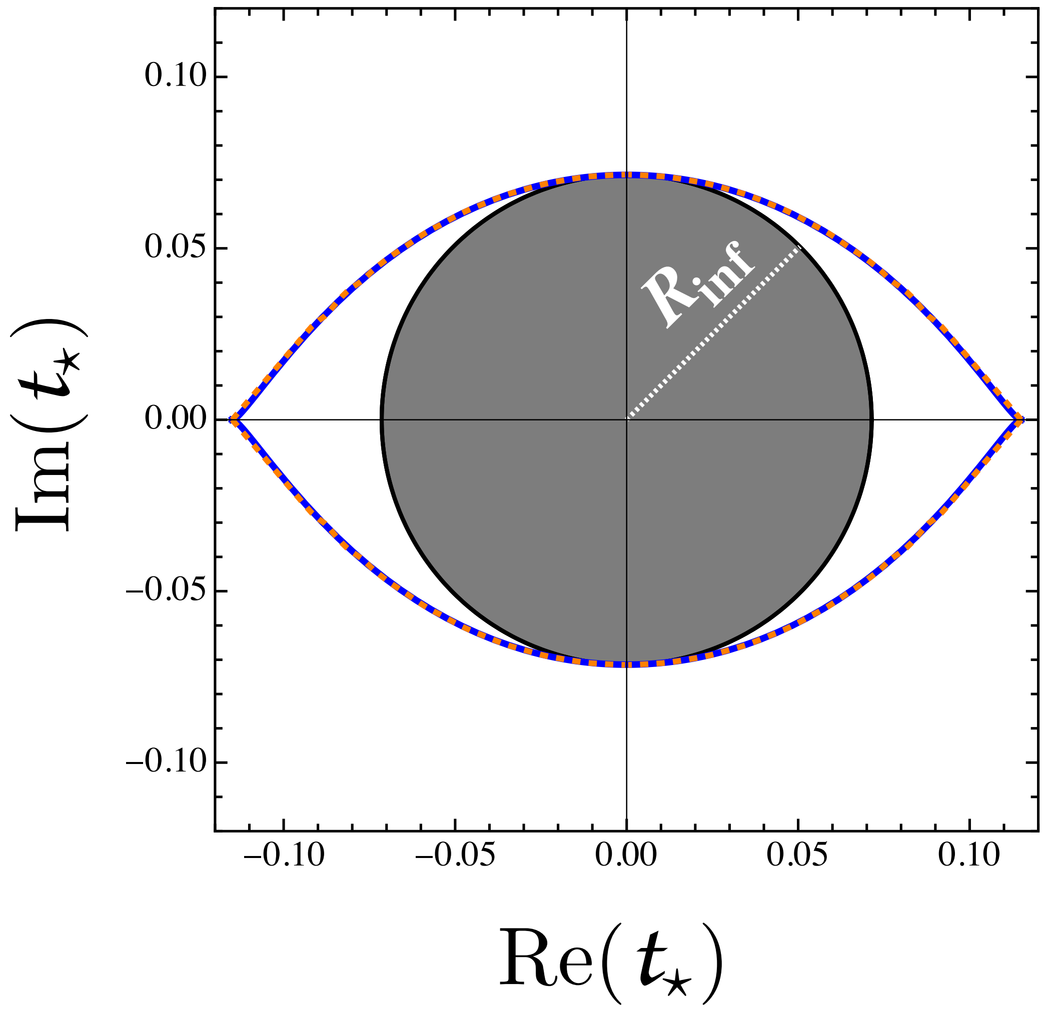
Eye of the Tyger: early-time resonances & singularities in the inviscid Burgers equation

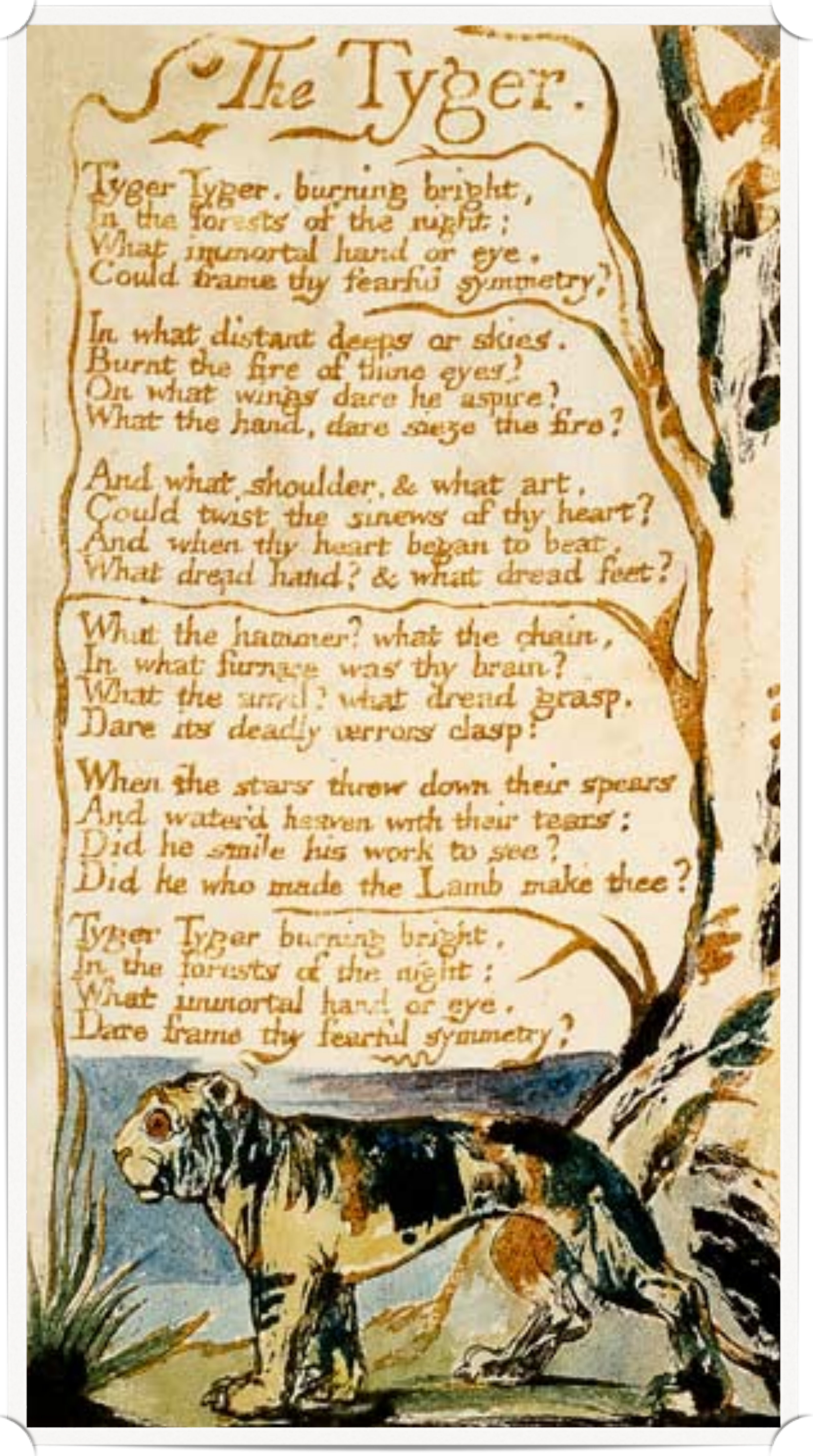


based on arXiv:2207.12416 by CR, Uriel Frisch and Oliver Hahn; submitted to Phys. Rev. Fluids

13th Plasma Kinetics Working Meeting, Wolfgang Pauli Institute

Cornelius Rampf, U Vienna

2nd August 2022



Context and overview (1 of 2)

- **Burgers' equation** occurs in various areas of applied mathematics, such as fluid mechanics (reduced model for turbulence), gas dynamics, etc.

- In one-space dimension with non-zero viscosity ν, Burgers' equation is

$$\partial_t u + u \nabla_x u = \nu \nabla_x^2 u \qquad\qquad u(x,0) = u_0$$

- solutions exist
  - **for $\nu \neq 0$ at all times,** obtained from exploiting the Hopf-Cole transformation [Hopf '50, Cole '51]
  - **for $\nu \to 0$ at all times,** via convex-hull construction / Legendre transformation [e.g. Noullez & Vergassola '94]
  - **for $\nu = 0$ until the first real singularity (= pre-shock),** through the method of characteristics, a.k.a. Lagrangian coordinates a

- **In today's talk, I focus exclusively on the $\nu = 0$ case, and work mostly in Eulerian coordinates x**

Context and overview (2 of 2)

1D inviscid Burgers equation $\partial_t u + u \nabla_x u = 0\,, \qquad u(x{,}0) = u_0$

Why focus “only” on $\nu = 0$ and until pre-shock time ?

- **Numerical simulations** (of Burgers, incompressible Euler, Navier-Stokes, …) **very often employ Eulerian coordinates**
- Eulerian coordinates are in general not optimal for resolving advection (the term $u \nabla_x u$); thus, one may be forced to live with the consequences, such as tygers in Burgers or incompressible Euler
- Many considerations, such as the blow-up problem, require high accuracy in the temporal regime until the first real singularity (if existent)

Outline of today's talk

- we detect so far unknown complex-time singularities in the 1D inviscid Burgers equation
- analysed by two complementary and independent means:
  1. asymptotic analysis by means of a Taylor-series representation for the velocity in Eulerian coordinates
  2. singularity theory in Lagrangian coordinates (which may be transferred to other fluids)
- for certain implementations, such as for the Taylor-series of u,
  loss of convergence is accompanied by initially localised resonant behaviour
- these resonances are highly related to the tyger phenomena reported in Galerkin-truncated implementations of inviscid fluids [e.g. Ray+ '11, Bardos+ '13, Pereira+ '13, Clark Di Leoni+ '18]
- finally, we apply two methods that reduce the amplitude of early-time tygers.
  One removes Fourier modes near the Galerkin truncation, the other attempts an iterative UV completion for the Taylor series

Basic setup

- 1D inviscid Burgers equation $$\partial_t u + u \nabla_x u = 0\,, \qquad u(x,0) = u_0 \tag{1}$$
- one way to investigate the analytic structure is by considering a time-Taylor series representation for the velocity;

  plug the Ansatz $u = \sum_{n=0}^{\infty} u_n t^n$ into Burgers' equation and identifying the involved powers in t, one easily finds ($n \geq 0$)

$$u_{n+1} = \frac{-1}{n+1} \sum_{i+j=n} u_i \, \partial_x u_j \tag{2}$$

- let's focus first on the simple single-mode model with initial data $u_0 = -\sin x$. Using (2) one finds

$$u_1 = (-1/2)\ \sin(2x)$$
$$u_2 = (1/8)[\sin x - 3\sin(3x)]$$
$$u_3 = 1/6[\sin(2x) - 2\sin(4x)]$$
$$\vdots$$
$$u_N = \cdots + c_N \sin[(N+1)x]$$

coefficient

In Fourier space, the Nth-order time-Taylor coefficient has maximum Fourier mode $k = \pm(N+1)$ and thus, u_N is **band limited**.
Such truncations play an important role for triggering tygers

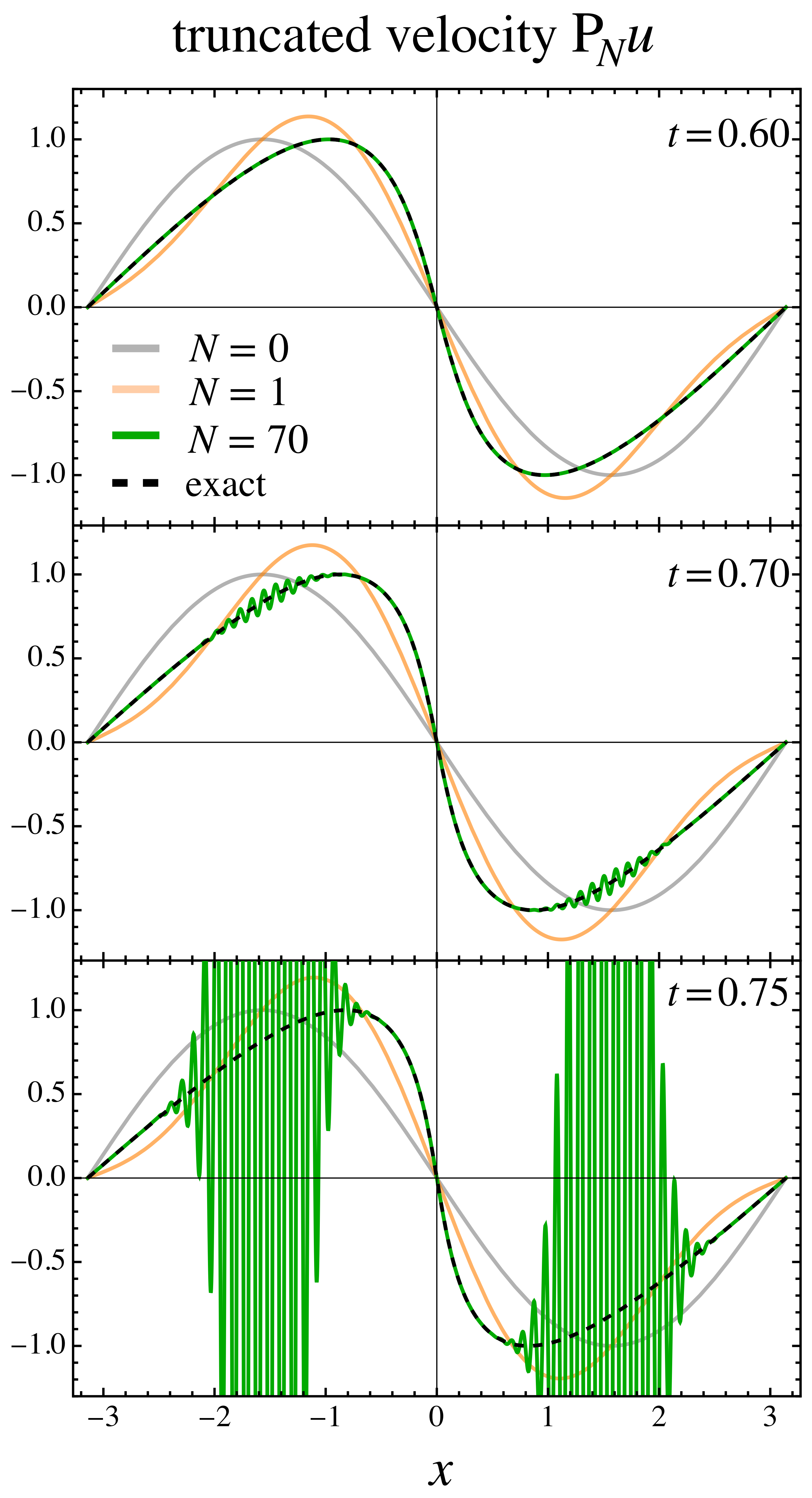
And here are some tygers
(see next slides for asymptotic analysis)

shown results for Taylor-truncated velocity $\mathrm{P}_N u := \sum_{n=0}^{N} u_n t^n$

with initial data $u_0 = -\sin x$,

for which **pre-shock** occurs at $t = 1$

$$\partial_x u\,|_{x=0} \to \infty$$



exact solution obtained in Lagrangian coordinates

Loss of convergence at seemingly boring locations, and at times well before the pre-shock.

These tyger resonances occur at much earlier times than those observed in Galerkin-truncated implementations. (but origin is the same: non-analyticity; see later)

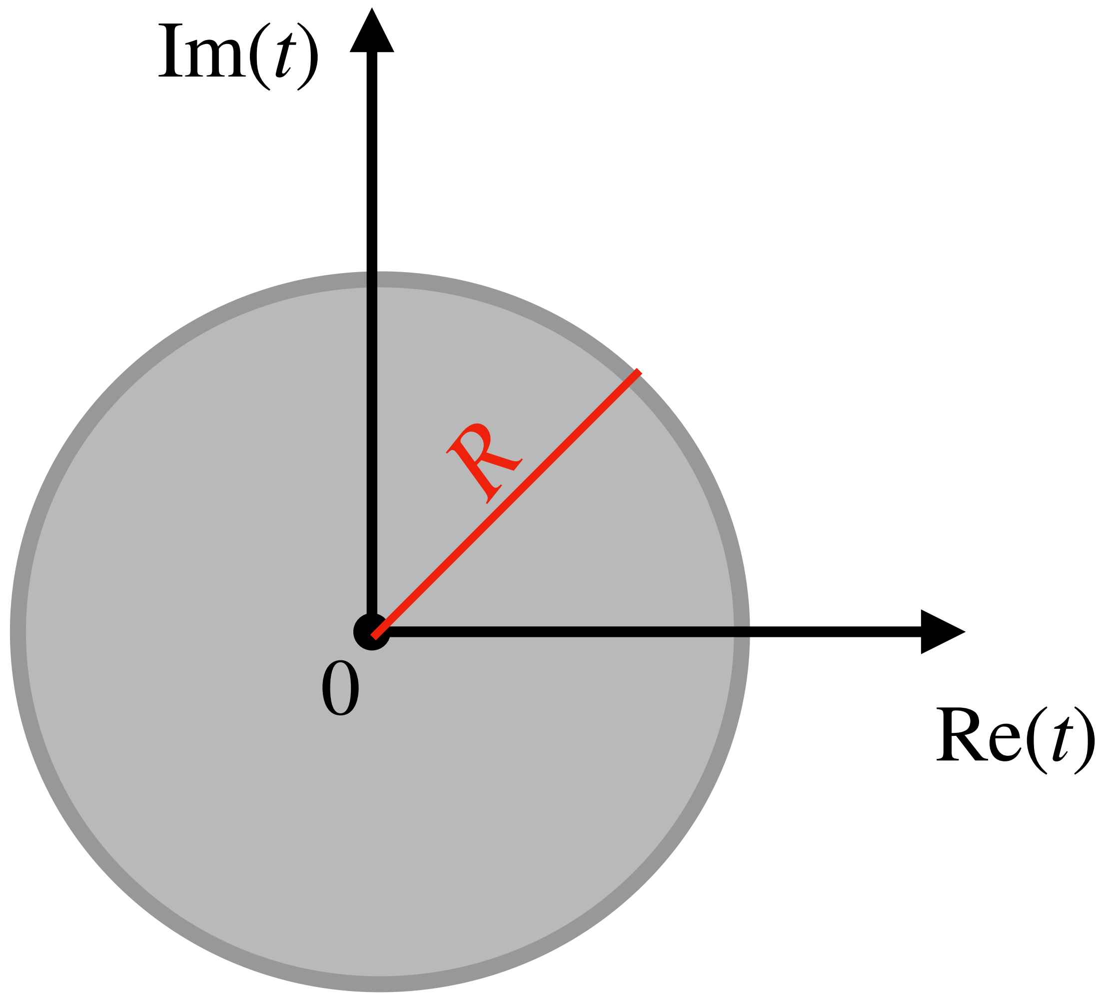
Asymptotic analysis of the time-Taylor series

- **initial attempt (too naive but constructive):** determine the radius of convergence R of the series $u = \sum_{n=0}^{\infty} u_n (t-0)^n$

  by **numerical extrapolation** of the ratio test $\frac{1}{R} = \lim_{n\to\infty} \frac{u_n}{u_{n-1}}$ (if the limit exists)

- Domb & Sykes (1957) suggest to draw subsequent ratios of u_n/u_{n-1} against $1/n$.
  For many problems, these ratios settle into a regular behaviour for $n \gg 1$,
  thereby allowing (linear) extrapolation to $1/n = 0$ (i.e., $n \to \infty$)

disc of convergence:



Asymptotic analysis of the time-Taylor series

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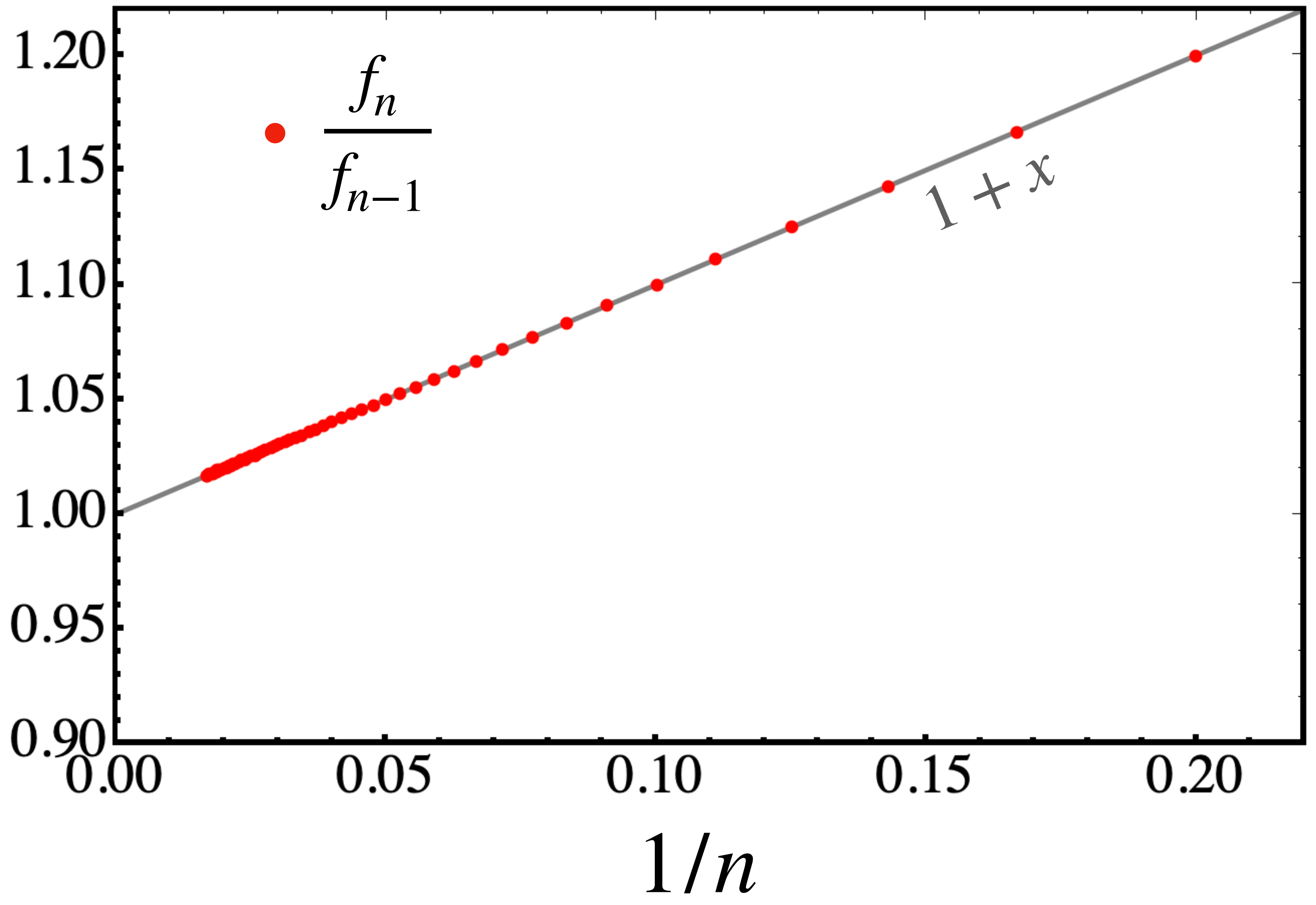
- **toy example**: consider the unrelated problem
  $f(t) = 1/(1-t)^2$
  and its time-Taylor series around $t=0$:
  $f(t) = \sum_{n=0}^{\infty} f_n t^n$ with $f_n = n+1$



Asymptotic analysis of the time-Taylor series

- for the present problem, the **Domb-Sykes** (DS) is however **not suitable** as the ratio u_n/u_{n-1} can swap sign.
  Thus, the limit in the ratio test does not exist, i.e., $\lim_{n\to\infty} u_n/u_{n-1} \neq 1/R$
- sign swap since the convergence-limiting singularity(ies) are at complex location(s)
- Mercer & Roberts (1990) have generalised the DS method to allow for a pair complex singularity (applied to Poisseuille flow), for which the **asymptotic behaviour** of u is modelled by

$$u(t) = \left(1 - \frac{t}{t_\star}\right)^\nu + \left(1 - \frac{t}{\bar{t}_\star}\right)^\nu, \qquad t_\star := R\mathrm{e}^{\mathrm{i}\theta}.$$

ν ….. singularity exponent

$t_\star$ ….. complex-time location of singularity

in generally x-dependent!

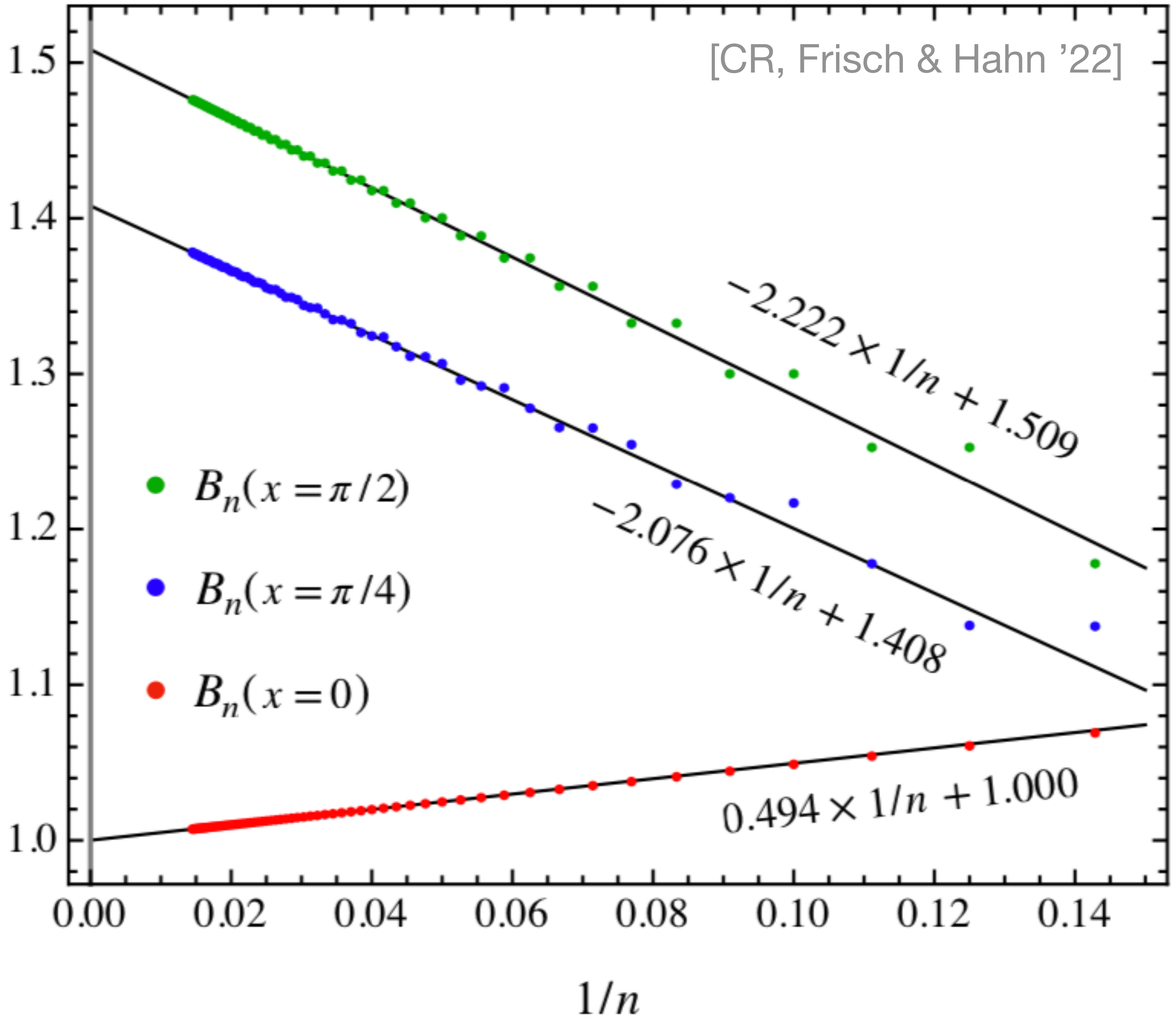
- By considering the Taylor expansion of the model function, one finds

[Mercer & Roberts 1990]

$$B_n^2 = \frac{u_{n+1}u_{n-1} - u_n^2}{u_n u_{n-2} - u_{n-1}^2}, \qquad \xrightarrow{n \to \infty} \qquad B_n = \frac{1}{R}\left(1 - (\nu+1)\frac{1}{n}\right) \times \left[1 + \frac{\nu+1}{2}\frac{\sin(2n-1)\theta}{\sin\theta}\frac{1}{n^2} + O(n^{-3})\right]$$

… and a similar estimator for the phase θ. Thus, all unknowns in model function can be obtained by graphical extrapolation (see next)

Mercer-Roberts extrapolation at three exemplary points x



only input needed: the time-Taylor coefficients u_n to sufficient high order (here up to $N = 70$)

$$B_n^2 = \frac{u_{n+1}u_{n-1} - u_n^2}{u_n u_{n-2} - u_{n-1}^2}\,,$$

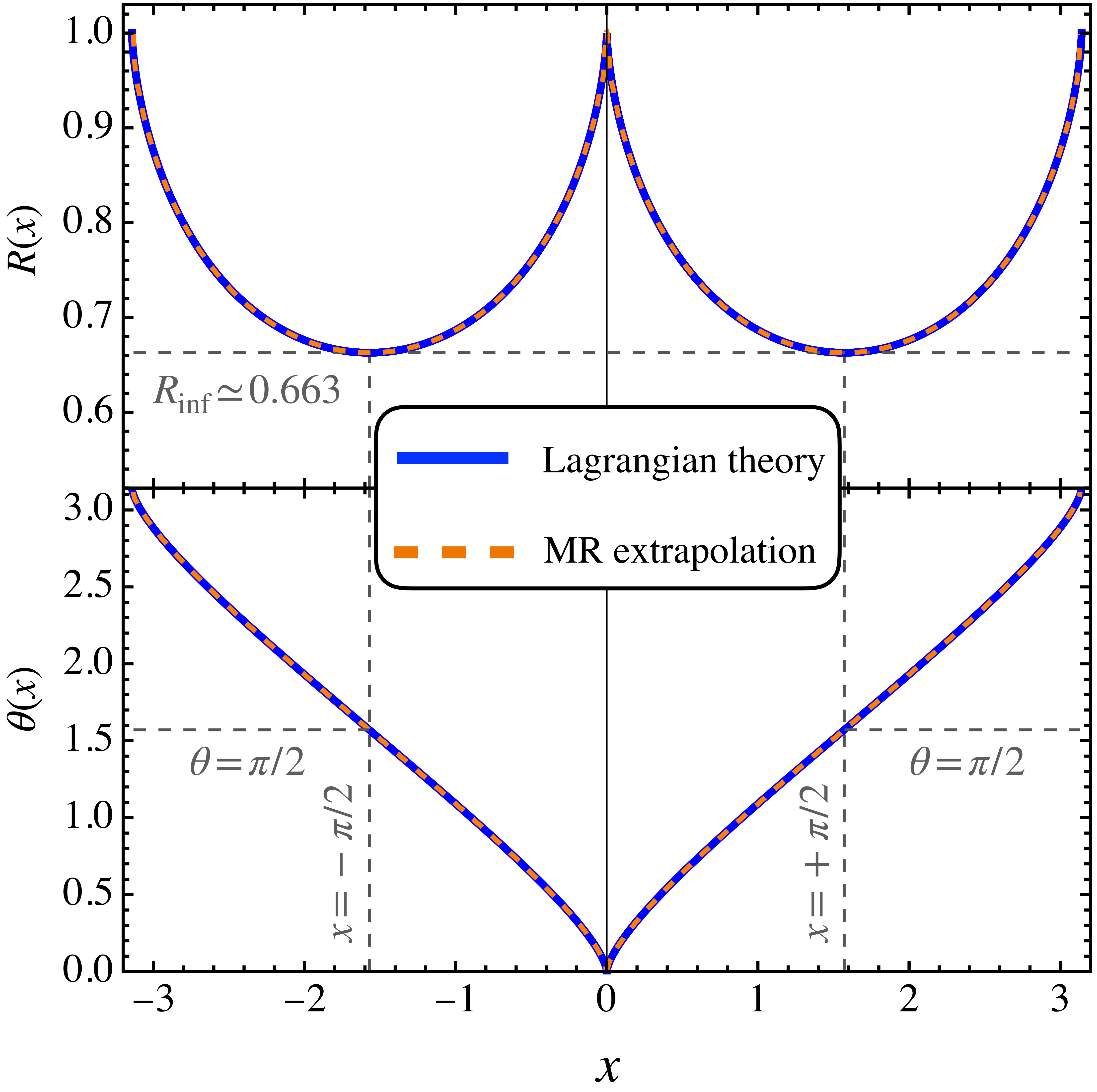
$n \to \infty$

$$B_n = \frac{1}{R}\left(1 - (\nu+1)\frac{1}{n}\right) \times \left[1 + \frac{\nu+1}{2}\frac{\sin(2n-1)\theta}{\sin\theta}\frac{1}{n^2} + O(n^{-3})\right]$$

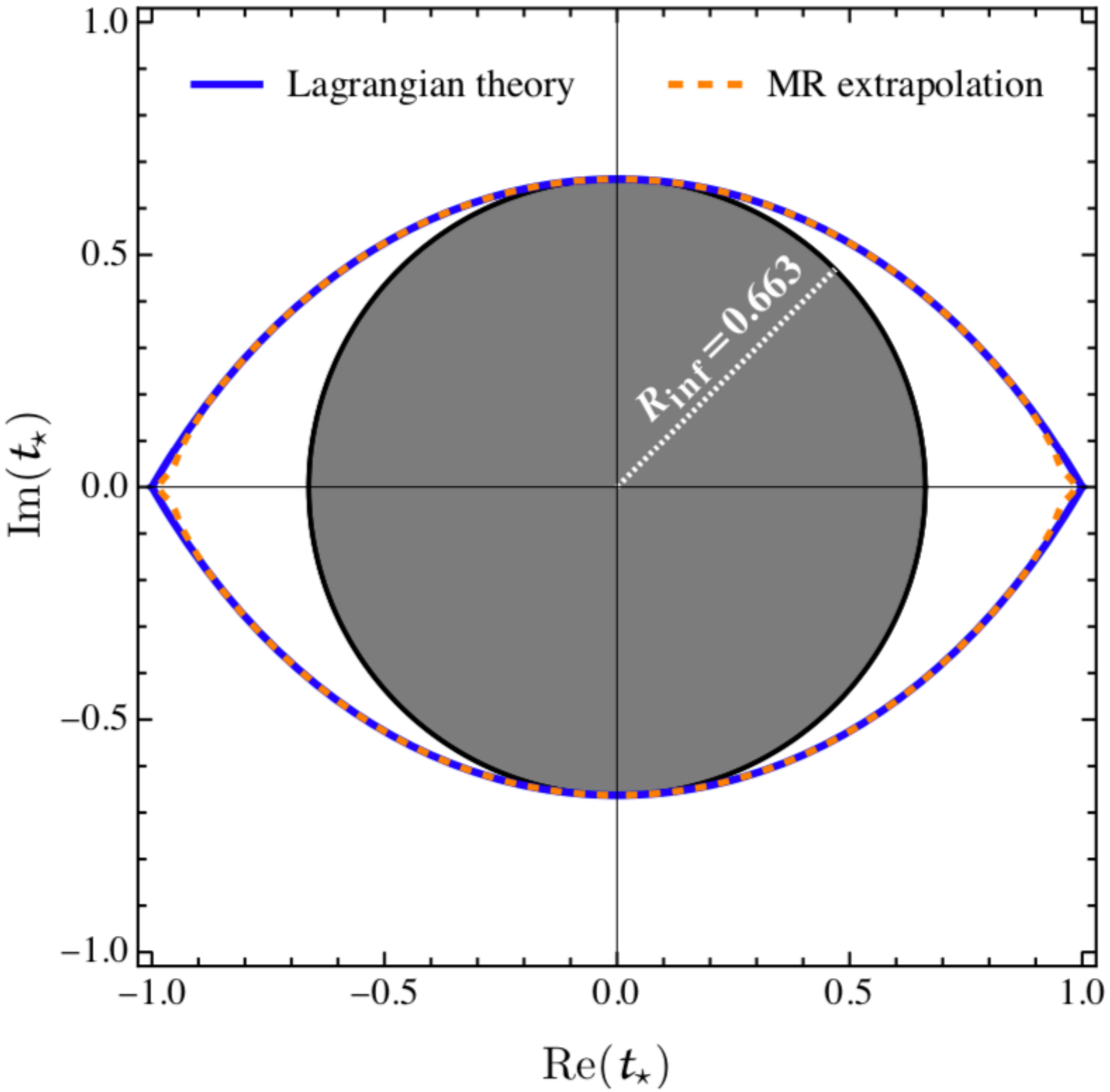
Mercer-Roberts (MR) extrapolation over whole space

[CR, Frisch & Hahn '22]

$$u(t) = \left(1 - \frac{t}{t_\star}\right)^\nu + \left(1 - \frac{t}{\bar{t}_\star}\right)^\nu, \qquad t_\star := R\mathrm{e}^{\mathrm{i}\theta}.$$



OR:



of course, same analysis can be done for multi-mode initial data; see CR+ '22

Lagrangian singularity theory in a nutshell (1 of 2)

[CR, Frisch & Hahn '22]

- Introduce direct Lagrangian map $a \mapsto x$ from initial position a at time $t=0$, to current position x at time t, defined through characteristic equation $u(x(a,t),t) = \dot{x}(a,t)$ where the dot denotes Lagrangian (total) time derivative

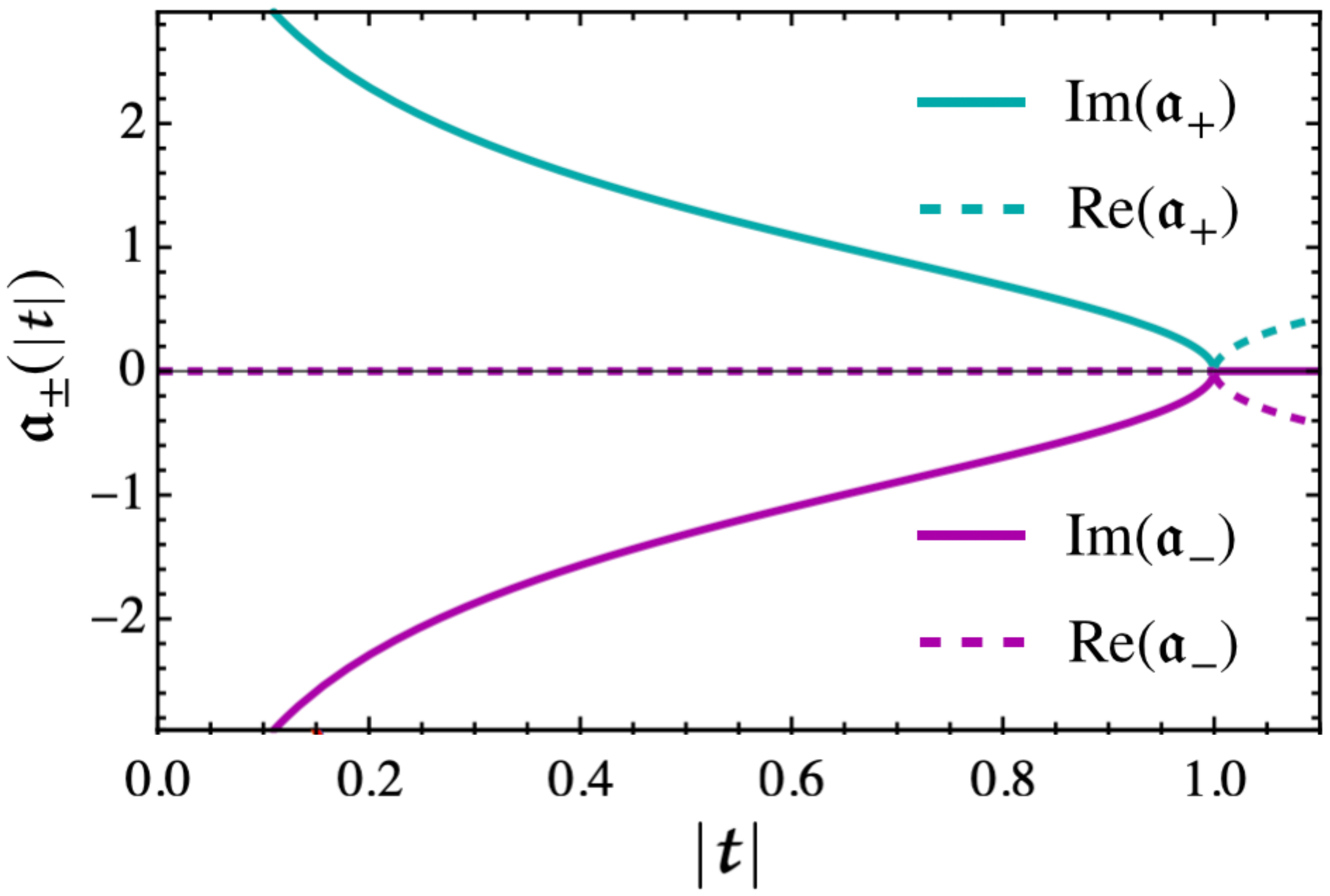
- Inviscid Burgers' equation becomes $\ddot{x} = 0$ which has well-known pre-shock solution $x(a,t) = a + t\,u_0(a)$

- Pre-shock occurs at real time $t = t_\star$ when Jacobian determinant $J := \partial_a x$ vanishes

- **Now we complexify both time and space, and search for complex Lagrangian locations $\mathfrak{a}_\pm$ for which $J = 0$**

  e.g. for the case $\boldsymbol{t} = |\boldsymbol{t}|$ (i.e., for vanishing phase) :

single-mode case for which pre-shock is at $t = 1$

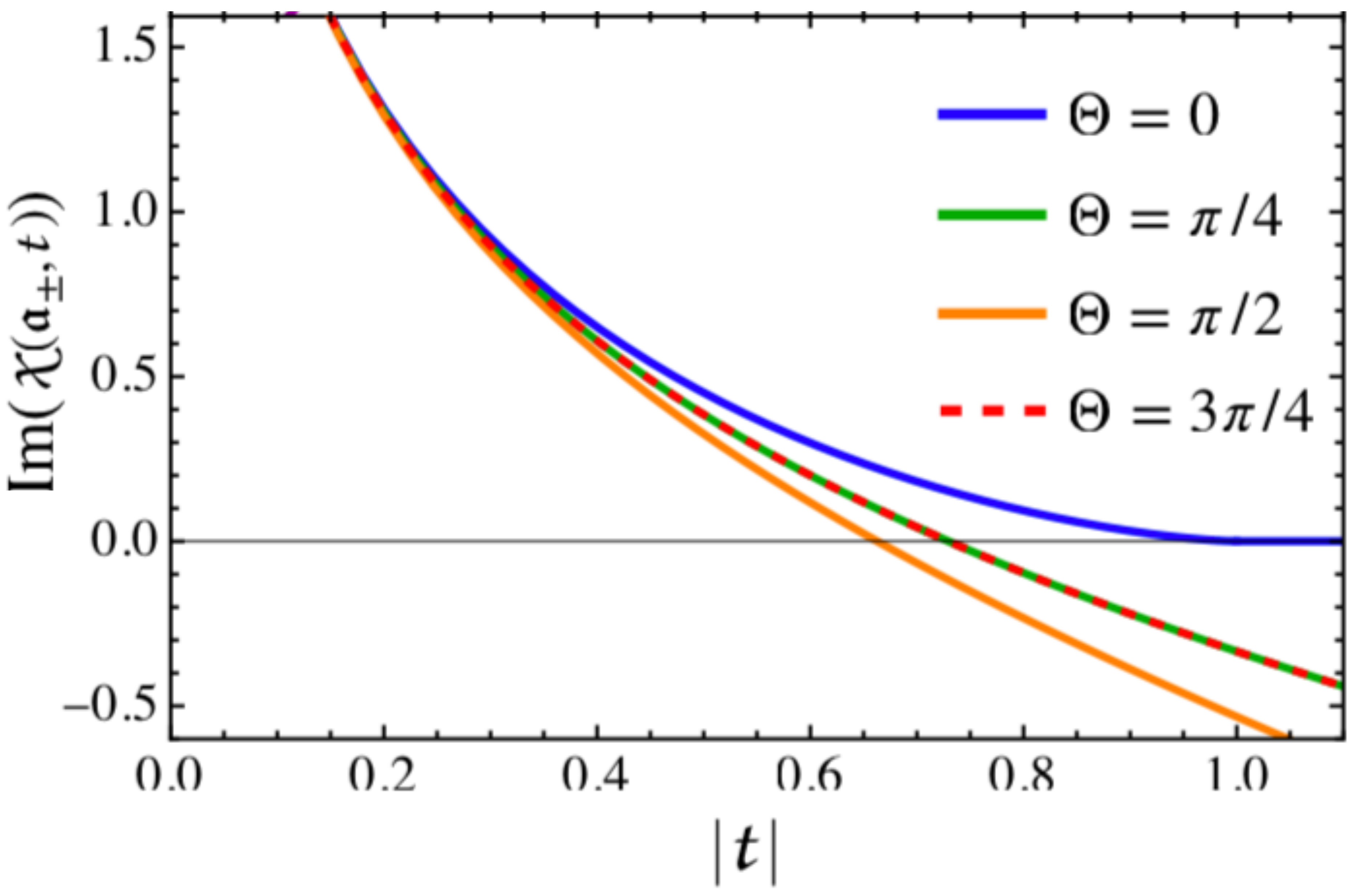


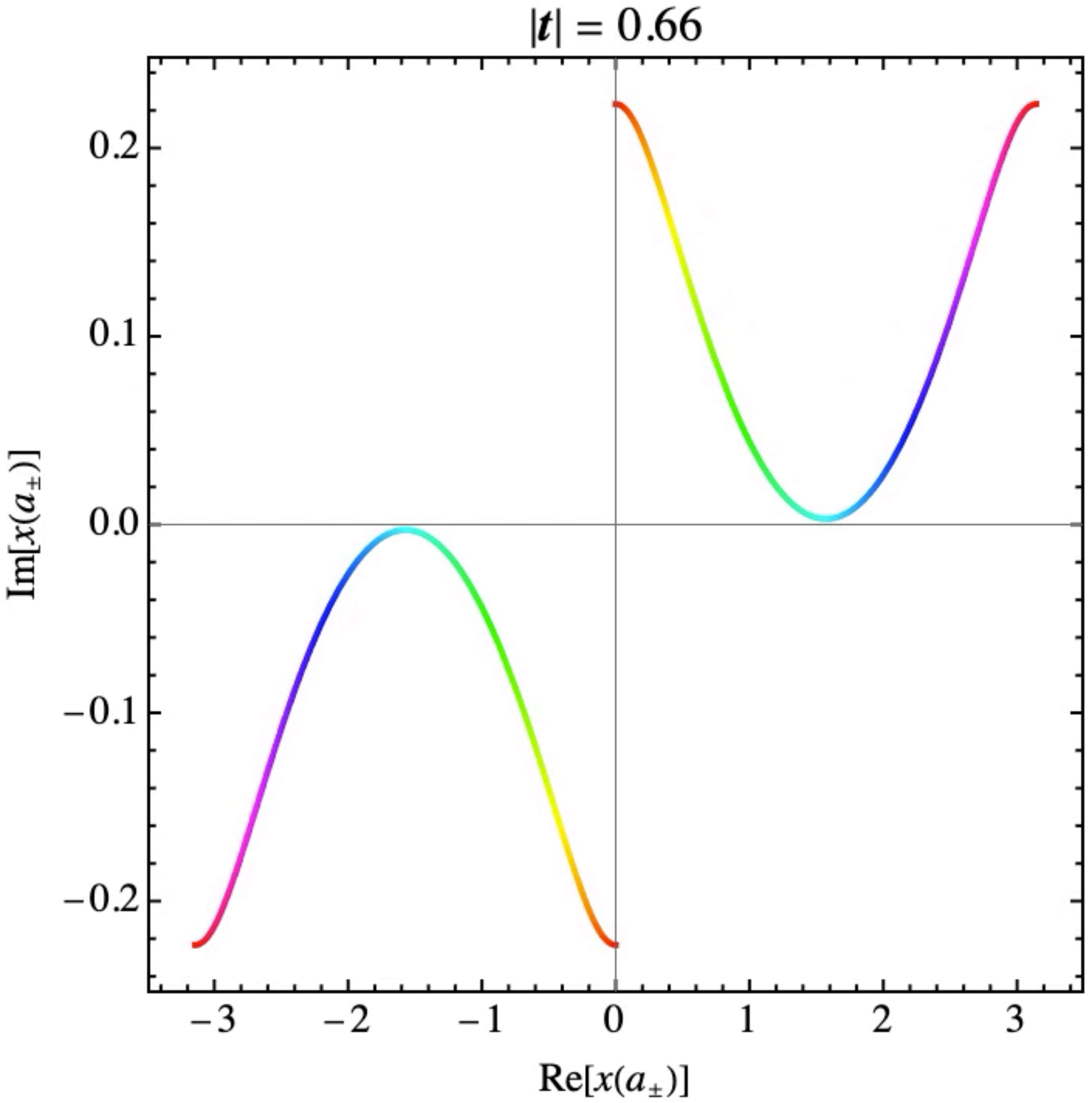
Lagrangian singularity theory in a nutshell (2 of 2)

[CR, Frisch & Hahn '22]

- Necessary condition for the complex Lagrangian roots of $J = 0$ to become relevant at the real-valued Eulerian position: the imaginary part of $x(\mathfrak{a}_\pm)$ has to vanish!

complex current location of pre-shock singularity





color hue is phase Θ of complex time

Two ways to suppress the growth of tygers

Tyger purging (an adapted technique of Murugan+ ’20)

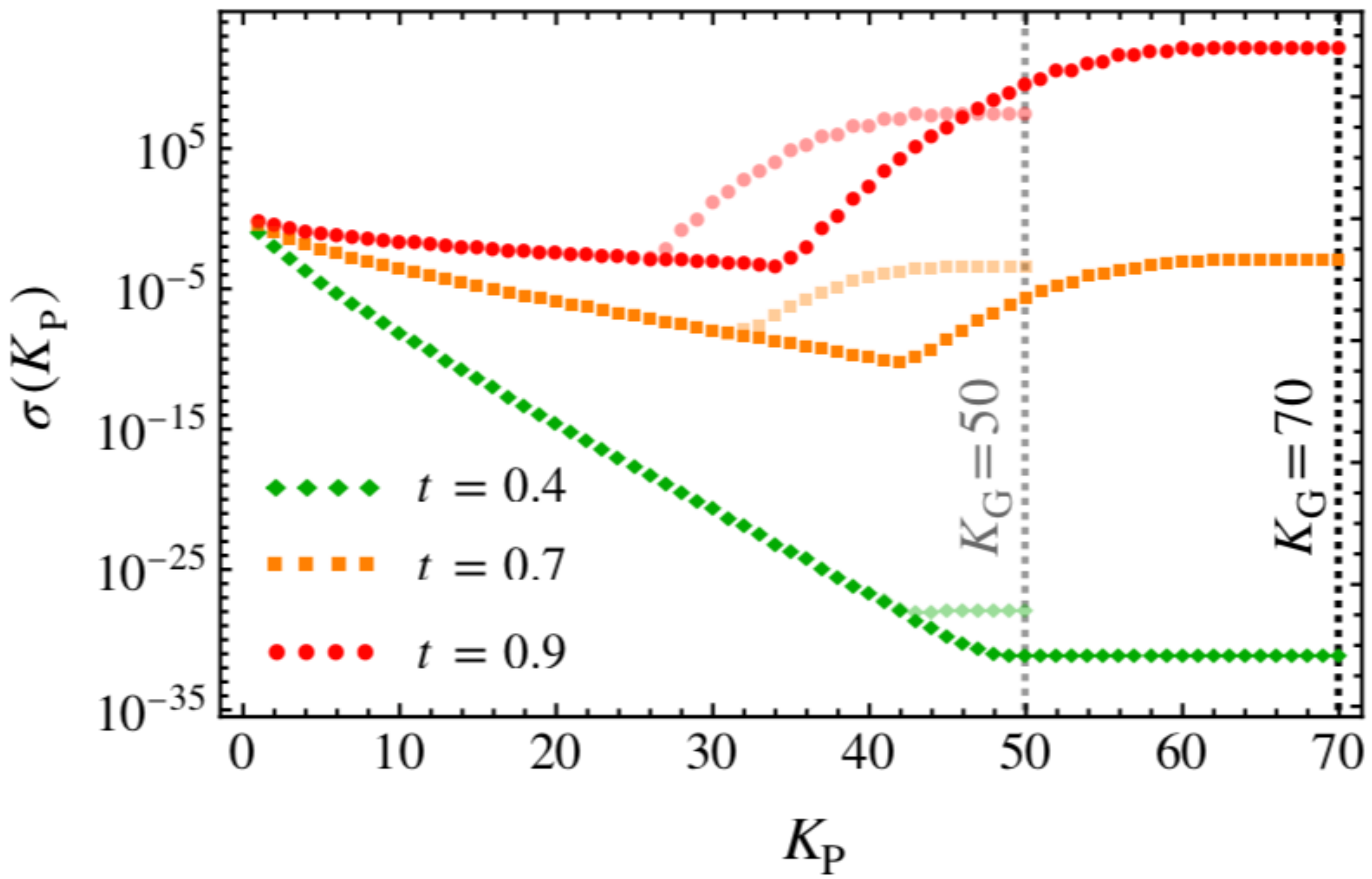
[CR, Frisch & Hahn ’22]

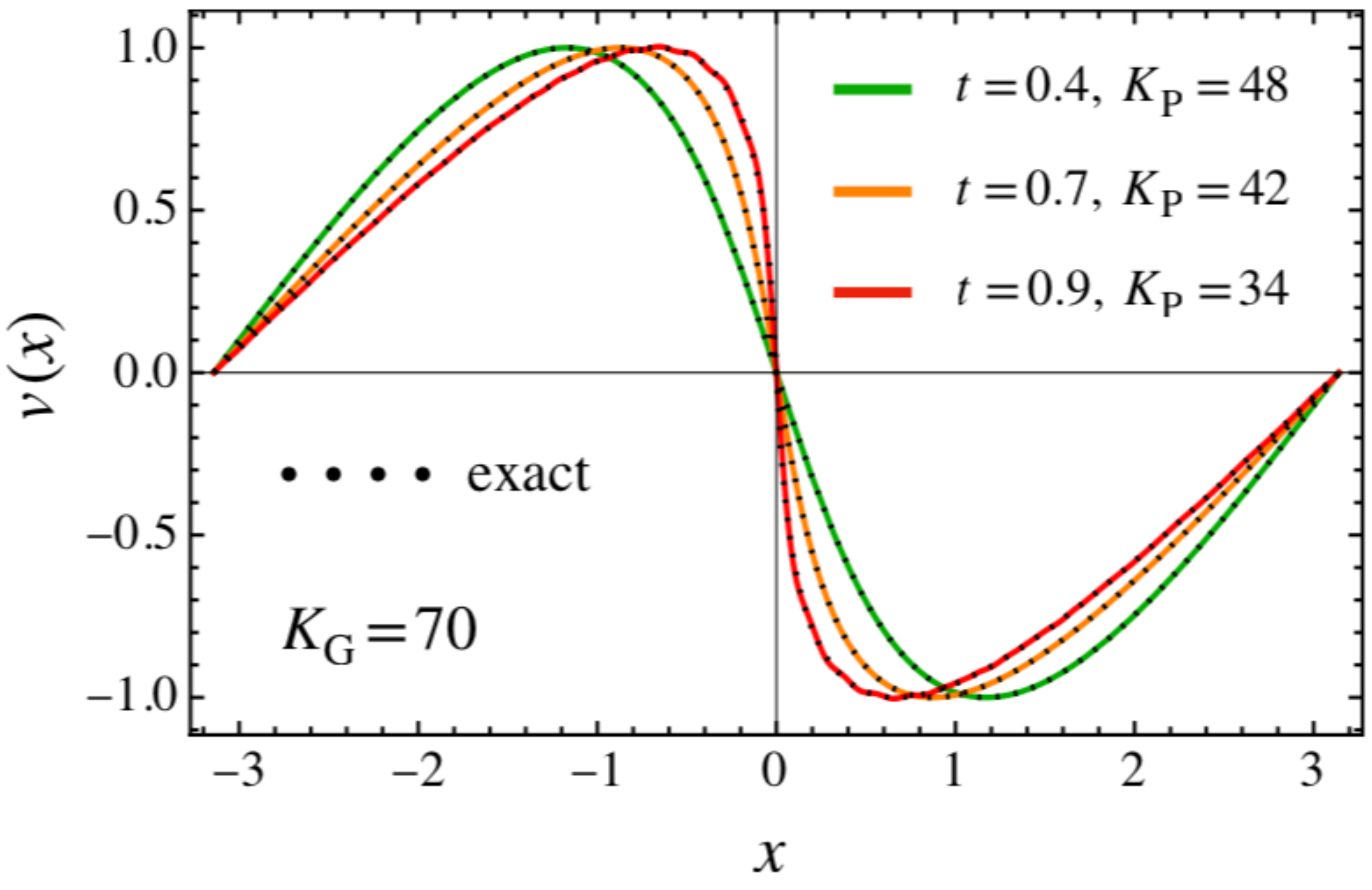
- main idea, remove Fourier modes below the (Galerkin/Taylor) truncation
- define $v := \mathrm{P}_N u = \sum_{k=0,\pm1,\pm2,\ldots,\pm\underbrace{(N+1)}_{=:\,K_\mathrm{G}}} \hat{u}_k \mathrm{e}^{\mathrm{i}kx}$

and purging operator $\mathrm{P}_{K_\mathrm{P}} v(x) = \sum_{|k|\le K_\mathrm{P}} \hat{u}_k\, \mathrm{e}^{\mathrm{i}kx}$

removes Fourier modes for $K_\mathrm{p} < |k| \le K_\mathrm{G}$

- integrated error $\sigma(t, K_\mathrm{P}) := \int_{-\pi}^{+\pi} \left[\mathrm{P}_{K_\mathrm{P}} v(x(a,t),t) - u_0(a)\right]^2 \mathrm{d}a$

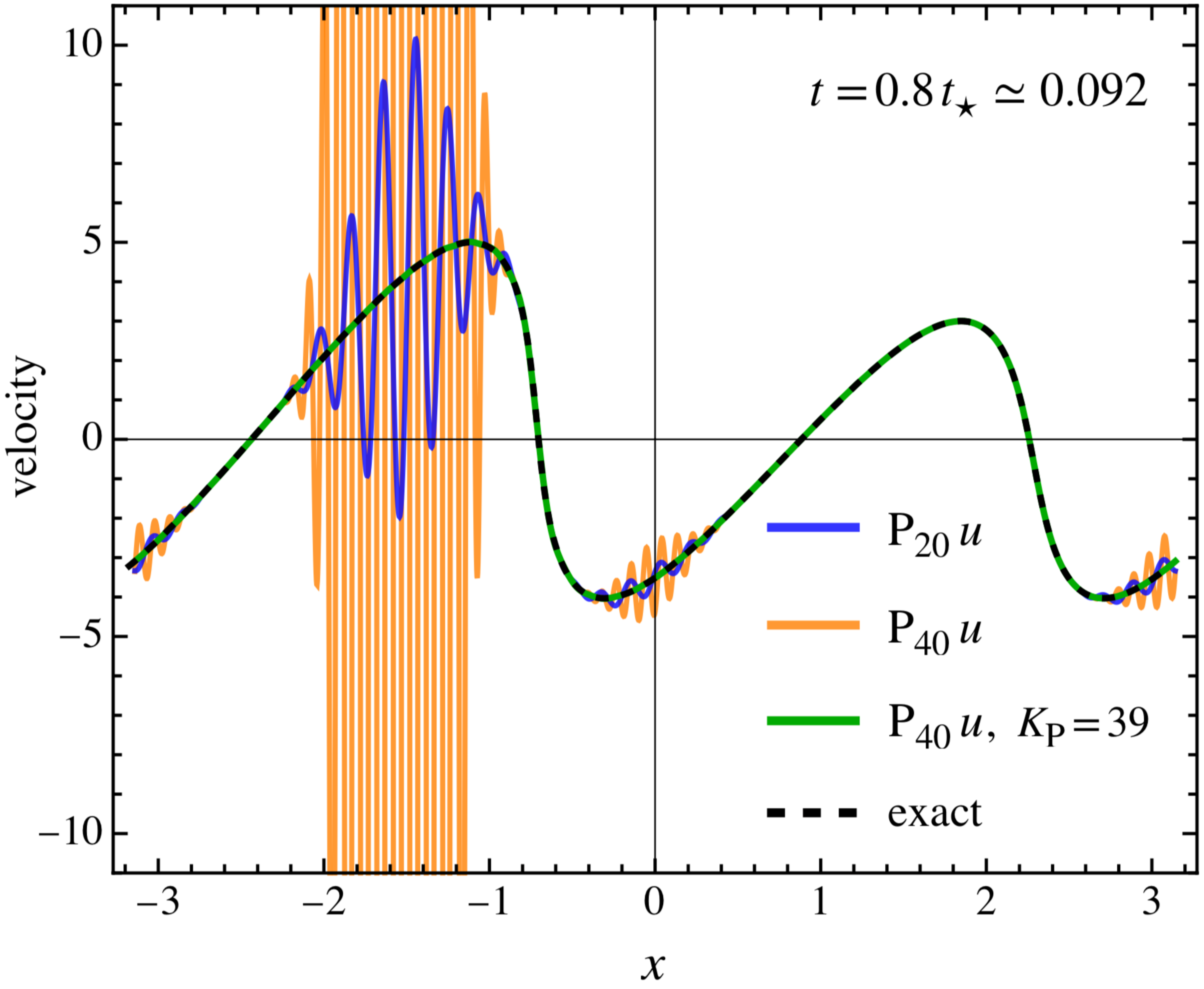




Tyger purging (an adopted technique of Murugan+ ’20)

[CR, Frisch & Hahn ’22]

- works also well for multi-mode initial conditions, e.g. for $u_0 = -\sin x - 4\cos(2x)$:



Finally, the “opposite” idea: iterative UV completion

Iterative UV completion

[CR, Frisch & Hahn ’22]

- basic idea: add efficiently Fourier modes instead of discarding modes, as in purging
- integrating Burgers equation $\partial_t u = -(1/2)\partial_x u^2$ in time, one obtains in the smooth case

$$u = u_0 - \frac{1}{2}\partial_x \int_0^t u^2(\tau)\,\mathrm{d}\tau$$

- Now let's approximate on the RHS $u^2 = (\mathrm{P}_N u)^2$ where $\mathrm{P}_N u = \sum_{n=0}^{N} u_n\, t^n$
- the resulting approximation on the velocity is called $v_{\{1\}}$ and is governed by

$$v_{\{1\}} = v_0 - \frac{1}{2}\partial_x \int_0^t [\mathrm{P}_N u(\tau)]^2\,\mathrm{d}\tau\,, \qquad v_0 = u_0\,.$$

note: depends on truncation order N

- perform an iterative bootstrapping (à la Duhamel's principle)

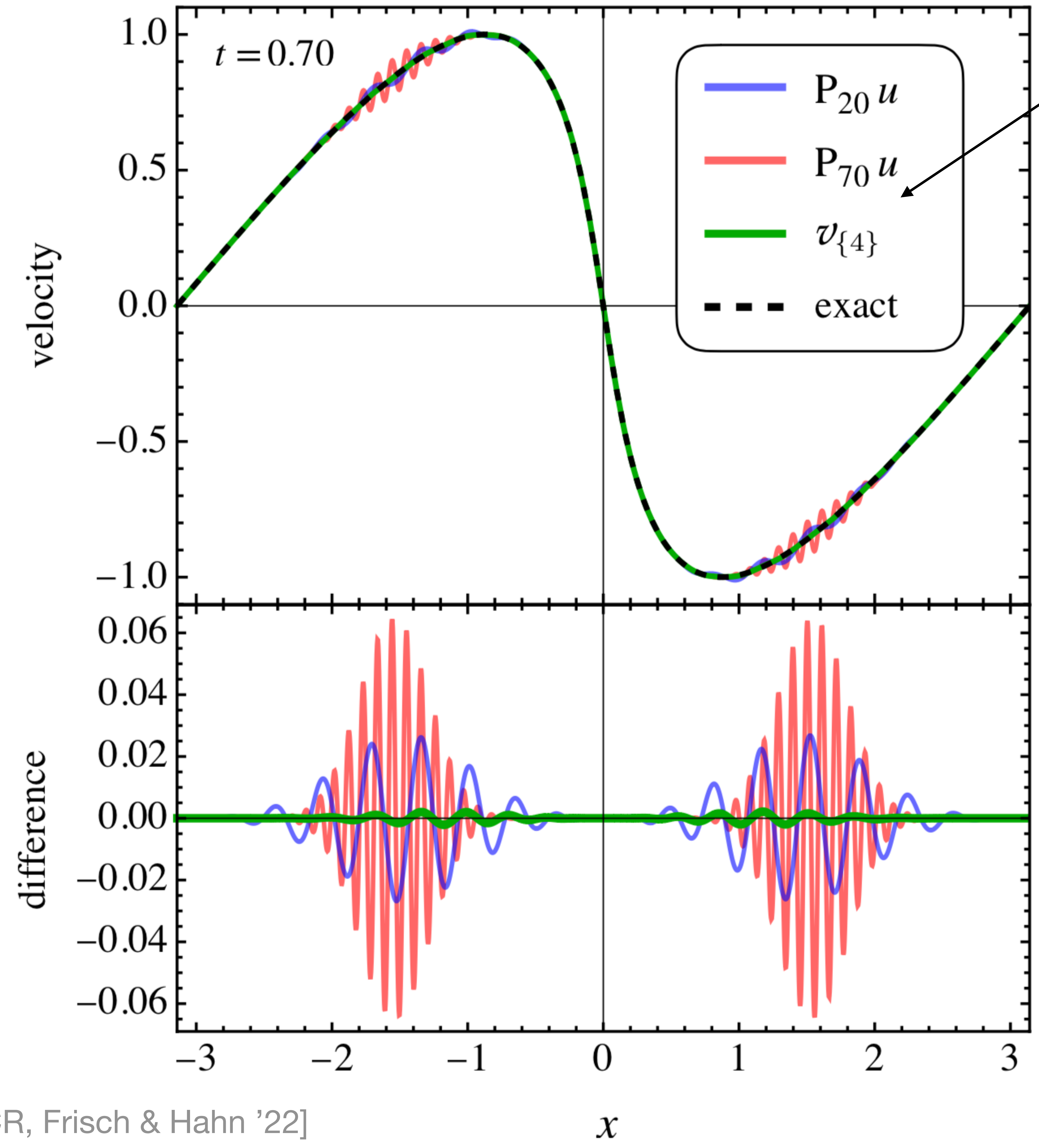
$$v_{\{2\}} = v_0 - \frac{1}{2}\partial_x \int_0^t v_{\{1\}}^2(\tau)\,\mathrm{d}\tau$$

… and so on. At each iteration, and for single- or multi-mode ICs, number of non-zero Fourier modes is roughly doubled

Iterative UV completion

single-mode case

recall $\mathrm{P}_N u := \sum_{n=0}^{N} u_n t^n$



4th iteration in the bootstrapping, with $\mathbf{P_{20}}\,u$ as input

bootstrapping reduces the tyger amplitude once convergence is lost (here: $t > 0.66$)

[CR, Frisch & Hahn ’22]

Iterative UV completion

single-mode case

recall $\mathrm{P}_N u := \sum_{n=0}^{N} u_n t^n$

violation on energy conservation, once convergence of the Taylor series is lost

$$\delta E(\mathcal{U}) := \frac{2}{\pi} \int_{-\pi}^{+\pi} \frac{\mathcal{U}^2(x,t)}{2} \mathrm{d}x - 1$$

$\mathcal{U} = \mathrm{P}_{70}\, u, \mathrm{P}_{20}\, u,\ v_{\{1\}}, \ldots$

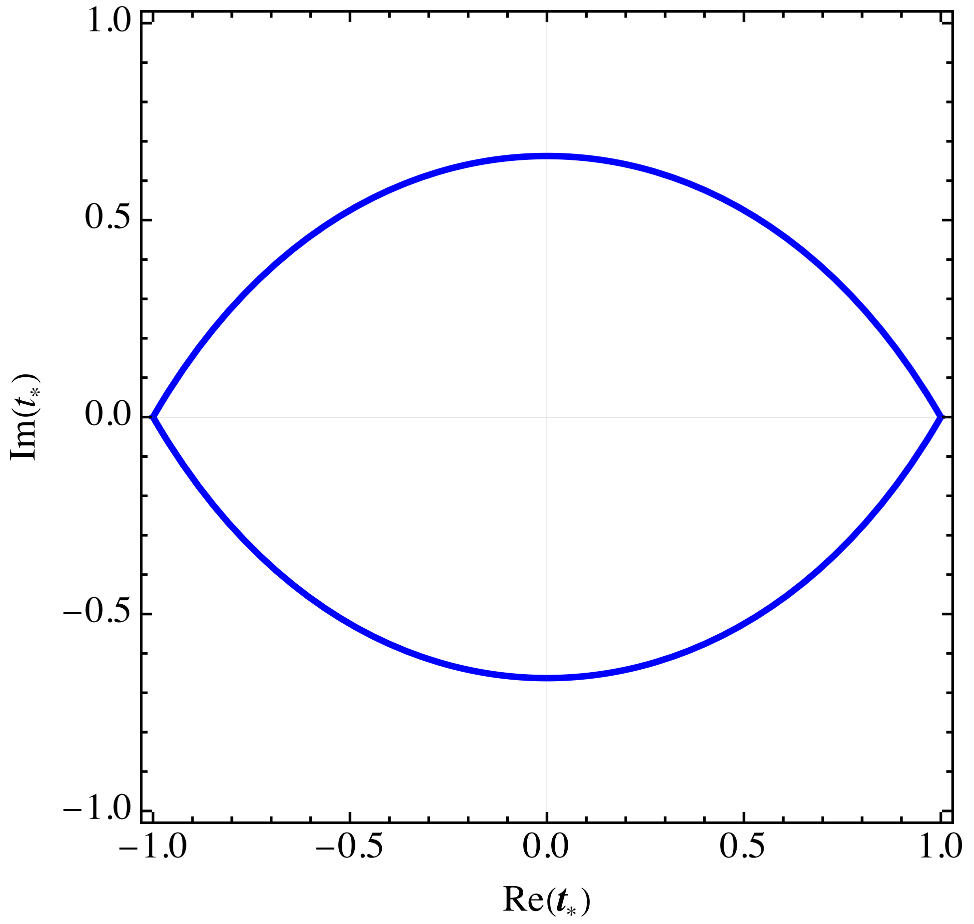
$\delta E(\mathcal{U})$ is exactly zero if energy is conserved

Time	$\delta E(\mathrm{P}_{70} u)$	$\delta E(\mathrm{P}_{20} u)$	$\delta E(v_{\{1\}})$	$\delta E(v_{\{2\}})$	$\delta E(v_{\{3\}})$	$\delta E(v_{\{4\}})$
0.70	6.14e-4	2.03e-4	5.52e-5	1.49e-5	4.06e-6	1.10e-6
0.75	8.93e+0	3.00e-3	9.09e-4	2.77e-4	8.56e-5	2.65e-5
0.80	6.97e+4	3.70e-2	1.25e-2	4.26e-3	1.47e-3	5.12e-4
0.85	3.14e+8	3.90e-1	1.56e-1	6.09e-2	2.15e-2	8.24e-3

energy conservation iteratively restored via bootstrapping

[CR, Frisch & Hahn ’22]

Conclusions & Outlook



- main cause for early-time tygers to appear: non-analyticity

  [cf. Bardos & Tadmor '13 on the “old” tygers in (pseudo)-spectral methods]

- Lagrangian singularity theory in complex space and time:
  origin of the singular landscape is the pre-shock, which is in Lagrangian coordinates a *localised* complex-time singularity

- both tyger purging or iterative UV completion work well for taming tygers

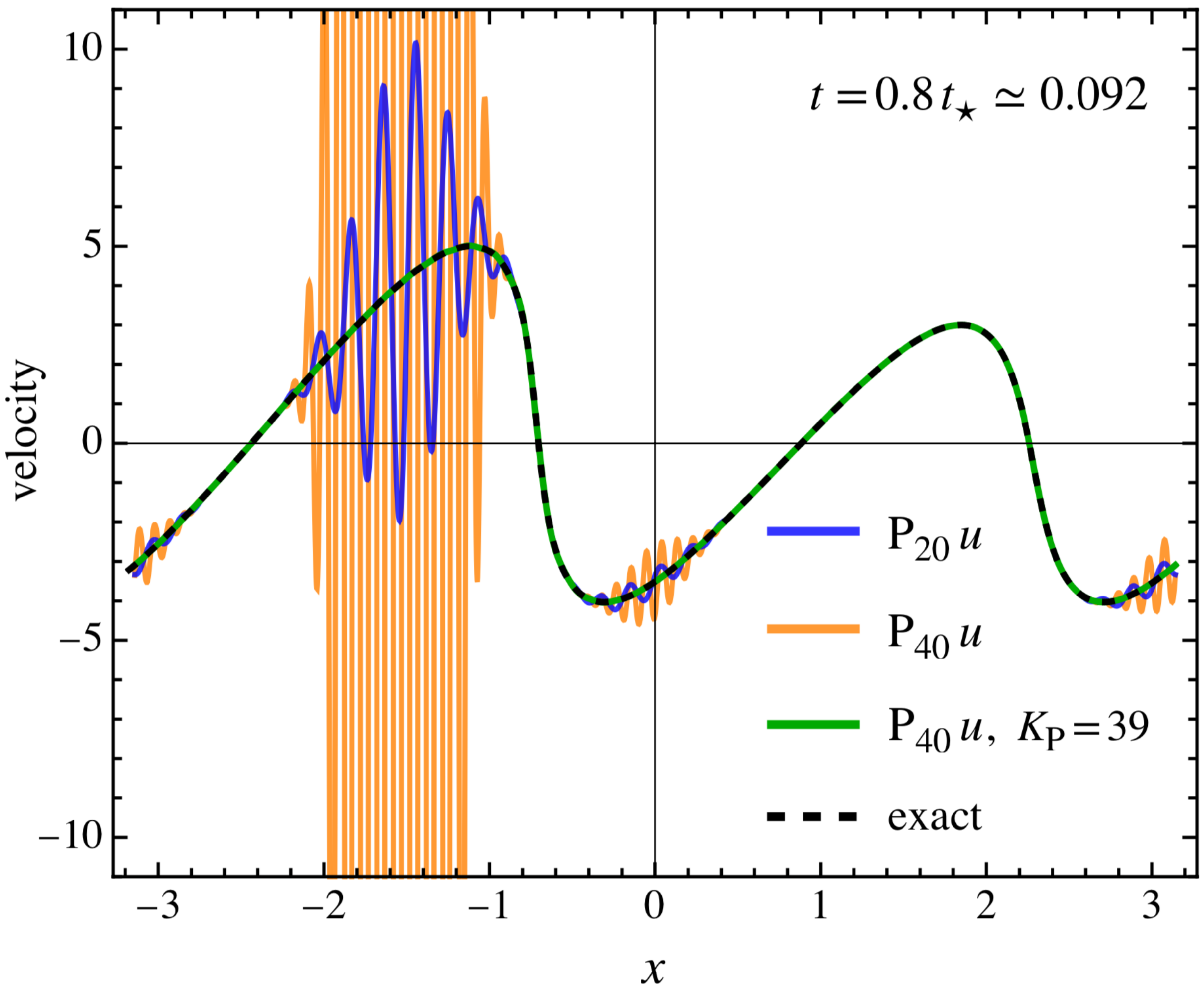
- precise mechanism of UV completion not yet understood; also, is there a resummation of the iterative method? (cf. Dyson series)

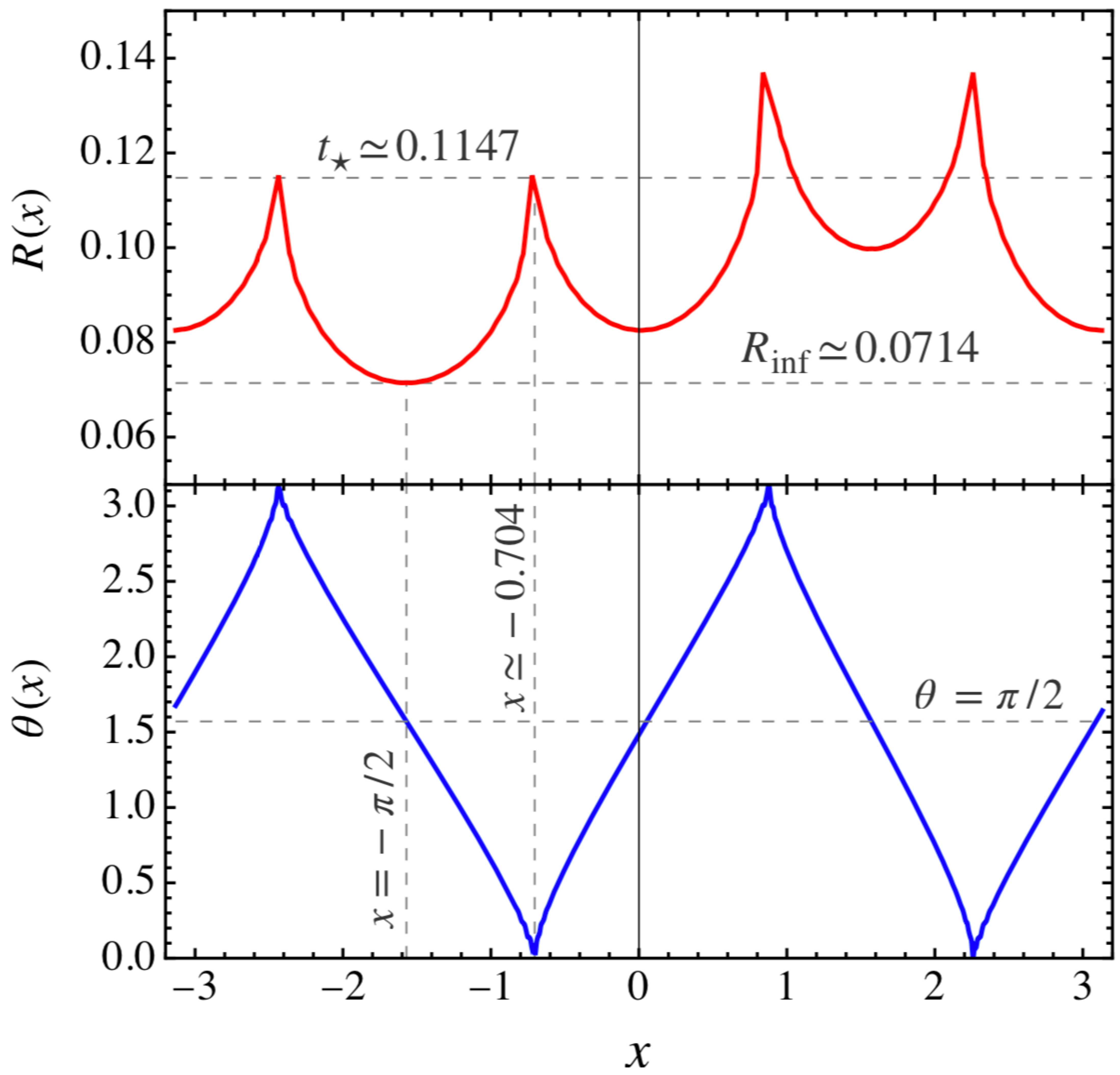
- UV completion for sub-grid scale modelling in general fluids?
  Method does not require specifically Taylor-series input; also weak formulations of the method may be feasible

- Apply methods to other fluids, such as incompressible Euler, cosmological Euler-Poisson, etc.

see arXiv:2207.12416 for more details

Backup slide 1





Backup slide 2

described by a theory that at its heart employs the method of characteristics (see section IV for the theory applied to multi-mode initial conditions).

For this we employ the direct Lagrangian map $a \mapsto x$ from initial ($t = 0$) position a to the current/Eulerian position x at time t. The velocity is defined through the characteristic equation $u(x(a,t),a) = \dot{x}(a,t)$, where the overdot denotes the Lagrangian (convective) time derivative. Employing Lagrangian coordinates, the inviscid Burgers equation (1) reduces to $\ddot{x}(a,t) = 0$, which has the well-known solution

$$x(a,t) = a + t u_0(a) = a - t \sin a \tag{15}$$

(see e.g. [2, 18]). The Jacobian of the transformation

$$J(a,t) := \frac{\partial x}{\partial a} = 1 - t \cos a \tag{16}$$

vanishes at pre-shock time $t = t_\star = 1$ at location $a = a_\star = 0 = x_\star$ (modulo 2π-periodic repetitions).

In section II B we have seen that singularities appear in Eulerian space at times well before $t_\star = 1$. To assess this scenario within the present description, we must allow the fluid variables to also take complex values. Thus, we *complexify the Lagrangian and Eulerian locations* and denote them respectively with $\mathfrak{a}$ and χ. Additionally, as in section II B, we employ the complexified time denoted with $\boldsymbol{t}$.

Now, let us consider complex times $\boldsymbol{t}$ with $|\boldsymbol{t}| \leq t_\star = 1$, and search for the complex Lagrangian roots, dubbed $\mathfrak{a}_\pm$, for which the Jacobian of the Lagrangian map vanishes, i.e.,

$$\mathfrak{a} = \mathfrak{a}_\pm : \qquad \mathcal{J} = \frac{\partial \chi}{\partial \mathfrak{a}} = 0\,. \tag{17}$$

One easily finds the two exact roots

$$\mathfrak{a}_\pm = \pm \arccos\left(\frac{1}{\boldsymbol{t}}\right)\,, \tag{18}$$

which imply the current/Eulerian locations

$$\chi(\mathfrak{a} = \mathfrak{a}_\pm, \boldsymbol{t}) = \pm \left[\arccos\left(\frac{1}{\boldsymbol{t}}\right) - \boldsymbol{t}\sqrt{1 - \frac{1}{\boldsymbol{t}^2}}\right]\,. \tag{19}$$

In the upper panel of Fig. 4, we show the evolution of the complex roots as a function of $\boldsymbol{t} = |\boldsymbol{t}|$. For $\boldsymbol{t} = |\boldsymbol{t}| < 1$, these roots are purely imaginary, but if $\boldsymbol{t}$ is not aligned along the real time axis, the roots are in general complex (not shown). Could these complex roots of $\mathcal{J} = 0$, evaluated at complex locations in time and space, lead to singularities in Eulerian coordinates before the pre-shock?

To address this question, we show in the lower panel of Fig. 4 the evolution of $\pm\mathrm{Im}(\chi(\mathfrak{a}_\pm, \boldsymbol{t}))$ as a function of $|\boldsymbol{t}|$ for

$$\boxed{\boldsymbol{t} = \boldsymbol{t}_\star : \quad \mathrm{Im}\left[\arccos\left(\frac{1}{\boldsymbol{t}_\star}\right) - \boldsymbol{t}_\star\sqrt{1 - \frac{1}{\boldsymbol{t}_\star^2}}\right] = 0}$$

Backup slide 3

Here we apply the Lagrangian singularity theory of section II C to the two-mode initial data (32); the generalization to the multi-mode case is straightforward and discussed at the end of the section. Employing the direct Lagrangian map $a \mapsto x$, one finds

$$x(a,t) = a - t\left[\sin a + 4\cos(2a)\right] , \qquad (34)$$

which implies the Jacobian determinant

$$J(a,t) = 1 + t\left[8\sin(2a) - \cos a\right] . \qquad (35)$$

Physically, the most relevant singularity is the one that is closest to the origin in time (for a Taylor expansion around $t=0$, this is the singularity that sets the radius of convergence). Thus, within a two-step process, we first define the critical times $t_{\star 1,2,3,4}$ corresponding to the roots $\mathfrak{a}_{1,2,3,4}$, for which

$$\boxed{t = t_{\star i} : \quad \mathrm{Im}\left[\chi(\mathfrak{a}=\mathfrak{a}_i, t=t_{\star i}\right] = 0} \qquad (38a)$$

is satisfied. Then, as a second and final step, we select

$$R := \inf\{|t_{\star 1}|, |t_{\star 2}|, |t_{\star 3}|, |t_{\star 4}|\} , \qquad (38b)$$

which is the physically relevant radius of convergence R for fixed phase $\Theta = \theta$. This methodology is not only valid for the