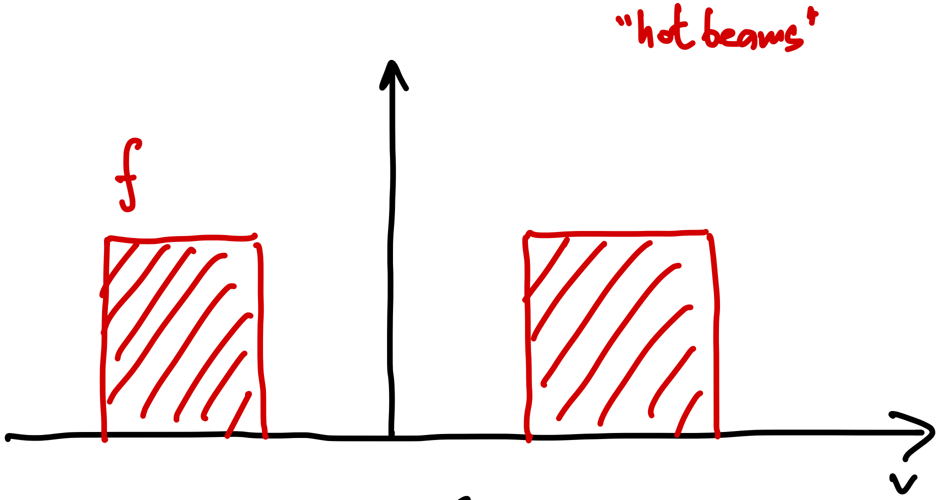
$$\int dQ\,\frac{mv^2}{2}\bar{f} = K$$

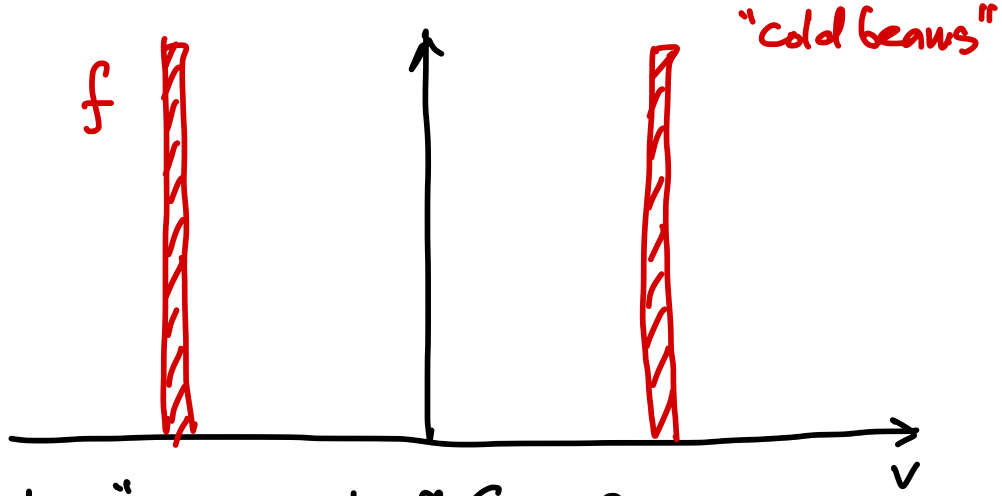
get Fermi-Dirac distribution

$$\bar{f} = \frac{\eta}{e^{\beta\left(\frac{mv^2}{2}-\mu\right)}+1}$$ ← get β and μ from

[NB: Assuming perfect mixing in phase space!]

STATISTICAL MECHANICS OF COLLISIONLESS PLASMA





More "degenerate" ($\beta \to \infty$):
f occupies more of the available phase space

given K

Less "degenerate" ($\beta \to 0$):
f occupies less of the available phase space

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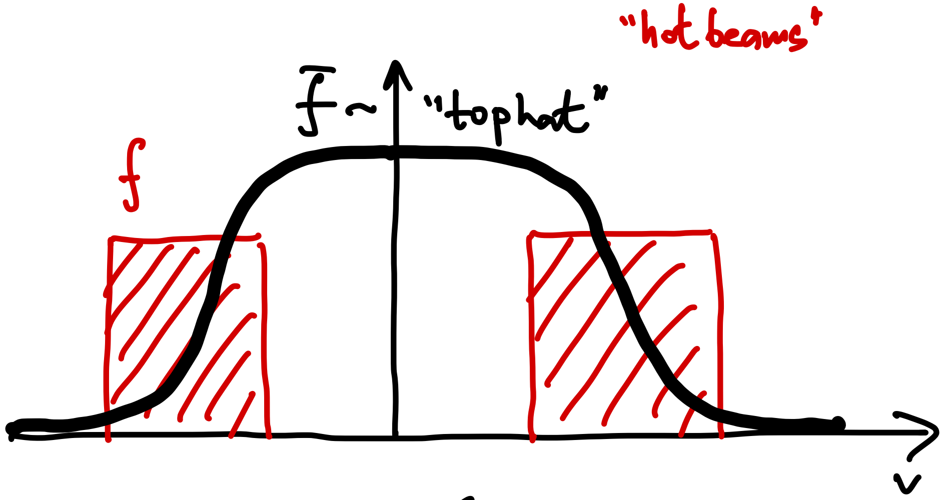

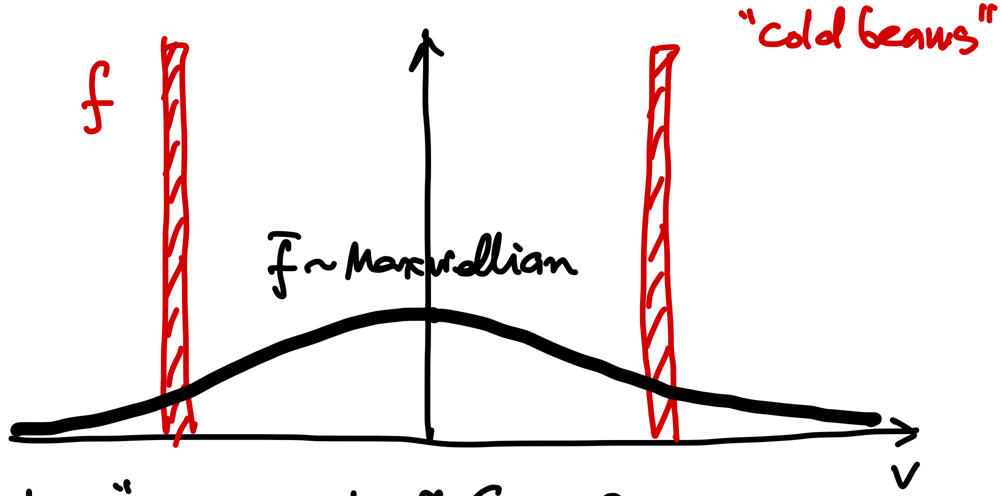

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[Interestingly, this is rather similar to what Skoutnev+ 2019 report...]

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Easy generalisation: "multi-waterbag model": $f(Q) = \sum_J \underbrace{f_J(Q)}_{= \,\eta_J \text{ or } 0}$

Treat each waterbag as a "species", get

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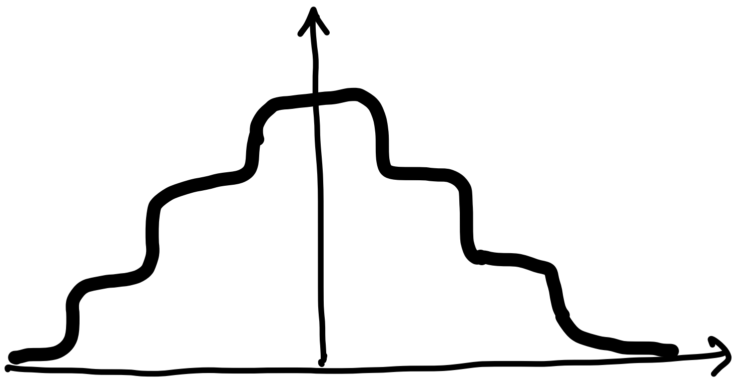

(sharper corners when more degenerate)

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$$\bar{f} = \sum_J \eta_J p_J(Q) \rightarrow \int d\eta\, \eta\, P(Q,\eta)$$

"hyperkinetic" distribution

$$P = \frac{e^{-\beta\eta\left[\frac{mv^2}{2} - \mu(\eta)\right]}}{\int d\eta' e^{-\beta\eta'\left[\frac{mv^2}{2} - \mu(\eta')\right]}}$$

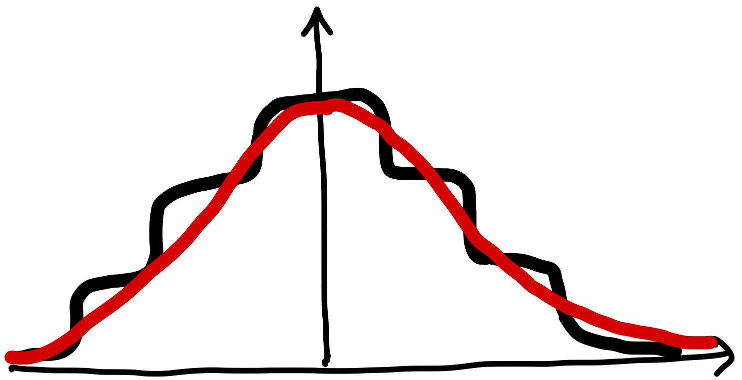
and again get β and $\mu(\eta)$ from energy and phase volume conservation

$$\int dQ \frac{mv^2}{2} \bar{f} = K$$

$$\int dQ\, P(Q,\eta) = \rho(\eta)$$

a continuum of "Casimir invariants"

cf.



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This is a kind of partition function

$$Z(\psi) = \int d\eta\, e^{-\eta\psi + \beta\eta\mu(\eta)}$$

$\psi = \beta \frac{mv^2}{2}$

$$\bar{f} = -\frac{\partial \ln Z}{\partial \psi}$$ ... this proves to be a useful formalism ...

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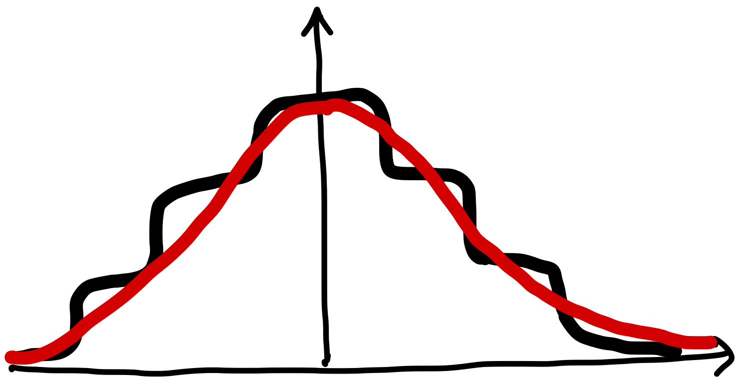

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This captures a wide class of equilibria, including, intriguingly, ones with power-law tails, which can be argued to emerge organically.

- Power-law distributions are observed in many astro contexts: electrons in solar wind, cosmic rays etc. They are associated with "non-thermal particle acceleration".
- Lynden-Bell equilibria develop power-law tails because of the exclusion principle - can't shove particles towards lower ("thermal") energies while conserving phase volume

[Ewart & AAS 2022]

We need KINETIC THEORY to find out if (and how quickly) collisionless plasmas relax to these (or other?) equilibria...

Objective: Derive "collisionless collision integral"

$$\frac{\partial \bar{f}}{\partial t} = C[\bar{f}] \leftarrow$$ equilibria are fixed points $C[\bar{f}]=0$, proved unique if there is an H-theorem, i.e., an "entropy" that always increases until maximized by the equilibrium distribution.

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$\vec{E} = -\nabla\varphi$ electrostatic

$\varphi_k = -\frac{4\pi e}{k^2}\int d^3\vec{v}'\,\delta f_k(\vec{v}')$

Poisson's law $-\nabla^2\varphi = 4\pi\sigma$

$C_k(\vec{v},\vec{v}')$ correlation function of phase-space turbulence

Thus, to have a theory of relaxation to collisionless equilibria, we need a theory of phase-space correlations (cf. hydrodynamics: large-scale transport from small-scale turbulence)

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$$\hookrightarrow \frac{\partial \bar f}{\partial t} = -\frac{e}{m}\frac{\partial}{\partial \vec v}\cdot\langle(\nabla\varphi)\delta f\rangle = \frac{e}{m}\frac{\partial}{\partial \vec v}\cdot\sum_{\vec k} i\vec k\langle\varphi_k^*\delta f_k\rangle = -\frac{4\pi e^2}{m}\frac{\partial}{\partial \vec v}\cdot\sum_{\vec k}\frac{i\vec k}{k^2}\int d^3\vec v'\underbrace{\langle\delta f_k^*(\vec v')\delta f_k(\vec v)\rangle}_{= C_k(\vec v,\vec v')}$$

$\vec E = -\nabla\varphi$ electrostatic

$\varphi_k = -\frac{4\pi e}{k^2}\int d^3\vec v'\,\delta f_k(\vec v')$ Poisson's law $-\nabla^2\varphi = 4\pi\sigma$

$C_k(\vec v,\vec v')$ correlation function of phase-space turbulence

If there is perfect mixing in phase space,

$$\langle\delta f(\vec r,\vec v)\delta f(\vec r',\vec v')\rangle = \underbrace{\langle\delta f^2\rangle}_{\langle f^2\rangle-\bar f^2}\,\underbrace{\Delta\Gamma}_{\text{"correlation volume"}}\,\delta(\vec r-\vec r')\delta(\vec v-\vec v')$$ "microgranulation ansatz"

$$\hookrightarrow C_k(\vec v,\vec v') = \left[\langle f^2\rangle - \bar f^2\right]\frac{\Delta\Gamma}{V}\delta(\vec v-\vec v')$$

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only $\operatorname{Im} C_k(\vec v,\vec v')$ contributes

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$\varphi_k = -\frac{4\pi e}{k^2}\int d^3\vec v'\,\delta f_k(\vec v')$

Poisson's law $-\nabla^2\varphi = 4\pi\sigma$

$C_k(\vec v,\vec v')$ correlation function of phase-space turbulence

If there is perfect mixing in phase space,

$$\langle\delta f(\vec r,\vec v)\,\delta f(\vec r',\vec v')\rangle = \langle\delta f^2\rangle\,\Delta\Gamma\,\delta(\vec r-\vec r')\,\delta(\vec v-\vec v')$$

"micrогranulation ansatz"

$\langle\delta f^2\rangle$: $\langle f^2\rangle - \bar f^2$; $\Delta\Gamma$: "correlation volume"

$$C_k(\vec v,\vec v') = \left[\langle f^2\rangle - \bar f^2\right]\frac{\Delta\Gamma}{V}\,\delta(\vec v-\vec v')$$

This gives $\frac{\partial \bar f}{\partial t}=0$ because

$$C_k(\vec v,\vec v') = C_k(\vec v',\vec v) = C_k^*(\vec v,\vec v')$$

so $\operatorname{Im} C_k(\vec v,\vec v') = 0$

So we need to work out how mixing happens dynamically...

Thus, to have a theory of relaxation to collisionless equilibria, we need a theory of phase-space correlations (cf. hydrodynamics: large-scale transport from small-scale turbulence)

KINETIC THEORY OF PHASE MIXING

This is still the same, but
decompose $f = \bar{f} + \delta f$
and look for δf

Starting point:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \frac{e}{m} \vec{E} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

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or, in k space,

$$\frac{\partial \delta f_k}{\partial t} + \underbrace{i\vec{k}\cdot\vec{v}\,\delta f_k}_{\text{phase mixing}} = \underbrace{-i\frac{e}{m}\varphi_k\vec{k}\cdot\frac{\partial \bar{f}}{\partial \vec{v}}}_{\text{"source" of } \langle\delta f^2\rangle} \underbrace{-\,i\frac{e}{m}\sum_{k'}\varphi_{k'}\vec{k}'\cdot\frac{\partial \delta f_{k-k'}}{\partial \vec{v}}}_{\text{nonlinear coupling}}$$

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QUASILINEAR THEORY: drop nonlinearity, let

$$\delta f_k = \underbrace{g_k e^{-i\vec{k}\cdot\vec{v}t}}_{\text{initial distribution}} + \underbrace{\delta f_k^{QL}}_{\text{QL evolution from source + phase mixing (also depends on } g_k \text{ via } \varphi_k \text{ and Poisson's law)}}$$

and work out the asymmetric part of $C_k(\vec{v},\vec{v}')$

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g_k: initial distribution

δf_k^{QL}: QL evolution from source + phase mixing (also depends on g_k via φ_k and Poisson's law)

Use microgranulation ansatz to calculate $\langle g_k^*(\vec{v}')\, g_k(\vec{v})\rangle$ and clever collisionless closure for $\langle g^2 \rangle$

[Severne & Luwel 1980, Chavanis 2005, Ewart + 2022]

This leads to collisionless collision operators that are generalisations of Balescu–Lenard for phase-volume conserving plasmas, $V_{\text{eff}} \propto \Delta\Gamma$.

Their fixed points are Lynden-Bell equilibria!

[Ewart + 2022, arXiv: 2201.03376]

For kinetic theory fans: SOME DETAILS ON COLLISION INTEGRALS

$$\frac{\partial \bar{f}}{\partial t} = \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \vec{v}} \cdot \int d^3\vec{v}' \sum_{k} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta(\vec{k}\cdot(\vec{v}-\vec{v}'))}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} \cdot \left[\langle g^2\rangle(\vec{v}') \frac{\partial \bar{f}(\vec{v})}{\partial \vec{v}} - \langle g^2\rangle(\vec{v}') \frac{\partial \bar{f}(\vec{v}')}{\partial \vec{v}'} \right]$$

dielectric function

$\langle f^2\rangle - \bar{f}^2$

- for one waterbag, $\langle f^2\rangle - \bar{f}^2 = (\eta - \bar{f})\bar{f}$

Kadomtsev-Pogutse coll. integral (1970)

↳ Fermi-Dirac is fixed point

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  - ↳ Fermi-Dirac is fixed point
- non-degenerate: $\eta \gg \bar{f}$, so Balescu-Lenard coll. integral (1960)

$$\langle f^2\rangle - \bar{f}^2 = \eta \bar{f}$$

e.g. for true 2 particle collisions, $f = \sum_i \delta(\vec{r}-\vec{r}_i)\delta(\vec{v}-\vec{v}_i)$ (Klimontovich distr. function)

↳ $\eta\Delta\Gamma = 1$

"Collisionless collision frequency": $\nu_{eff} \sim \eta\Delta\Gamma\,\nu_{true} \gg \nu_{true}$ FAST RELAXATION

For kinetic theory fans: SOME DETAILS ON COLLISION INTEGRALS

$$\frac{\partial \bar f}{\partial t} = \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V}\frac{\partial}{\partial \vec v}\cdot\int d^3\vec v' \sum_k \frac{\vec k\vec k}{k^4}\frac{\delta(\vec k\cdot(\vec v-\vec v'))}{|\epsilon_{\vec k,\vec k\cdot\vec v}|^2}\cdot\left[\langle g^2\rangle(\vec v')\frac{\partial \bar f(\vec v)}{\partial \vec v} - \langle g^2\rangle(\vec v)\frac{\partial \bar f(\vec v')}{\partial \vec v'}\right]$$

dielectric function

$\langle f^2\rangle - \bar f^2$

- for one waterbag, $\langle f^2\rangle - \bar f^2 = (\eta - \bar f)\bar f$ Kadomtsev-Pogutse coll. integral (1970)
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- non-degenerate: $\eta \gg \bar f$, so
  $\langle f^2\rangle - \bar f^2 = \eta \bar f$ Balescu-Lenard coll. integral (1960)

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"Collisionless collision frequency": $\nu_{eff} \sim \eta\Delta\Gamma\,\nu_{true} \gg \nu_{true}$ FAST RELAXATION

- for many waterbags, upgrade to HYPERKINETICS:

$$\bar P(\eta,\vec v) = \langle \delta(f(\vec v,\vec r) - \eta)\rangle \quad \text{7D phase space } (\eta,\vec r,\vec v)$$

$$\frac{\partial \bar P}{\partial t} = \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V}\frac{\partial}{\partial \vec v}\cdot\int d^3\vec v' \sum_k \frac{\vec k\vec k}{k^4}\frac{\delta(\vec k\cdot(\vec v-\vec v'))}{|\epsilon_{\vec k,\vec k\cdot\vec v}|^2}\cdot\int d\eta'\,\eta'\Big[(\eta' - \bar f(\vec v'))\bar P(\vec v',\eta')\frac{\partial \bar P(\vec v,\eta)}{\partial \vec v}$$
$$- (\eta - \bar f(\vec v))\bar P(\vec v,\eta)\frac{\partial \bar P(\vec v',\eta')}{\partial \vec v'}\Big]$$

↳ Lynden-Bell equilibria are fixed points [Severne & Luwel 1980, Ewart+ 2022]

$$\bar f(\vec v) = \int d\eta\,\eta\,\bar P(\eta,\vec v)$$
$$V\int d^3\vec v\,\bar P(\eta,\vec v) = \rho(\eta) = \text{const}$$

↑ "waterbag content" conserved ("Casimir invariants")

KINETIC THEORY OF PHASE MIXING

This is still the same, but decompose $f = \bar{f} + \delta f$ and look for δf

Starting point:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \frac{e}{m}\vec{E}\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

$$\Rightarrow \quad \frac{\partial \delta f}{\partial t} + \vec{v}\cdot\nabla\delta f = \frac{e}{m}\vec{E}\cdot\frac{\partial \bar{f}}{\partial \vec{v}} + \frac{e}{m}\vec{E}\cdot\frac{\partial \delta f}{\partial \vec{v}},$$

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[Severne & Louvel 1980, Chavanis 2005, Ewart + 2022]

Why is this reasonable? Just as Maxwell did with Stoßzahlansatz, we assume that everything gets thoroughly stochasticized so we can "reset" the initial condition at each "time step" of $\frac{\partial \bar{f}}{\partial t}$. That is the effect of nonlinearity!

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$\Downarrow$

$$C_k(\vec{v},\vec{v}') = \langle \delta f_k^*(\vec{v}')\,\delta f_k(\vec{v})\rangle$$

$$\frac{\partial C_k}{\partial t} + i\vec{k}\cdot(\vec{v}-\vec{v}')C_k = S_k + N_k$$

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$\hookrightarrow C_{ks}$ "phase-space spectrum"

s is dual of $\vec{v}-\vec{v}'$

$$\frac{\partial C_{ks}}{\partial t} + \vec{k}\cdot\frac{\partial C_{ks}}{\partial \vec{s}} = S_{ks} + N_{ks}$$

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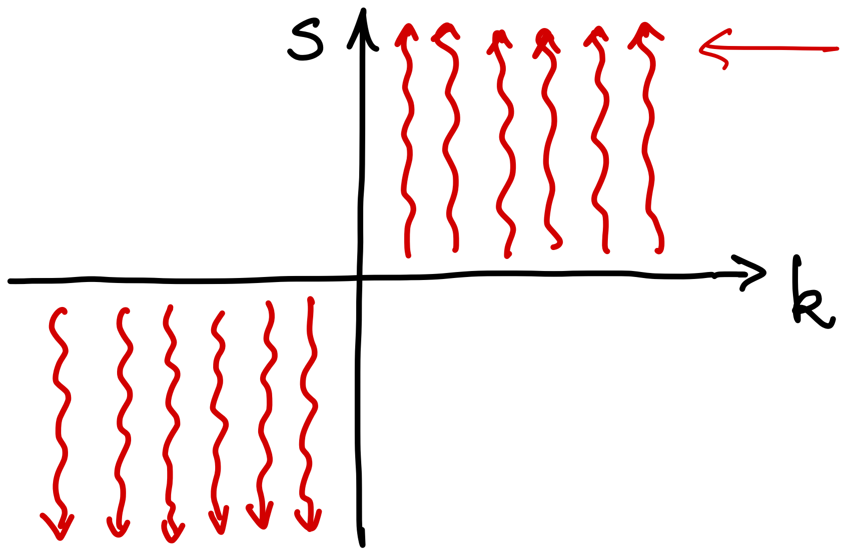
at low s

phase mixing



NB:

$C_{k,s} \neq C_{k,-s}$

so

$C_k(\vec{v},\vec{v}') \neq C_k(\vec{v}',\vec{v})$

asymmetric

(this is where QL coll. integrals come from)

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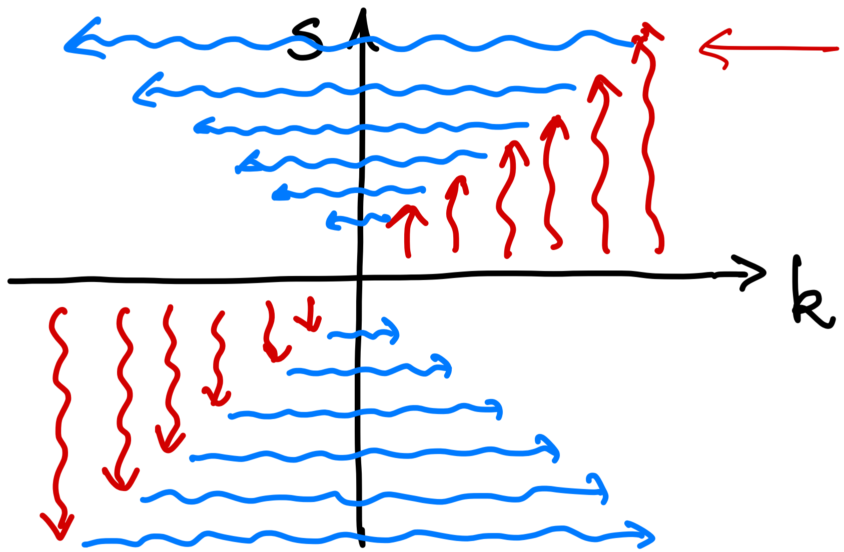
at low s

coupling between k's

phase mixing



KINETIC THEORY OF PHASE MIXING & UNMIXING

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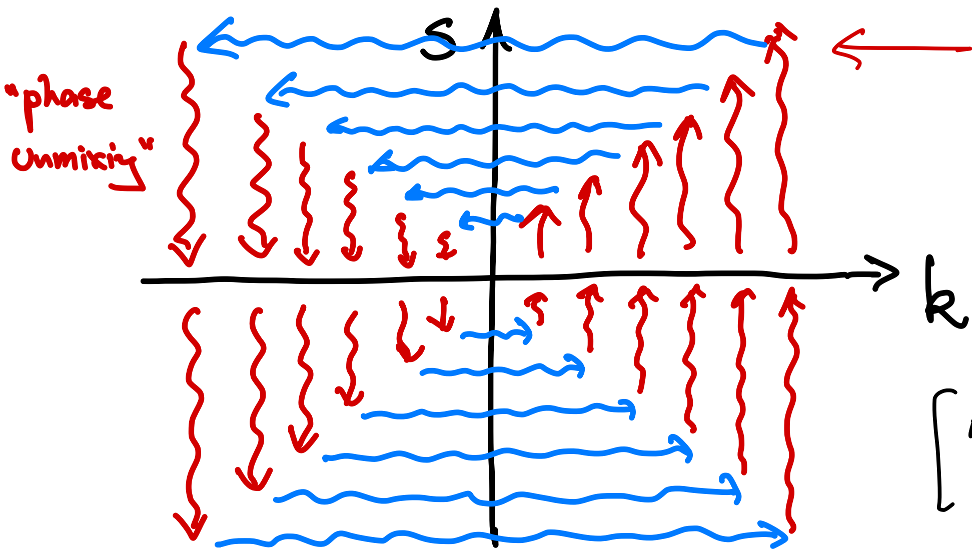
phase mixing

at low s

coupling between k's leading to "phase unmixing" (plasma echo)



[A simple solvable model of this:
Adkins & AAS JPP (2018)
Nastac+ (2022)]

KINETIC THEORY OF PHASE MIXING & UNMIXING

superficially, suppression of phase mixing (=Landau damping) by echoes seems to symmetrise C_{ks} and perhaps even produce a solution with zero flux of $\langle\delta f^2\rangle$ roughly compatible with microgranulation ansatz:

$$\langle \delta f_k^*(\vec{v}')\,\delta f_k(\vec{v})\rangle = \langle\delta f^2\rangle \frac{\Delta\Gamma}{V}\,\delta(\vec{v}-\vec{v}')$$

↑ C_{ks} = const in s

this might depend on k and be $\propto$ velocity scale^3 ($\sim s^{-3}$) where nonlinearity wins over phase mixing

↳ ENHANCED COLLISIONALITY

C_{ks} "phase-space spectrum"

s is dual of $\vec{v}-\vec{v}'$

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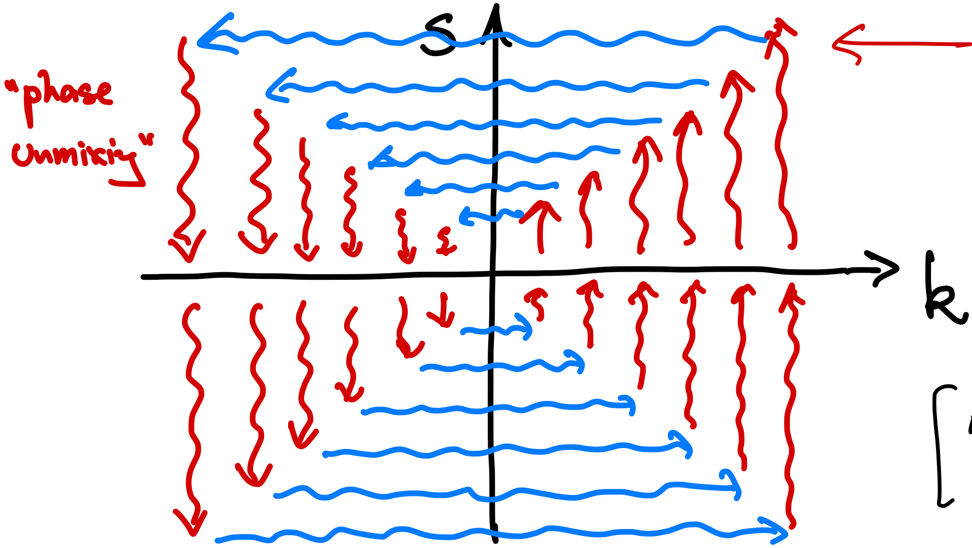

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IN FACT, this is a system into which $\langle\delta f^2\rangle$ is constantly injected by S_{ks} and this flux has to be carried to large s, to be thermalised by collisions!

C_{ks} "phase-space spectrum"
s is dual of $\vec{v}-\vec{v}'$

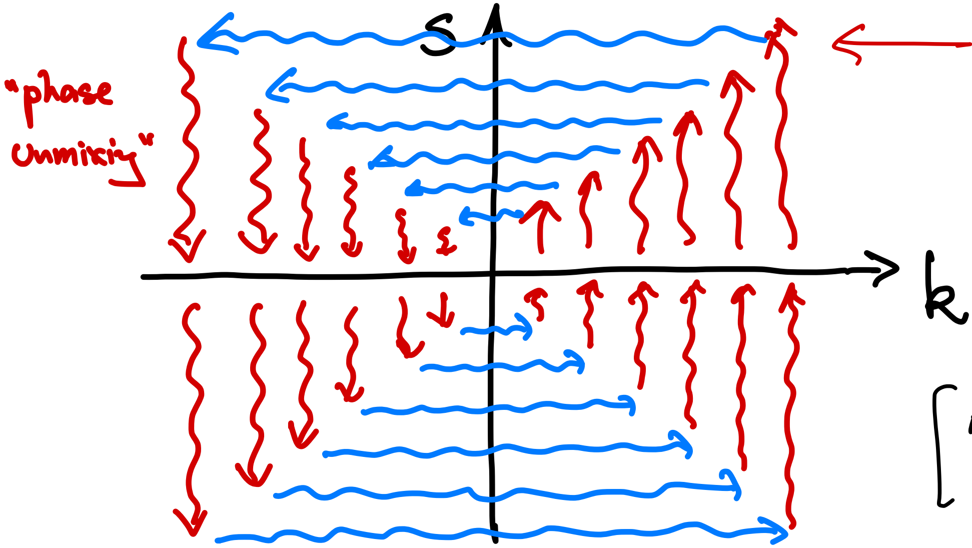
$$\frac{\partial C_{ks}}{\partial t} + \vec{k}\cdot\frac{\partial C_{ks}}{\partial \vec{s}} = S_{ks} + N_{ks} - \nu s^2 C_{ks}$$

$\nu s^2 \sim \frac{\partial^2}{\partial v^2}$

phase mixing

at low s

coupling between k's leading to "phase unmixing" (plasma echo)



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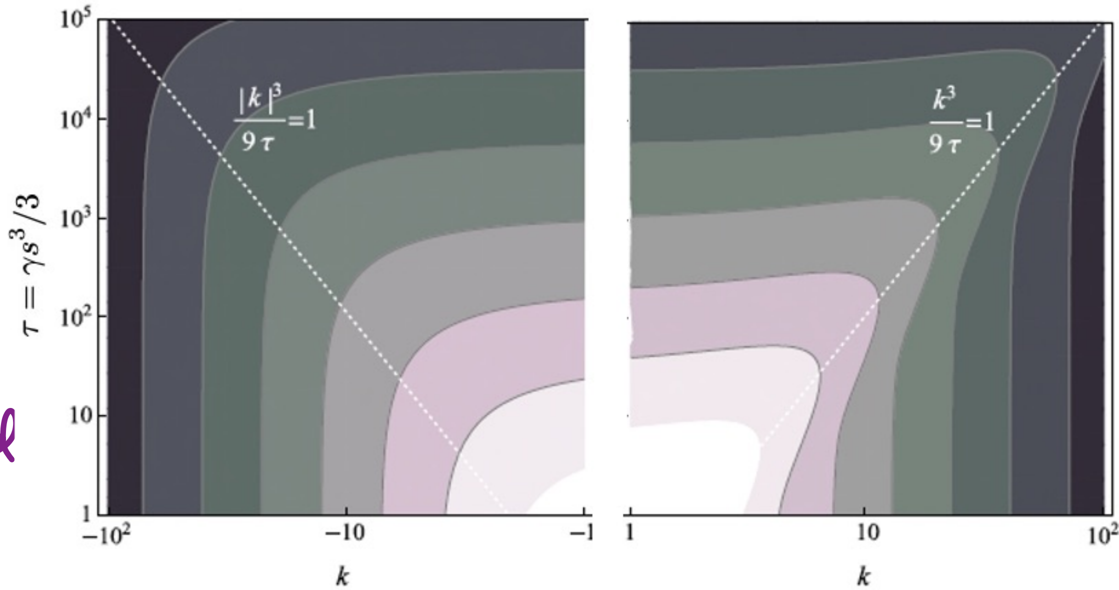
KINETIC THEORY OF PHASE-SPACE TURBULENCE

Phase-space spectrum found by Adkins & AAS was not symmetric and can be shown to carry constant flux [Nastac+ 2022]



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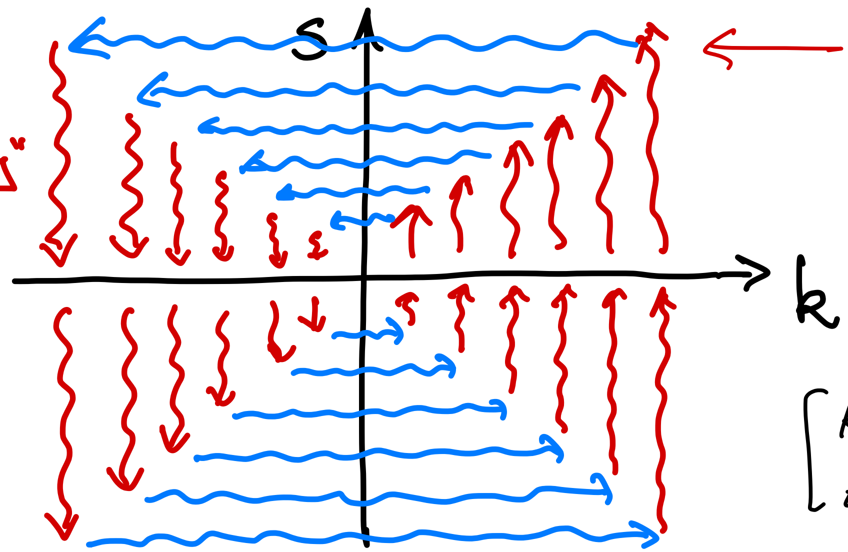
C_{ks} "phase-space spectrum"
s is dual of $\vec{v}-\vec{v}'$

$$\frac{\partial C_{ks}}{\partial t} + \underbrace{k \cdot \frac{\partial C_{ks}}{\partial s}}_{\text{phase mixing}} = \underbrace{S_{ks}}_{\text{at low } s} + \underbrace{N_{ks}}_{\text{coupling between } k\text{'s leading to "phase unmixing" (plasma echo)}} - \nu s^2 C_{ks}$$

$s^2 \sim \frac{\partial^2}{\partial v^2}$

"phase unmixing"



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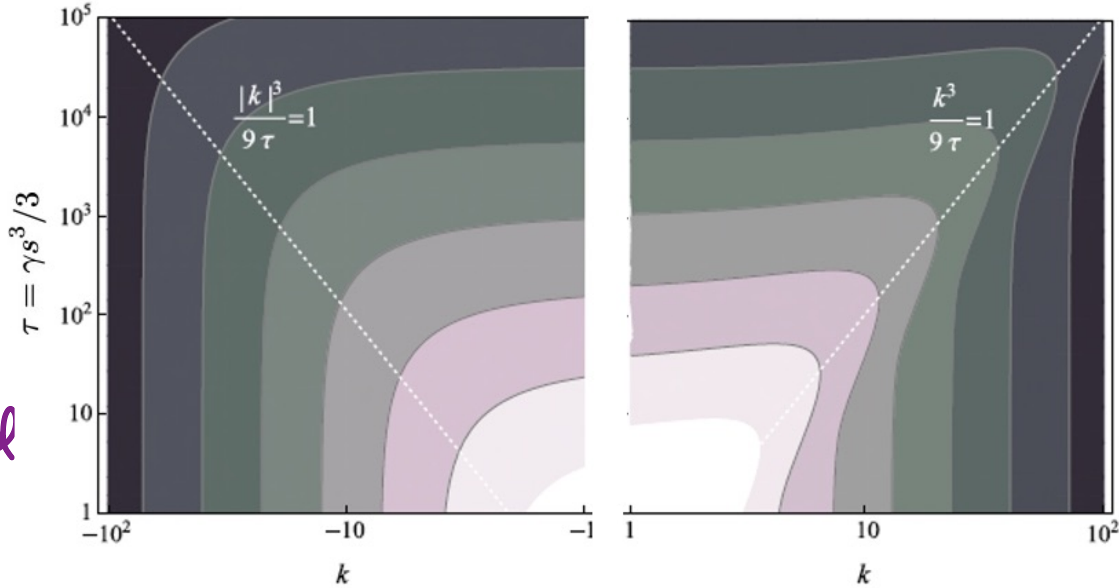
KINETIC THEORY OF PHASE-SPACE TURBULENCE

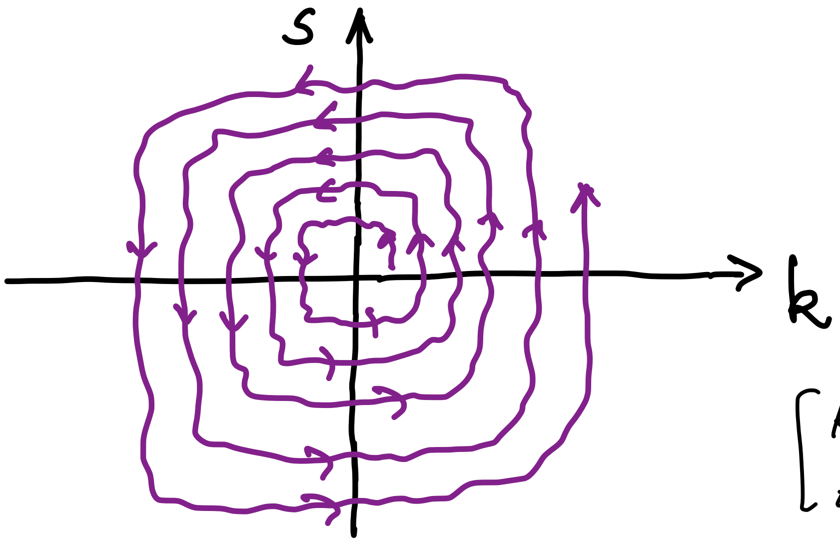
Phase-space spectrum found by Adkins & AAS was not symmetric and can be shown to carry constant flux [Nastac+ 2022]



IN FACT, this is a system into which $\langle \delta f^2 \rangle$ is constantly injected by S_{ks} and this flux has to be carried to large s, to be thermalised by collisions!

C_{ks} "phase-space spectrum"
s is dual of $\vec{v}-\vec{v}'$

$$\frac{\partial C_{ks}}{\partial t} + k \cdot \frac{\partial C_{ks}}{\partial s} = S_{ks} + N_{ks} - \underbrace{\nu s^2}_{\sim \frac{\partial^2}{\partial v^2}} C_{ks}$$



$\langle \delta f^2 \rangle$ injected at low k & s and spirals out by combined action of phase mixing and nonlinearity until thermalised by collisions

[A simple solvable model of this:
Adkins & AAS JPP (2018)]
Nastac+ (2022)

INTERESTING QUESTIONS FOR FURTHER RESEARCH:

- Collisionless collision integral in a field of constant-flux turbulence:

$$\frac{\partial \bar{f}}{\partial t} = -\frac{4\pi e^2}{m}\frac{\partial}{\partial \vec{v}}\cdot\sum_{\vec{k}}\frac{i\vec{k}}{k^2}\int d^3\vec{v}'\langle\delta f_{\vec{k}}^*(\vec{v}')\delta f_{\vec{k}}(\vec{v})\rangle \equiv C[\bar{f}] \neq 0$$

What is this? Does it have fixed points?
Is there an H theorem?
What is entropy?

- While $\bar{f}$ is collisionless, δf reaches collisional scales (small δv) fast (cf. "dissipative anomaly" in fluid turbulence) – see poster by M. Nastac.
Is this always true? What does it imply for Casimir constraints?
[cf. Zhdankin's recent papers]

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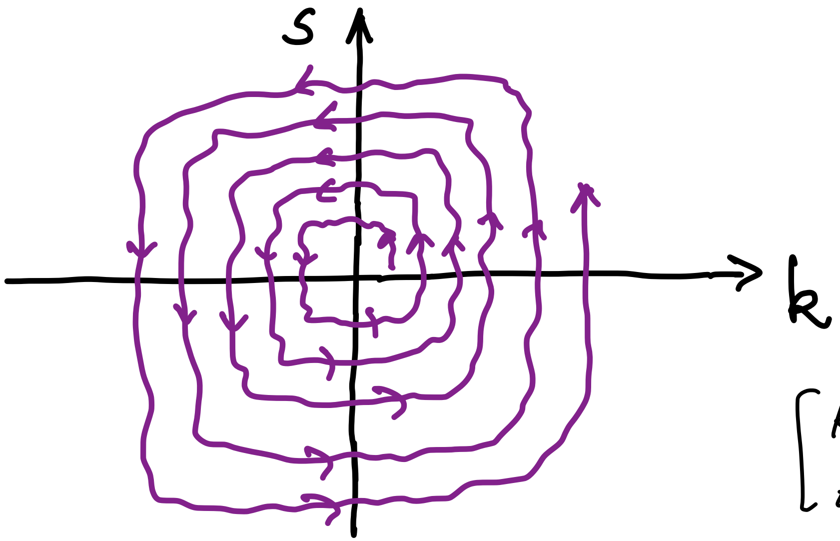

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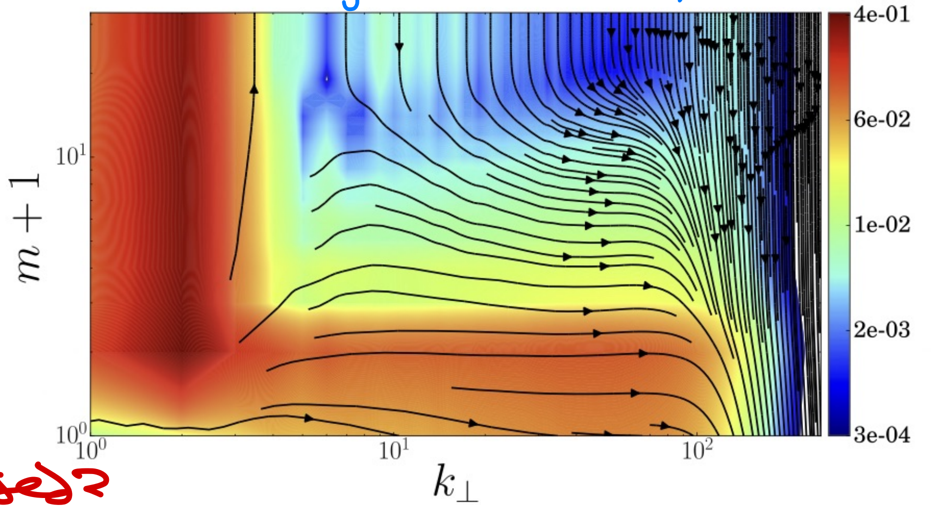
- How does all this work in magnetised systems?
Nonlinearity is "more space-like" – advection by E×B flow:

Maxwellian eq'm.

$$\frac{\partial \delta f}{\partial t} + \vec{u}_\perp\cdot\nabla_\perp\delta f + v_\parallel\nabla_\parallel\left(\delta f + \frac{e\varphi}{T}F_M\right) = 0$$

AAS+ JPP (2016) – theory for drift-kinetic turbulence
Meyrand+ PNAS (2019) – simulations for density turbulence in solar wind: flux entirely along k ("fluidisation" – no Landau damping in inertial range).

Meyrand+ PNAS 116, 1185 (2019)



Is phase mixing always suppressed?