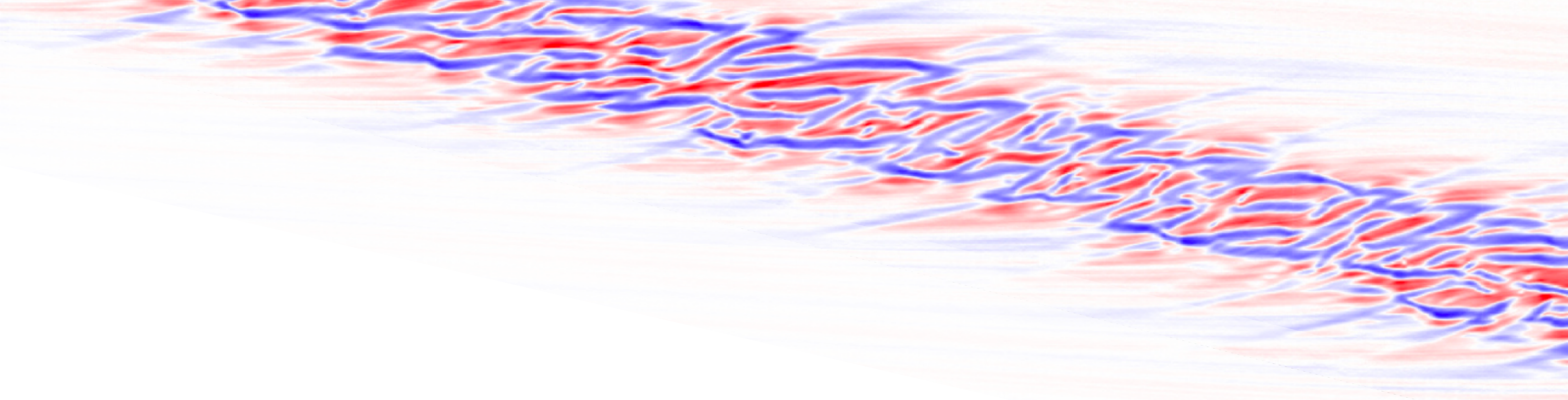
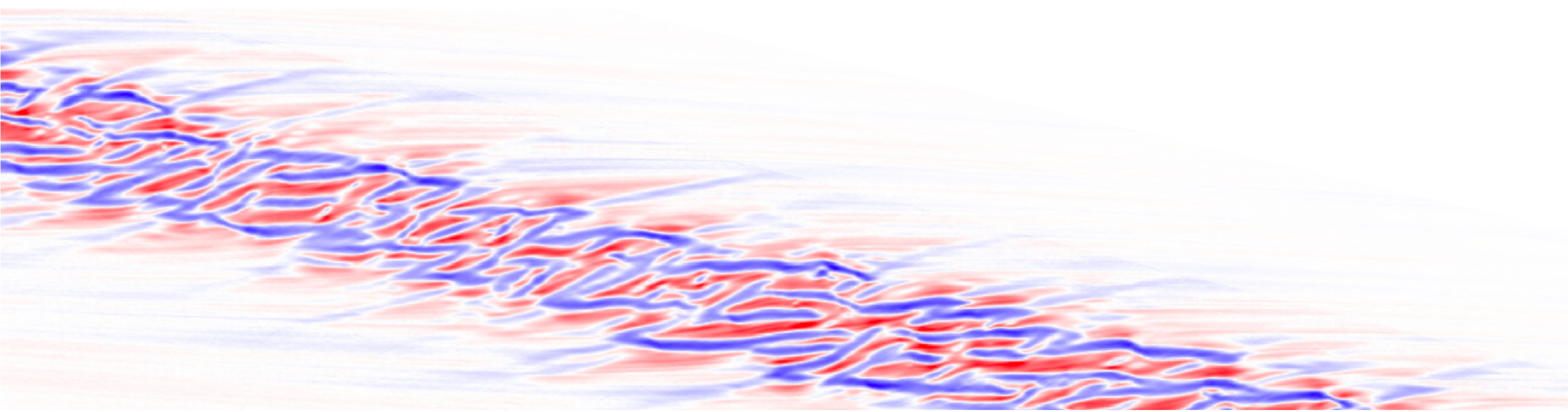


Microphysically modified magnetosonic modes in collisionless, high-β plasmas

Stephen Majeski, M. Kunz, J. Squire



Some motivation…

Density, pressure, and B fluctuations measured by WIND spacecraft at 1 AU

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→*Measured eigenmode relationships suggest most compressive modes are MHD-like as opposed to kinetic*

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→*Measured eigenmode relationships suggest most compressive modes are MHD-like as opposed to kinetic*

Coburn, Chen, Squire, '22:

→*Matched fluctuations from SW to linear KMHD with Krook collisions, finding* $\lambda_{mfp,eff}$ (*measured*) $\sim 10^{-3}\lambda_{mfp,coll}$ (*calculated*) *for protons*

***Collisionless waves conserve adiabatic invariants* (μ, J)**

Shear Alfvén waves:

δB oscillation generates Δ $(= p_\perp / p_\parallel - 1)$

Mirror $(\Delta\beta > 1)$ *and firehose* $(\Delta\beta < -2)$ *instabilities*

**Squire et al 2017, PRL*

Pitch-angle scattering, Braginskii-like behavior

δB rapidly decays until below instability thresholds

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Pitch-angle scattering, Braginskii-like behavior

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Landau damping interrupted, wave propagates undamped

Non-propagating (NP) modes:

δB *pressure near balances* $\delta p_{\perp,i} \rightarrow \Delta$

Fast modes:

δB, δn *oscillation generates* Δ

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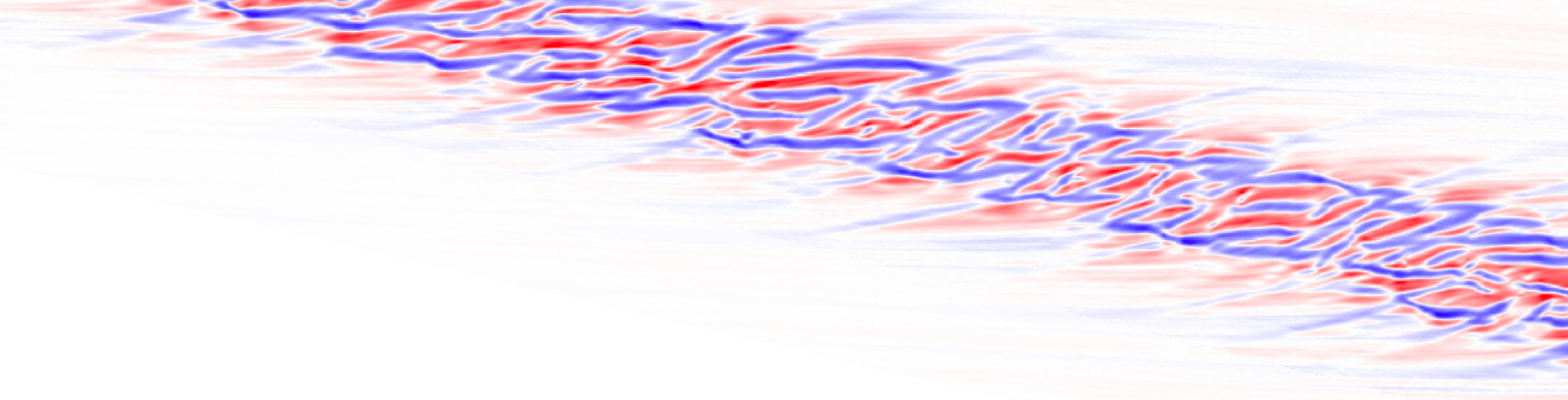
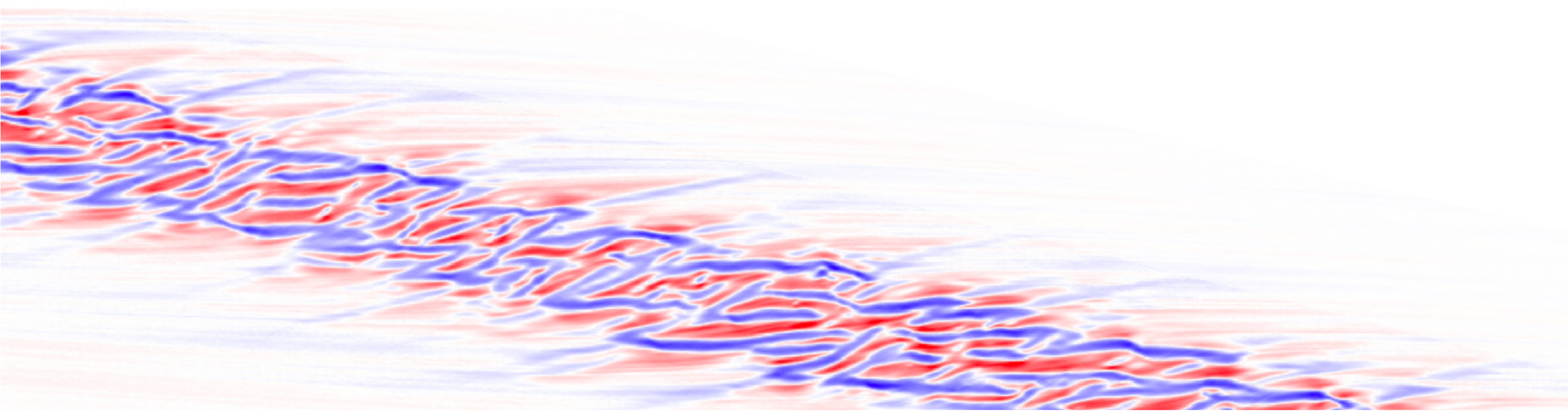
Kinetic microinstabilities, pitch-angle scattering

?



Non-propagating modes



Linear/Nonlinear behavior:

*δB **and** δn perturbations,*

Non-propagating,

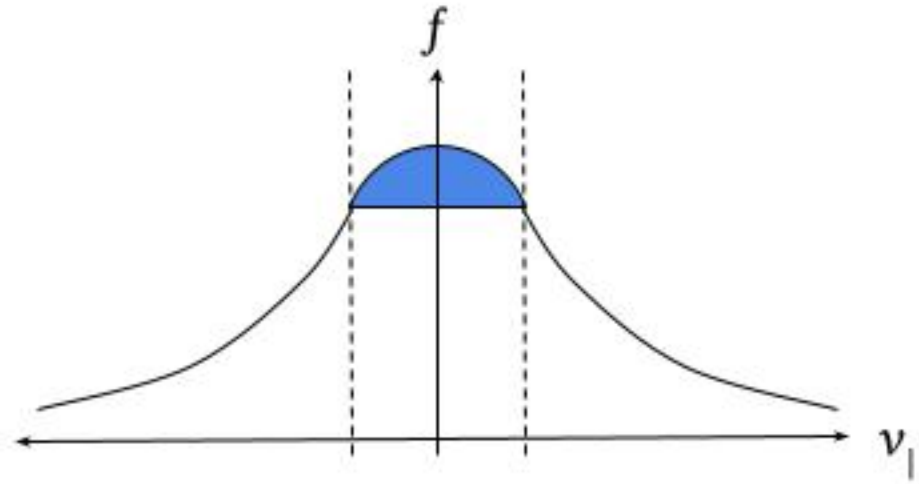
$$\omega \approx - i k_{||} v_{\mathrm{th,i}} / \beta \sqrt{\pi}$$

↓

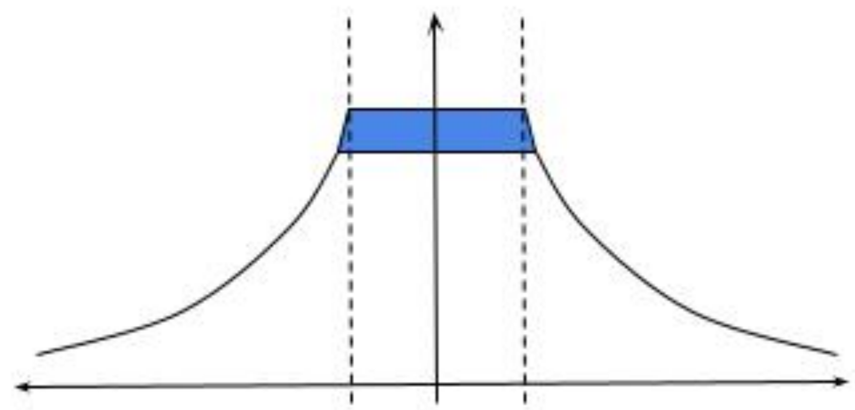
Transit time damping flattens distribution near $\omega / k_{||}$(nonlinear plateau) after $\sim \Omega_b^{-1}$

↓

Plateau eliminates $\partial f / \partial v_{||}$, reduces decay rate to ~ 0



$$\delta t \sim \Omega_b^{-1} \doteq \left(k_{||} v_{\mathrm{th,i}} \sqrt{\frac{\delta B_{||}}{4 B_0}} \right)^{-1}$$



$\Omega_b = \gamma_{NP}$ *when* $\delta B_{||} \sim \beta_{\mathrm{i}}^{-2}$

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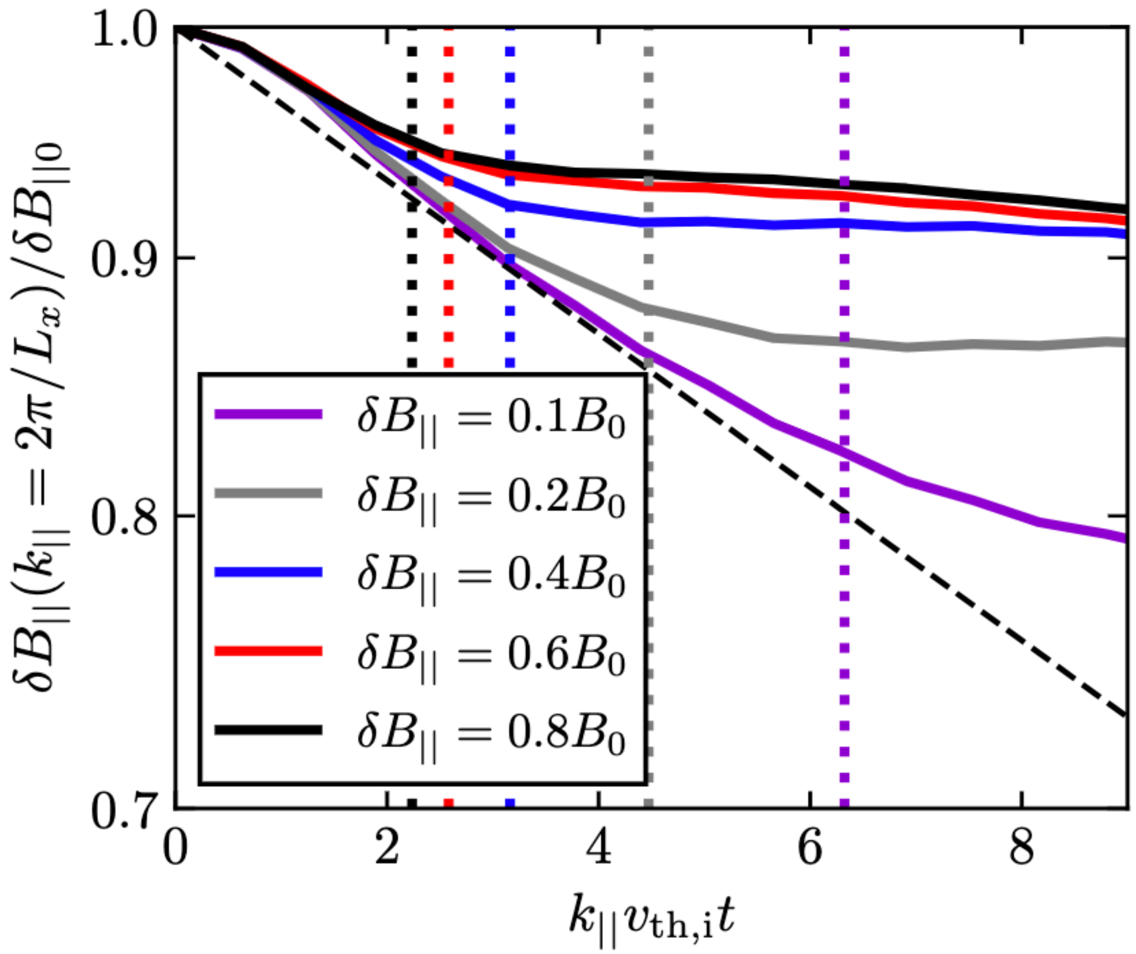
Hybrid-kinetic simulations: Pegasus++

$\beta = 16,\ k_{\perp}/k_{\parallel} = 4,\ \lambda_{\parallel} = 1000\rho_i$



(perturbation at the box wavenumber vs time, with dotted bounce times)

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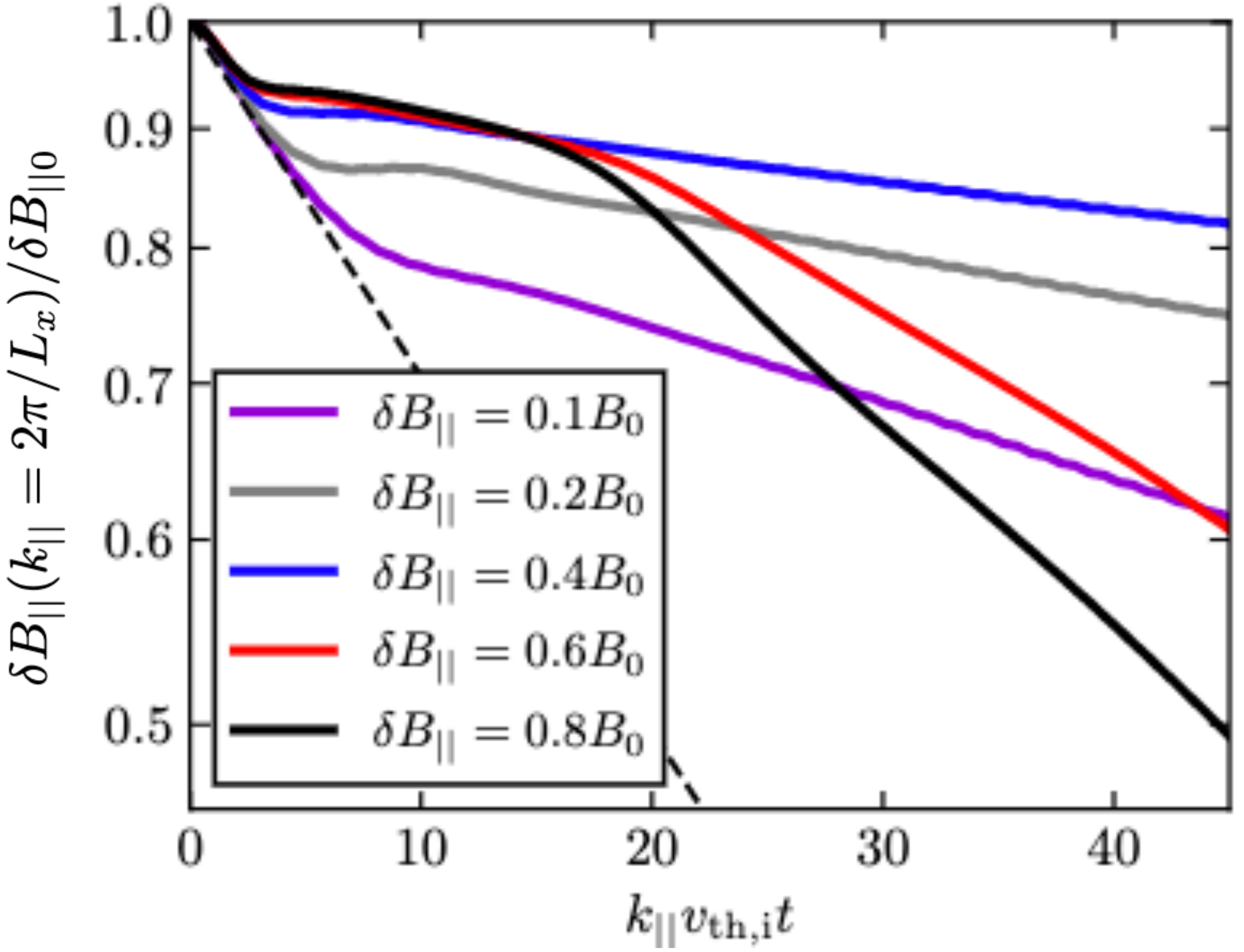

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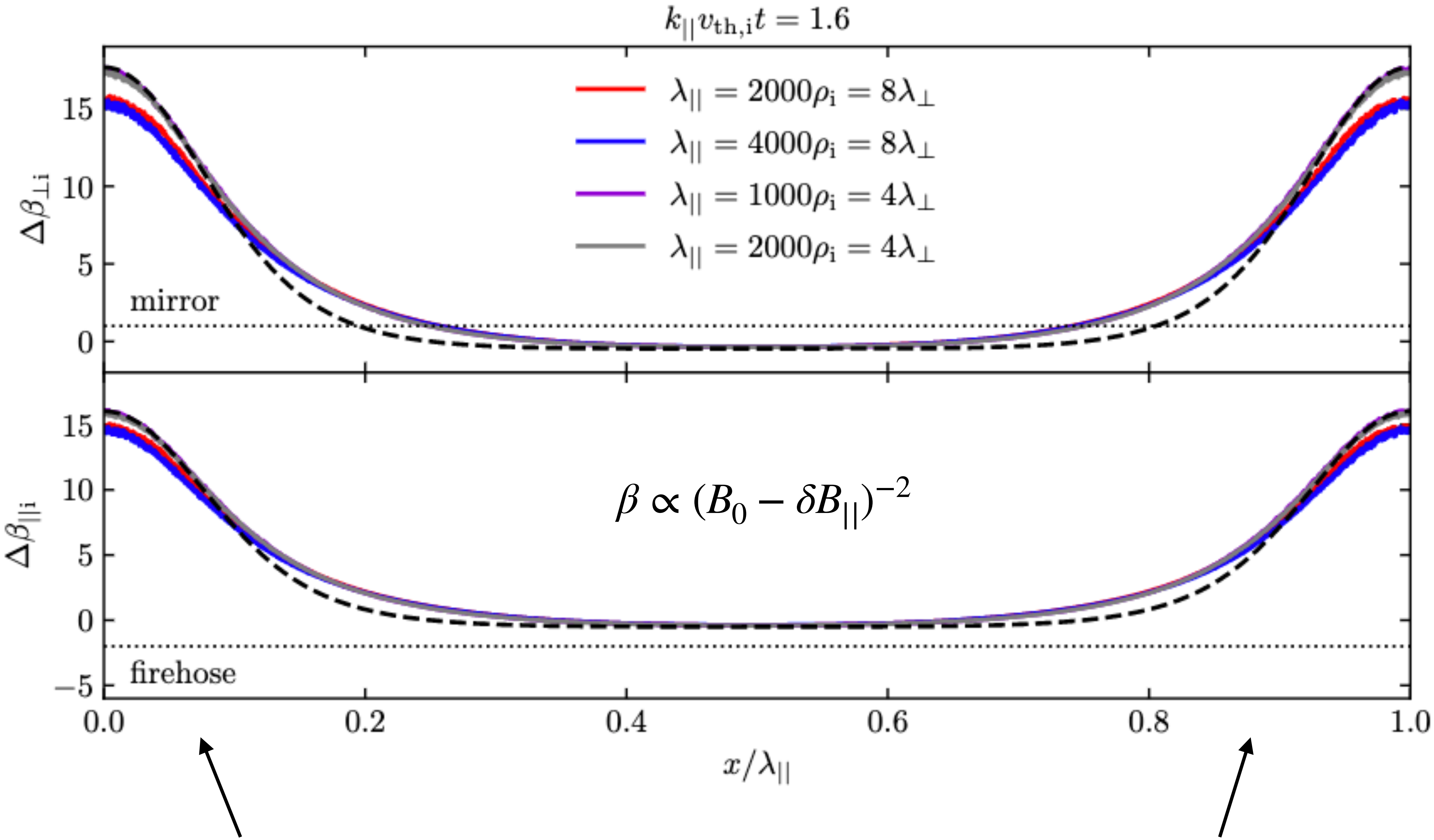
Amplitude dependence?

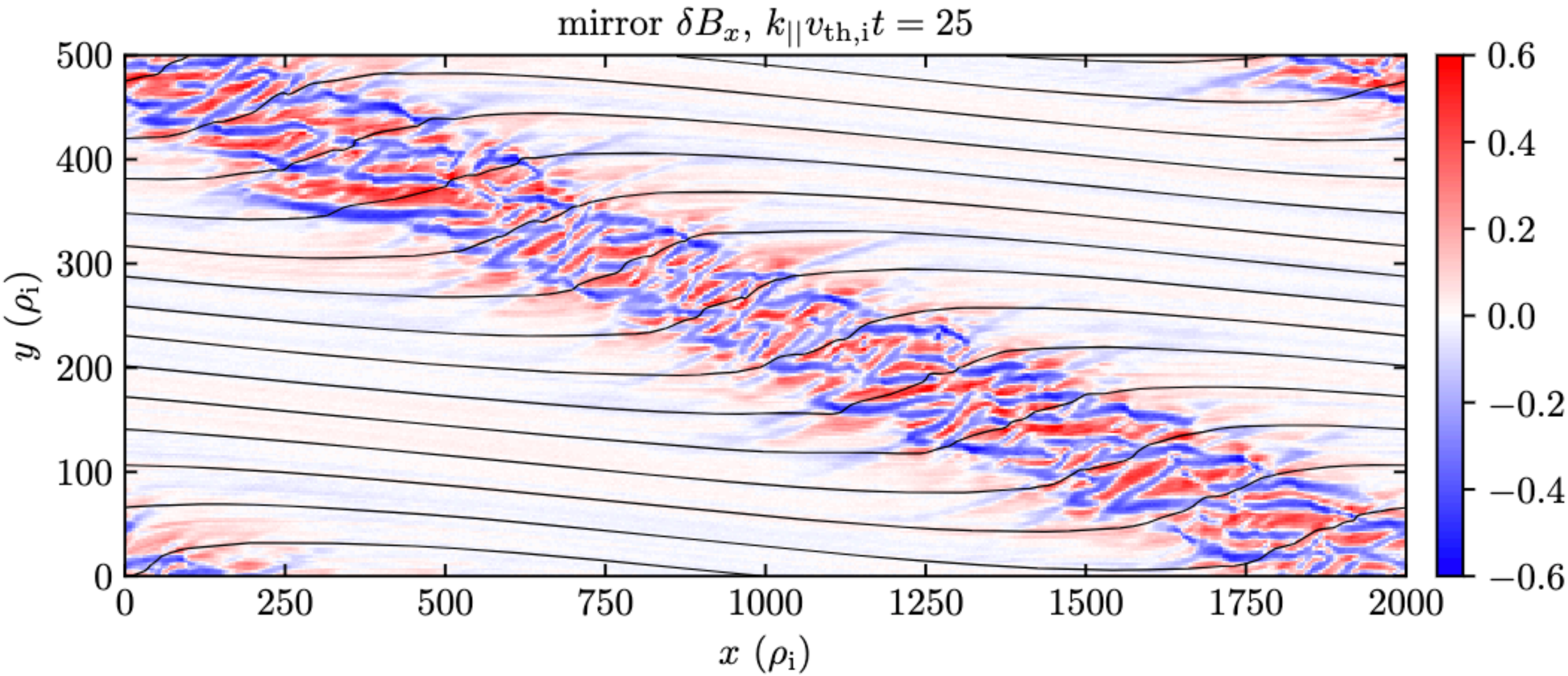
Perpendicular pressure balanced form of polarization:

$$\Delta \propto \delta n \propto -\delta B_{||} \longrightarrow \Delta \propto \delta\beta$$



Mirror unstable regions coincide with resonant particle locations

High positive anisotropy → *Mirror instability* → *Kinked field lines/scattering*



Simulation parameters:

$\beta = 16,\ k_\perp / k_\| = 4,\ \lambda_\| = 2000\rho_i,\ \delta B_{||} = 0.8 B_0$

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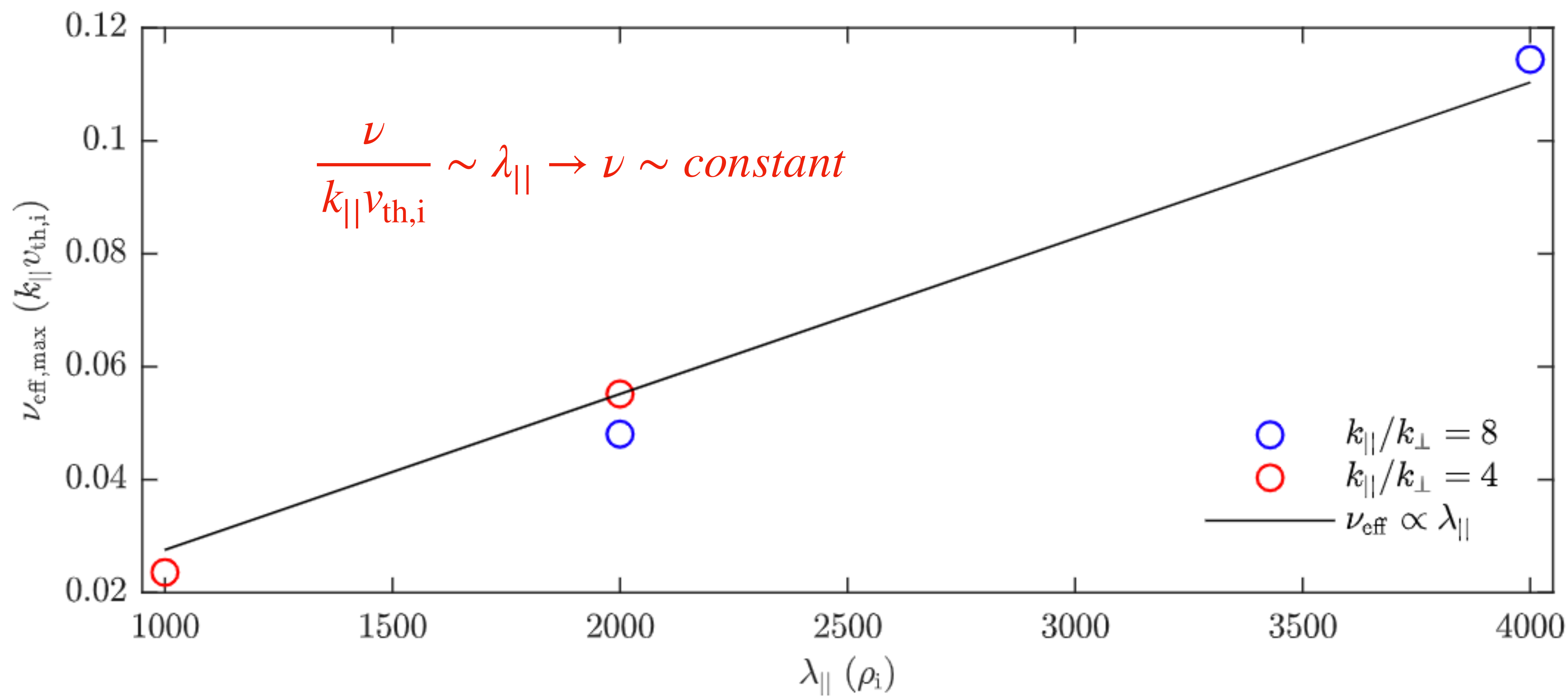

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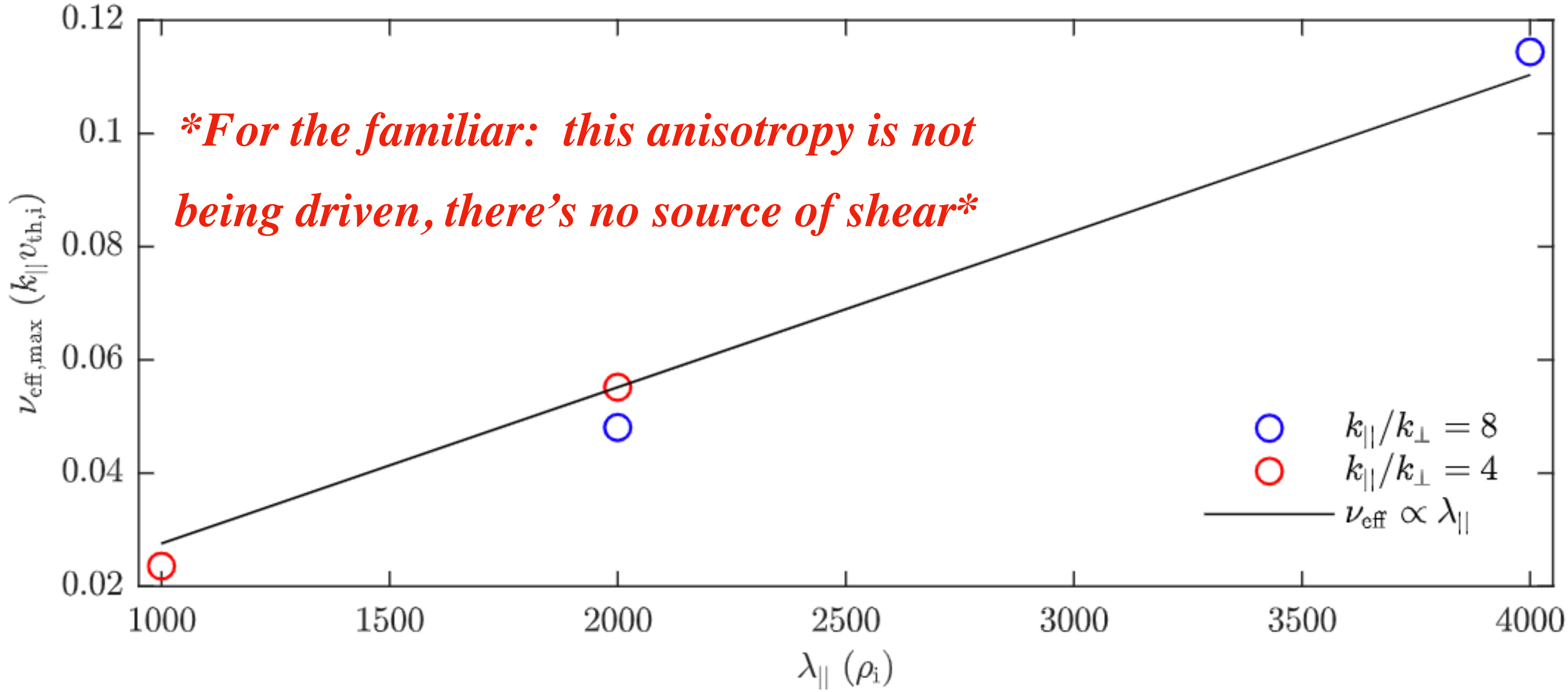

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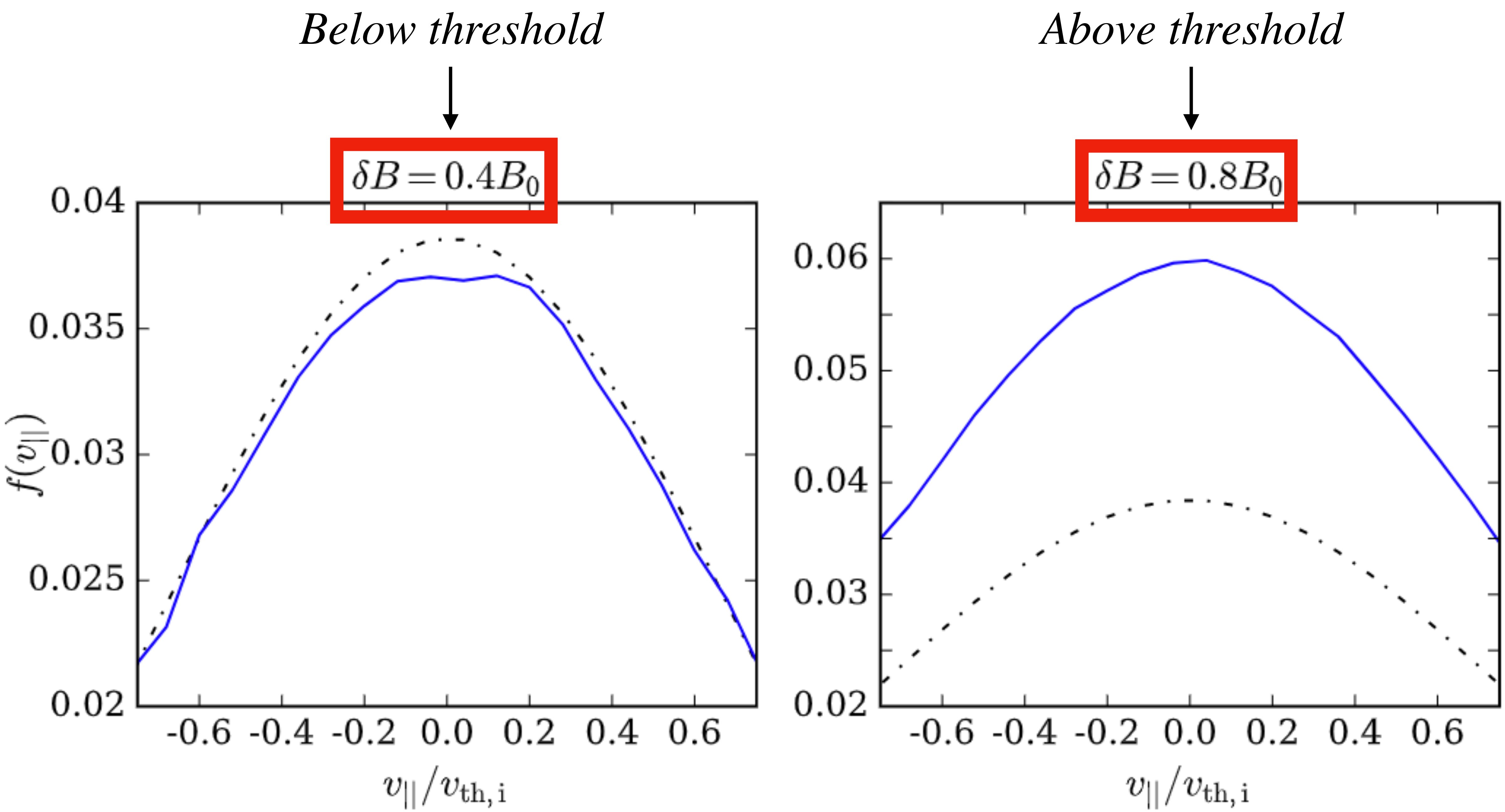

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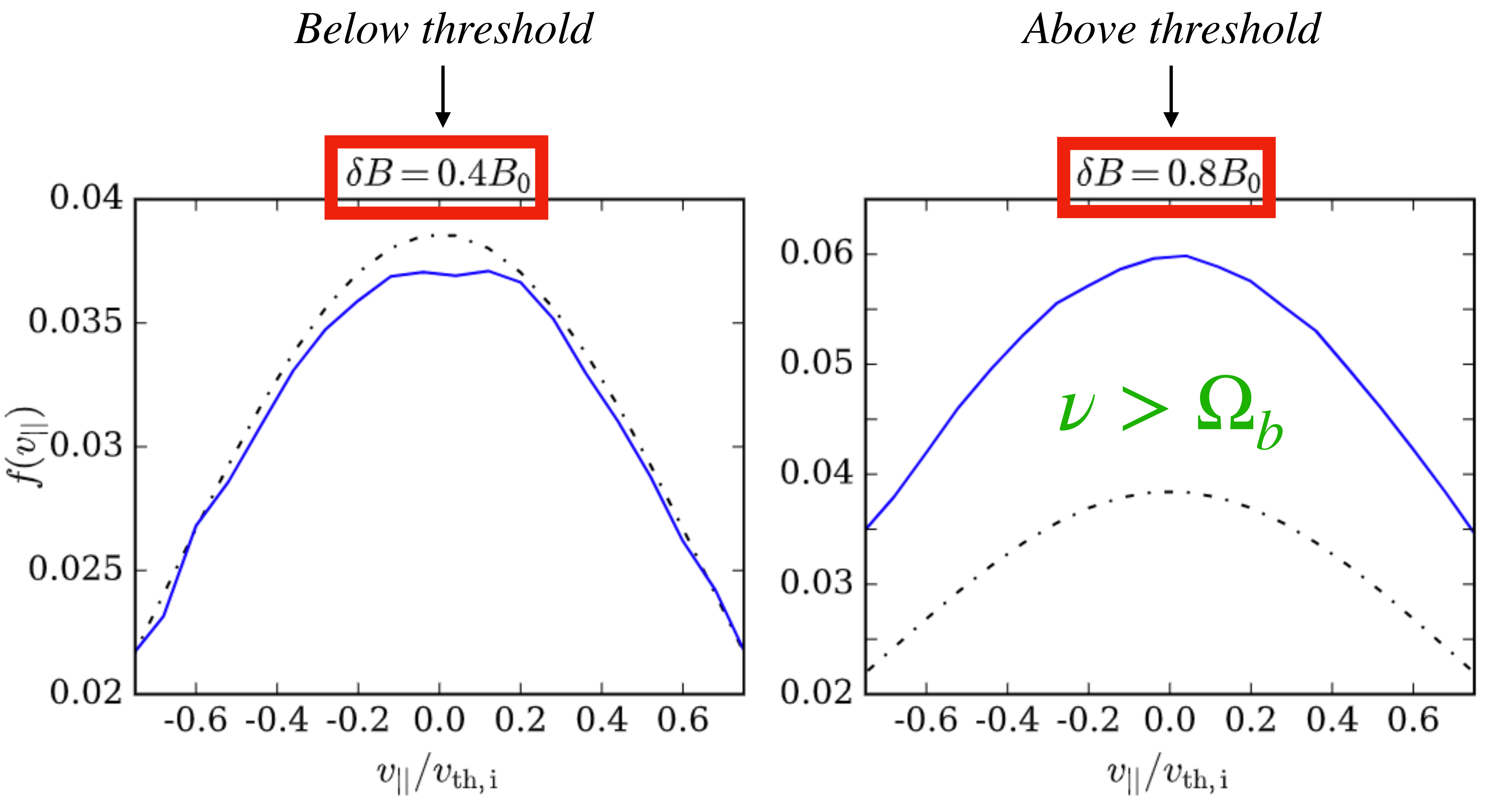
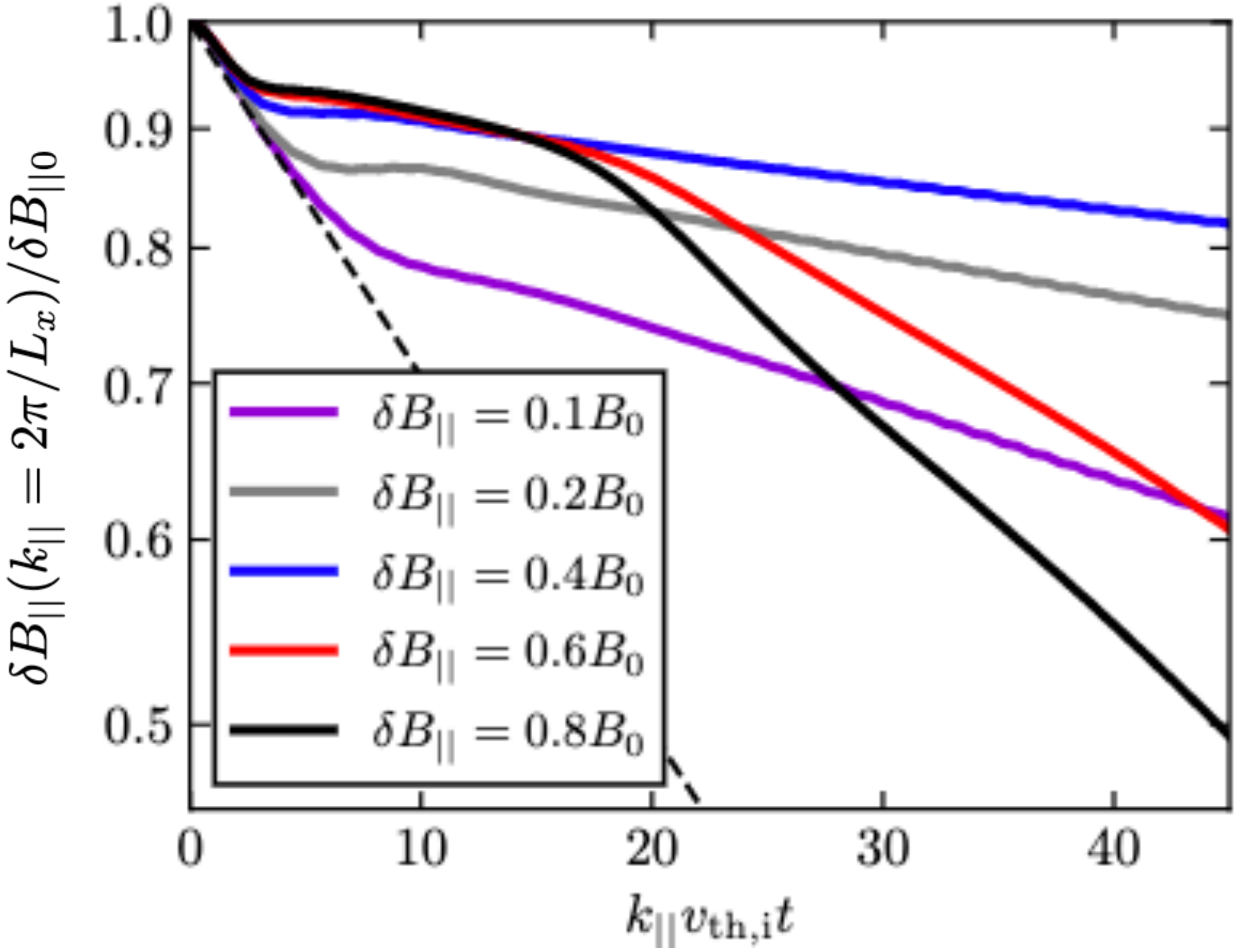


Eroding the plateau faster than it is generated re-establishes linear damping!

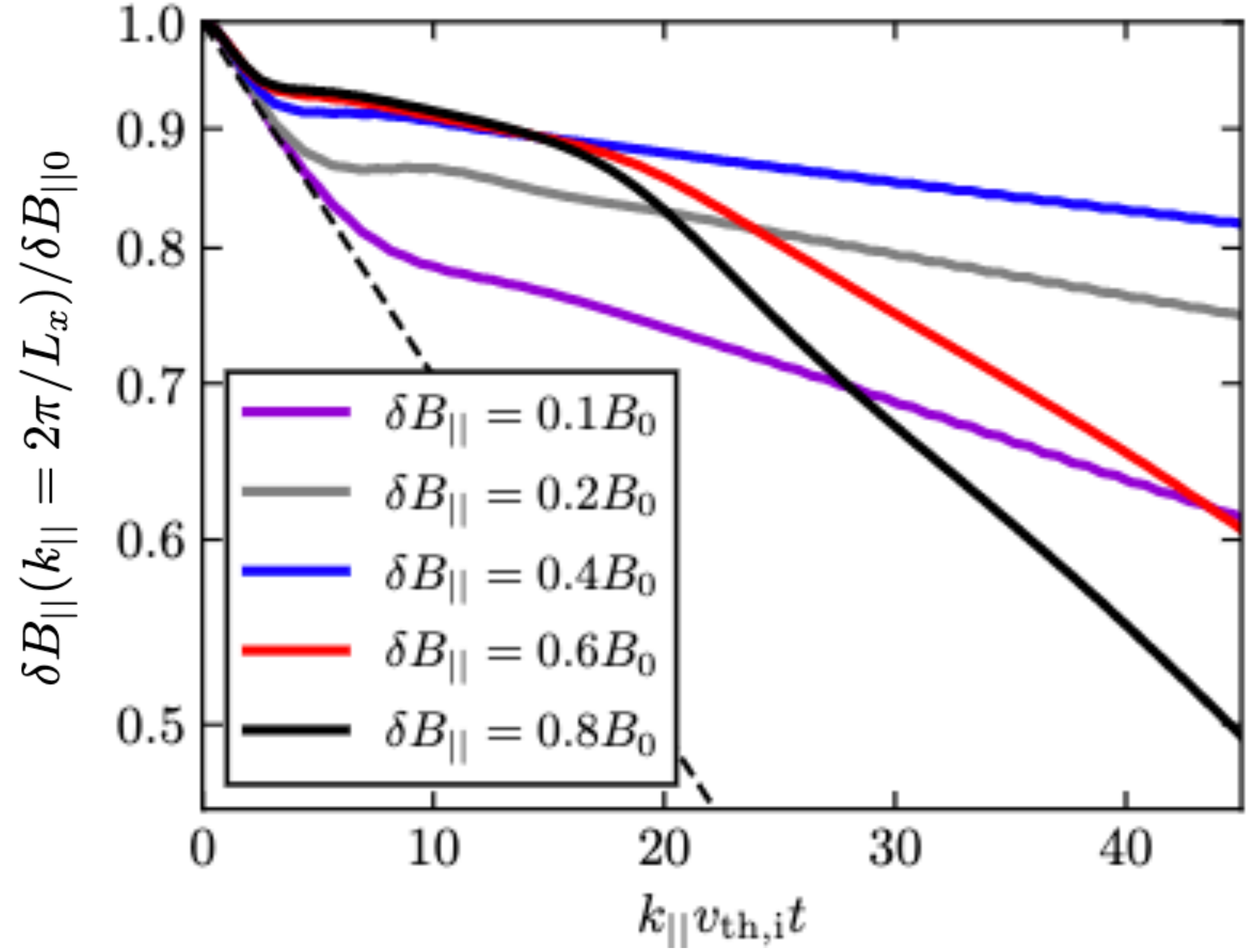
Amplitude dependence!



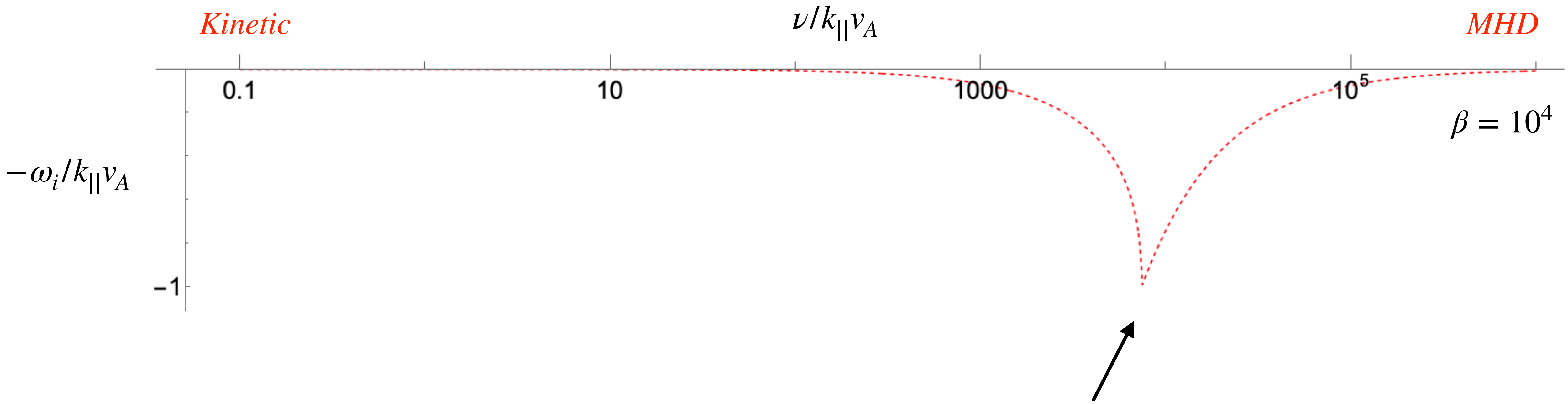
Amplitude dependence!

But linear decay only happens if ν not so large that we shut off damping…

So what ν is too high?



Asymptotically long wavelengths? *CGL dispersion relation with arbitrary ν:*



Collisionless → MHD transition occurs at $\nu \sim \frac{3}{4}\sqrt{\beta}k_{||}v_{\rm th,i} \quad (\beta \gg 1)$

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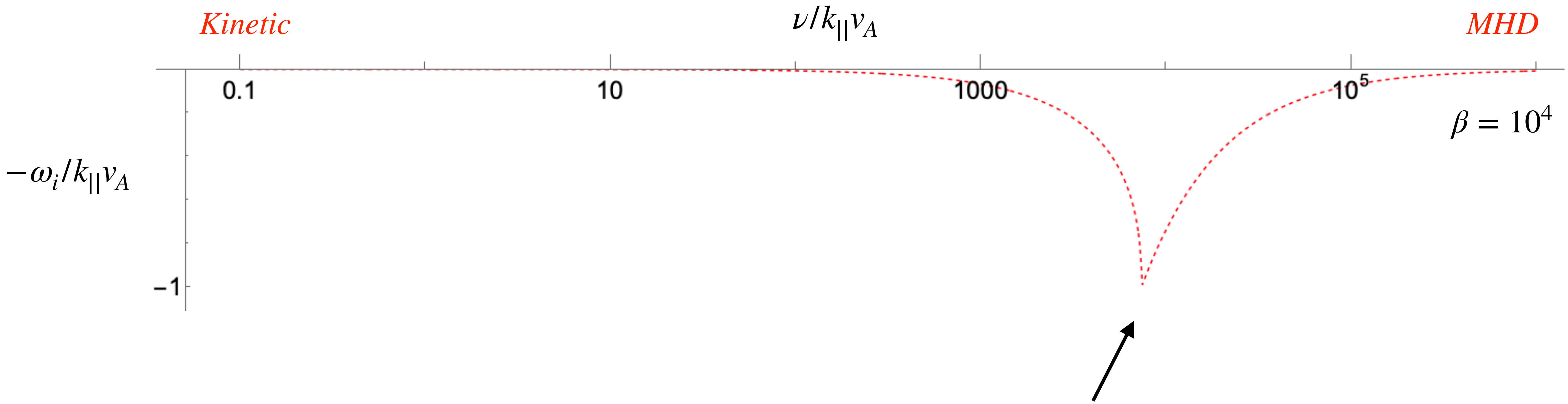

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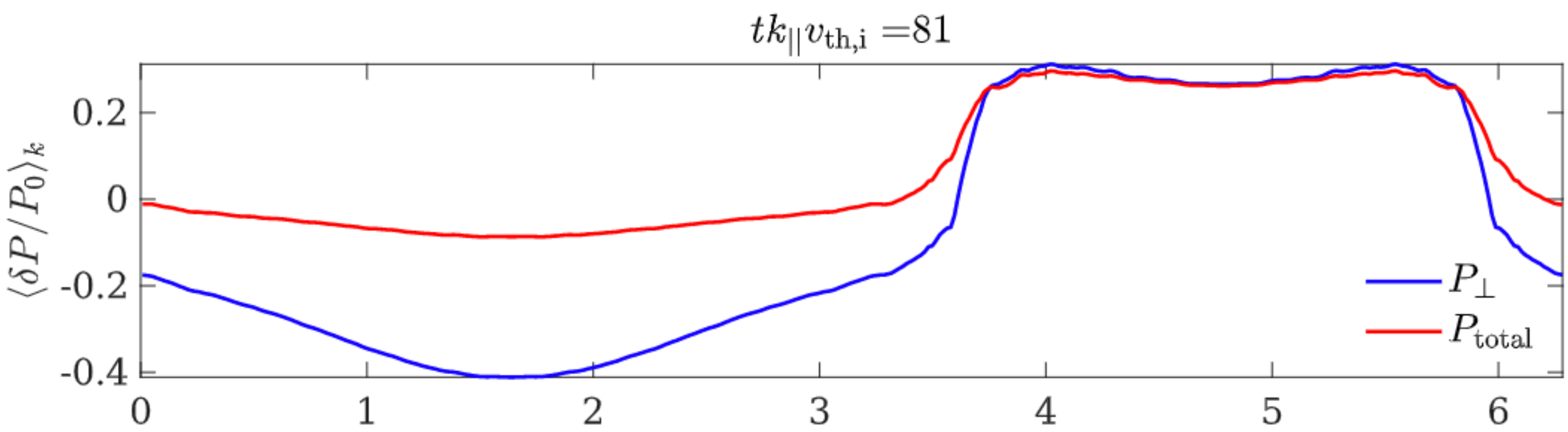
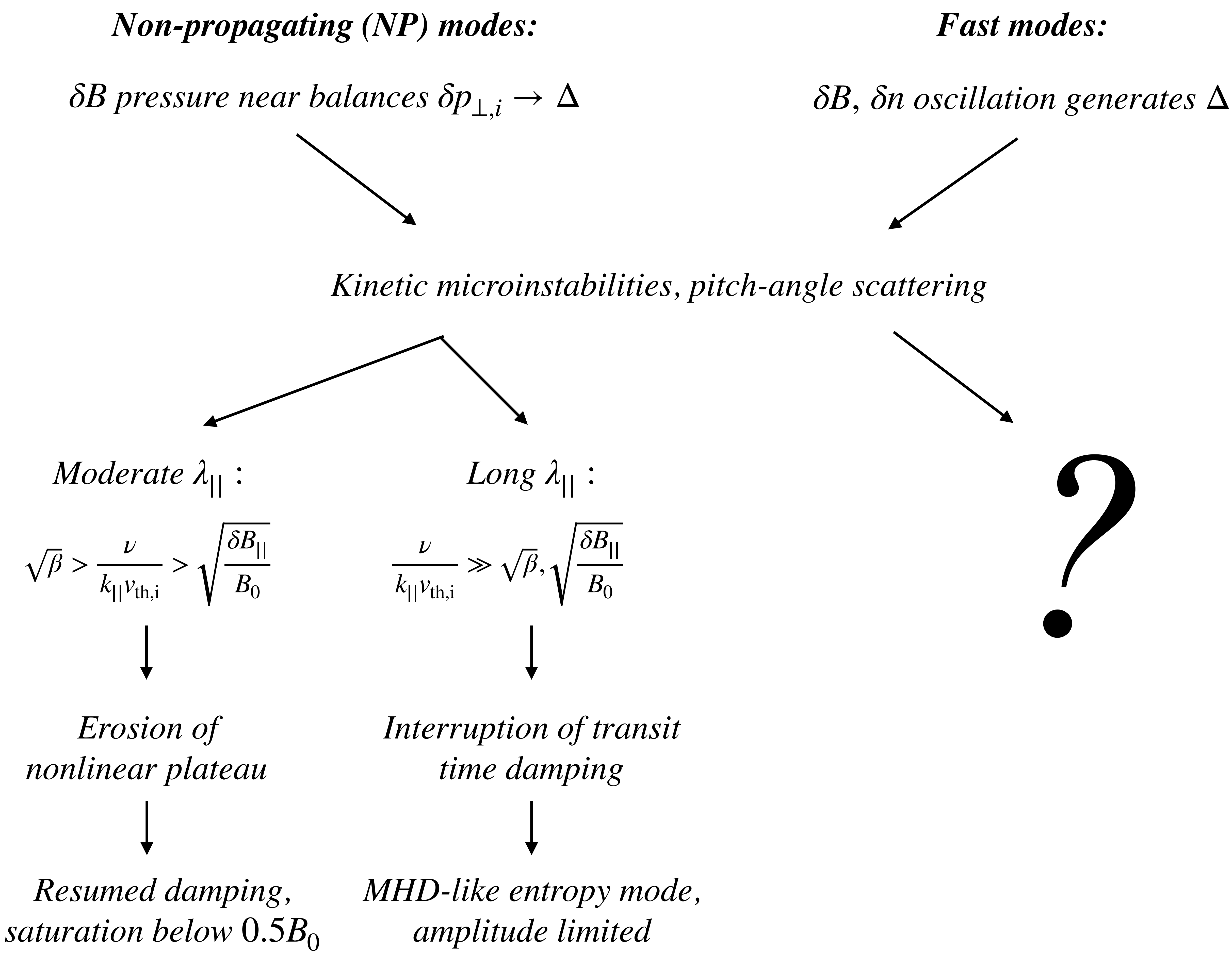


CGL simulation parameters: $\beta = 18$, $k_\perp/k_{||} = 8$, $\delta B_{||} = 0.8B_0$, $\nu = 10^3 k_{||}v_A$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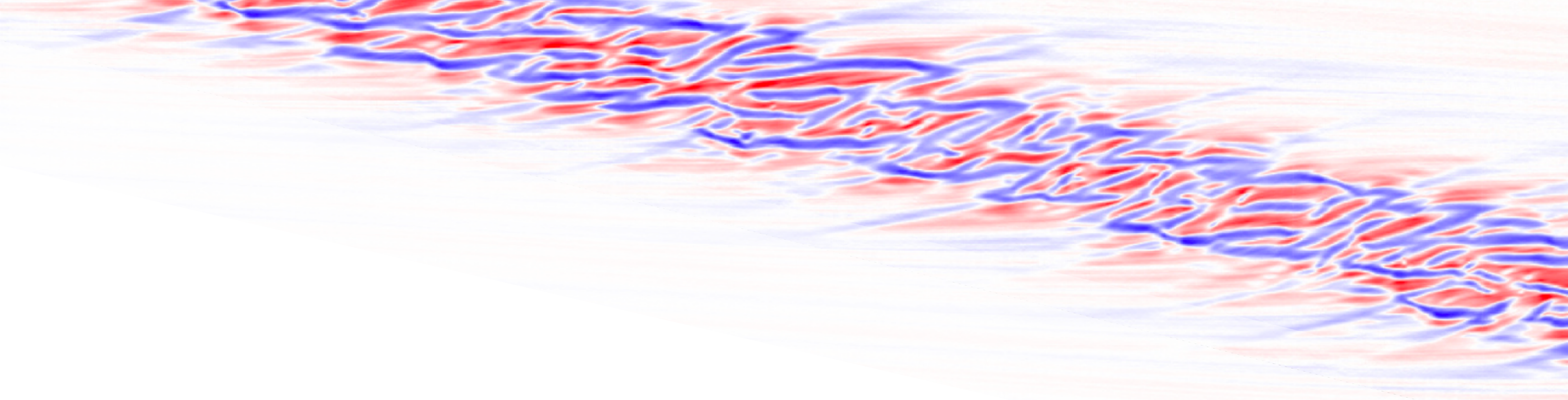
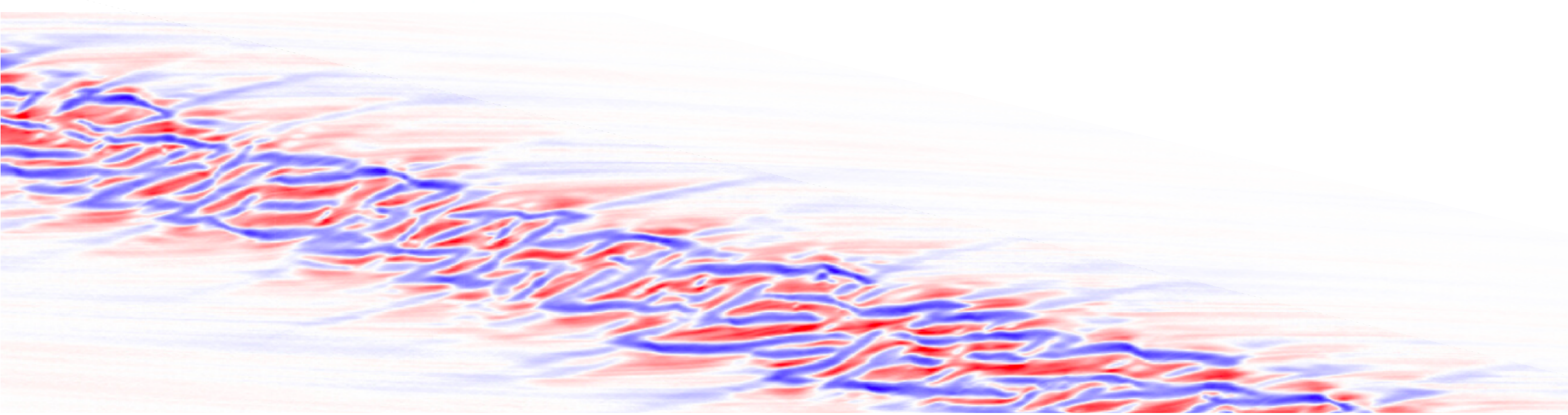
Erosion of nonlinear plateau
Interruption of transit time damping
Resumed damping, saturation below $0.5B_0$
MHD-like entropy mode, amplitude limited



Fast modes



Important fast mode characteristics for $k_{||} = 0$:

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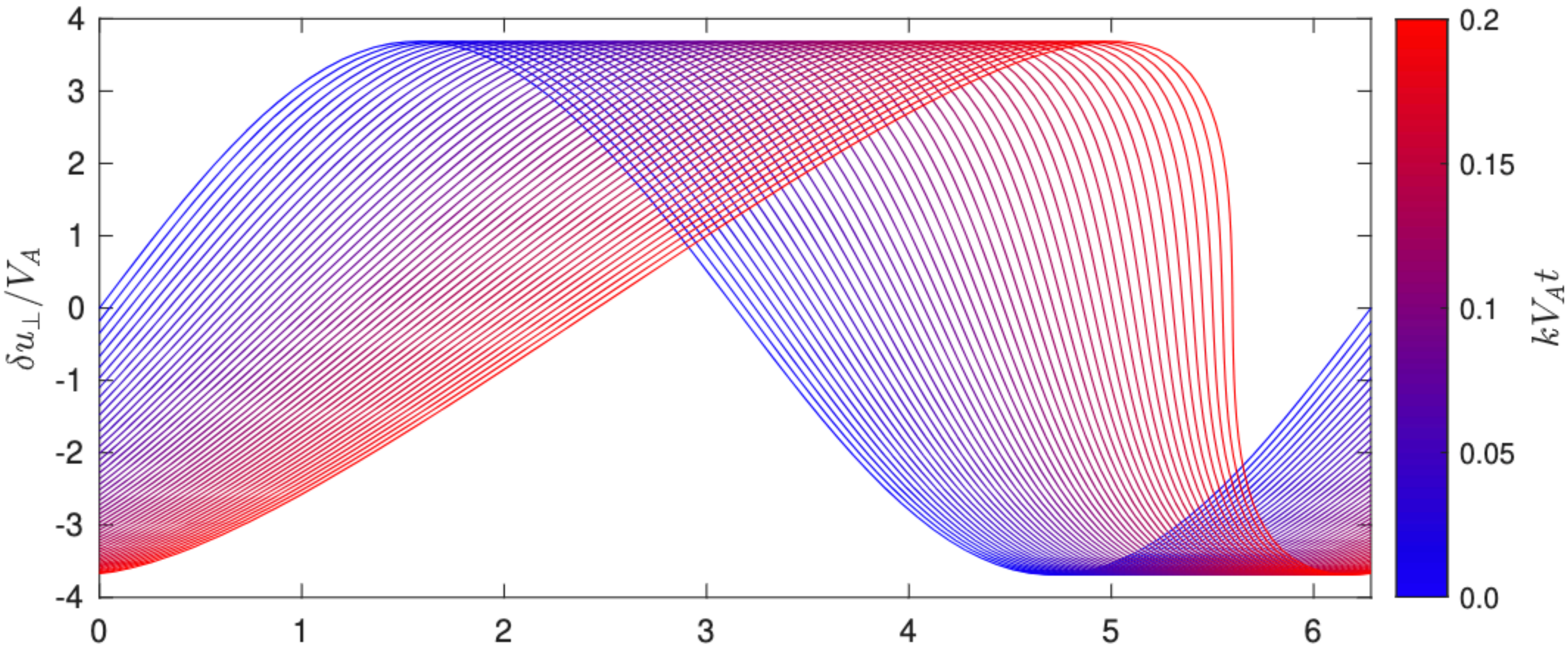
- Wave steepening!



*Shock time can be found through generalized Riemann *approximately* invariants*

Double adiabatic ($p_\perp / nB = const$)

$$t_s^{da} \approx \left[k \frac{\delta B_{||}}{B_0} \left(V_f^{da} + \frac{1+\beta}{2V_f^{da}} \right) \right]^{-1}$$

Single adiabatic ($p\rho^{-\gamma} = const$)

$$t_s^{sa} \approx \left[k \frac{\delta B_{||}}{B_0} \left(V_f^{sa} + \frac{1+(\gamma^2-\gamma)\beta/2}{2V_f^{sa}} \right) \right]^{-1}$$

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$$t_s^{sa} \approx \frac{3B_0}{4\delta B_{||}k\sqrt{5\beta_i/6}}$$

High β:

Double adiabatic model's direct connection between $p_\perp$ and B facilitates decrease in shock time by 23% from MHD

$$V_f^{da} = v_A\sqrt{\beta\left(1+\frac{T_e}{2T_i}\right)+1}$$

$$V_f^{sa} = v_A\sqrt{\beta\frac{\gamma}{2}+1}$$

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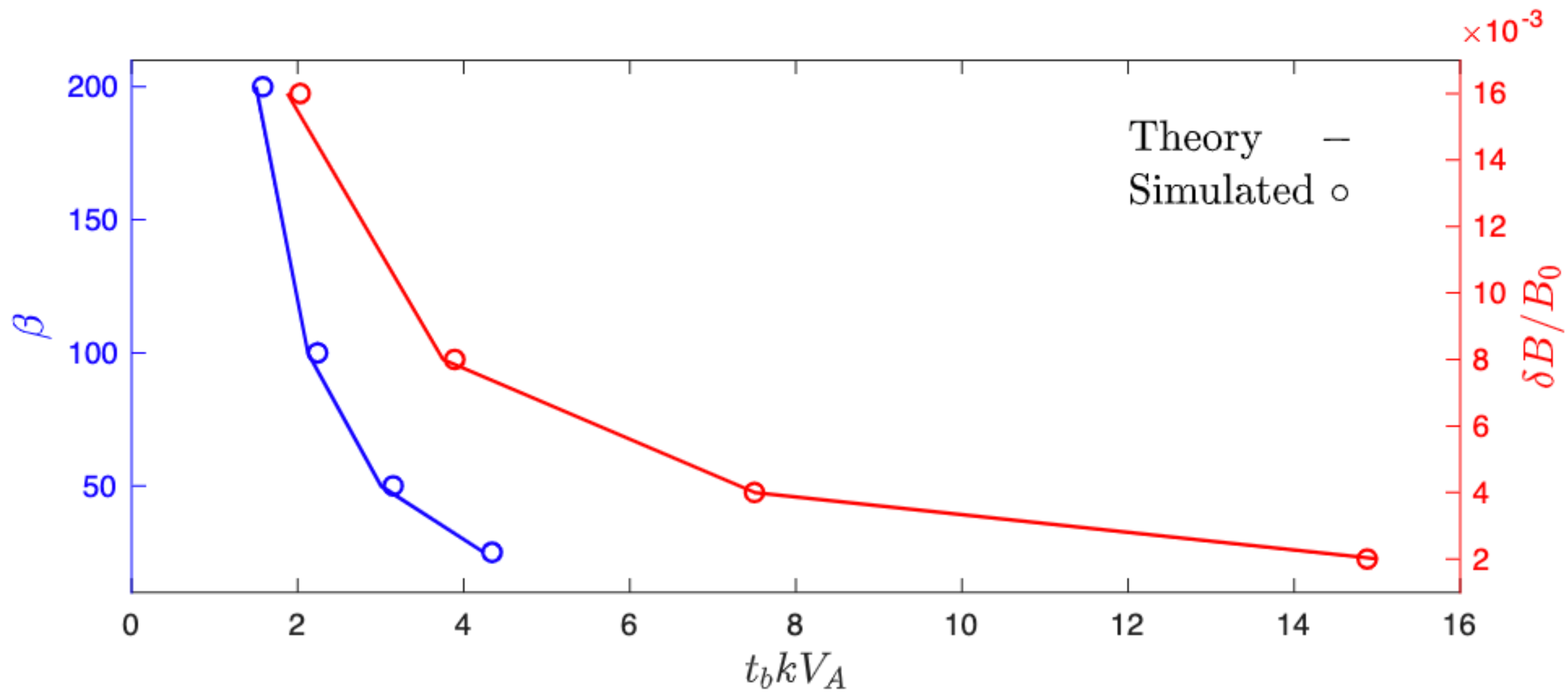

Perpendicular propagation is entirely described within double adiabatic theory:

$$\frac{\delta p_\perp}{p_0} = 2\frac{\delta B_{||}}{B_0} \qquad \textit{and} \qquad \frac{\delta p_{||}}{p_0} = \frac{\delta n}{n_0} = \frac{\delta B_{||}}{B_0}$$

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marginal firehose stability

$$\Delta\beta = -2$$

$$\boxed{\frac{\delta B_{||}}{B_0} > \frac{2}{\beta}}$$

Much smaller threshold for microinstabilities than NP, linear at high β!

Compression-generated collisions:

$$\frac{dp_\perp}{dt} = p_\perp \frac{d}{dt}\ln(Bn) - \nu(p_\perp - p)$$

$$\frac{dp_\parallel}{dt} = p_\parallel \frac{d}{dt}\ln(n^3/B^2) - \nu(p_\parallel - p)$$

$$\longrightarrow \quad \frac{d\Delta p}{dt} = p\frac{d}{dt}3\ln(B/n^{2/3}) - \nu\Delta p$$

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$\longrightarrow$

Braginskii: $\mathcal{O}(\epsilon^2)$ $\mathcal{O}(\epsilon)$ $\mathcal{O}(\epsilon)$

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(no shear, just compression)

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(no shear, just compression)

Balance anisotropy production with scattering, knowing marginal stability is $\Delta\beta \sim \mathcal{O}(1)$

$$\nabla\cdot\vec{u} \sim kV_f\frac{\delta B_{||}}{B_0} \longrightarrow kV_f\frac{\delta B_{||}}{B_0} \sim \nu\Delta \longrightarrow \boxed{\nu \sim \beta kV_f\frac{\delta B_{||}}{B_0}}$$

For $\beta = 25$, $\delta B_{||} = 0.1 B_0$, $T_e/T_i = 1$, *expect scattering rate of*
$\nu \sim \beta k V_f \delta B_{||}/B_0 \sim 15.5 k v_A$

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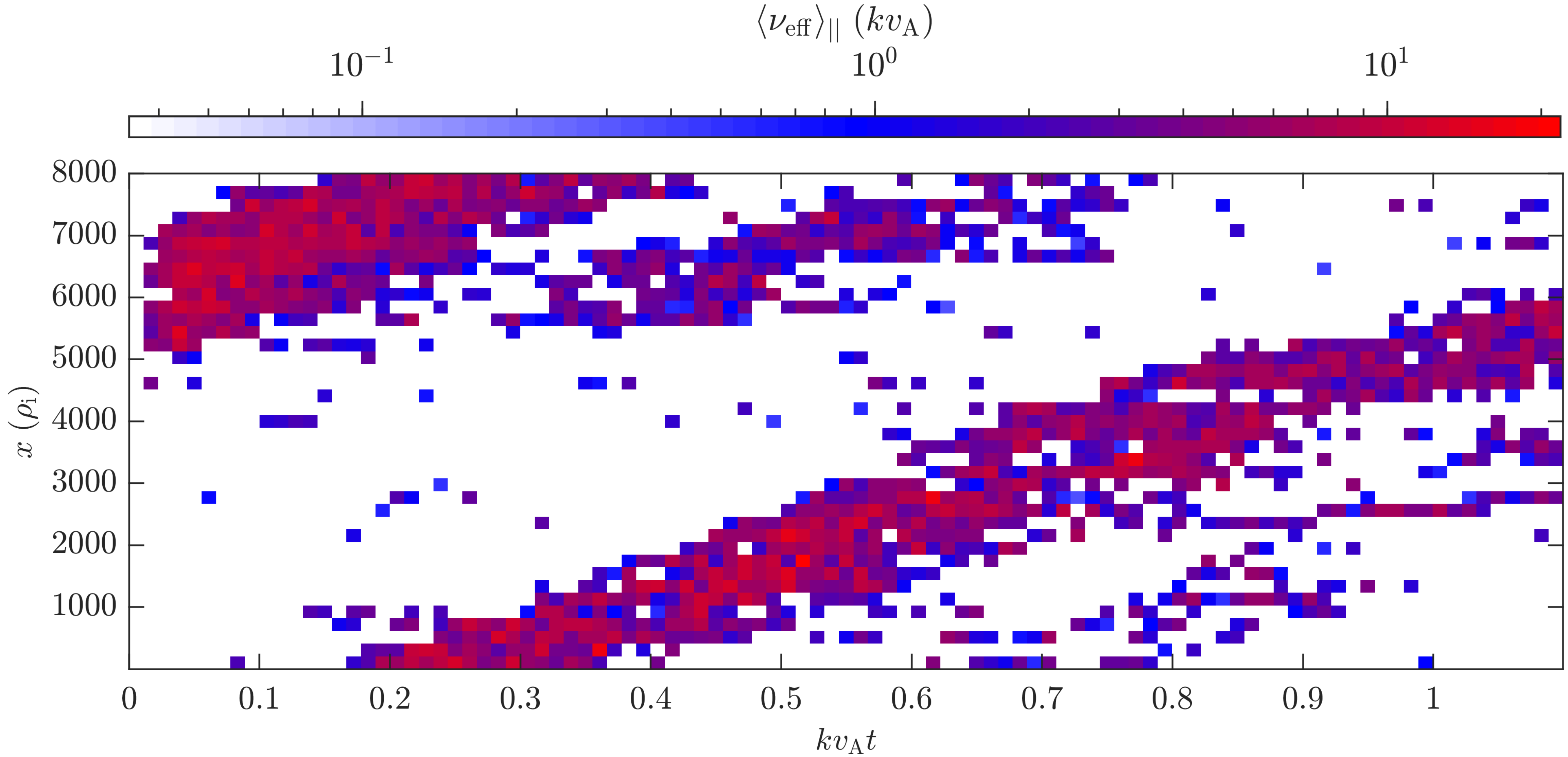

At sufficiently long wavelengths, both mirrors and firehoses generate this collision frequency from mode compression.

For arbitrary pitch-angle scattering:

$$\omega^3 - i\nu\omega^2 - \omega k^2 V_{f,da}^2 + i\nu k^2 V_{f,sa}^2 = 0$$

→*Transition from collisionless to MHD occurs where* $\nu \sim \omega \sim k v_{\mathrm{th,i}}$

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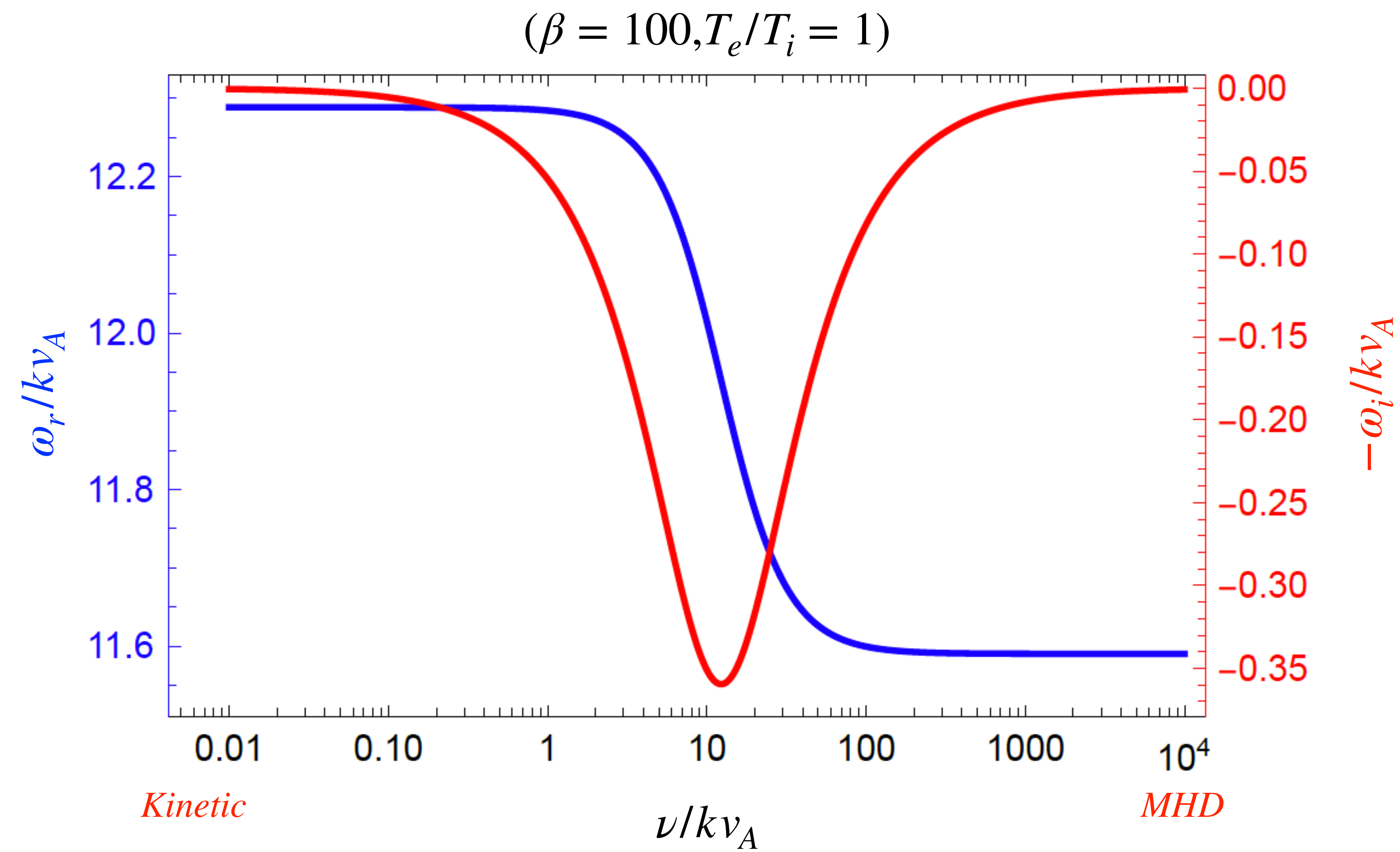
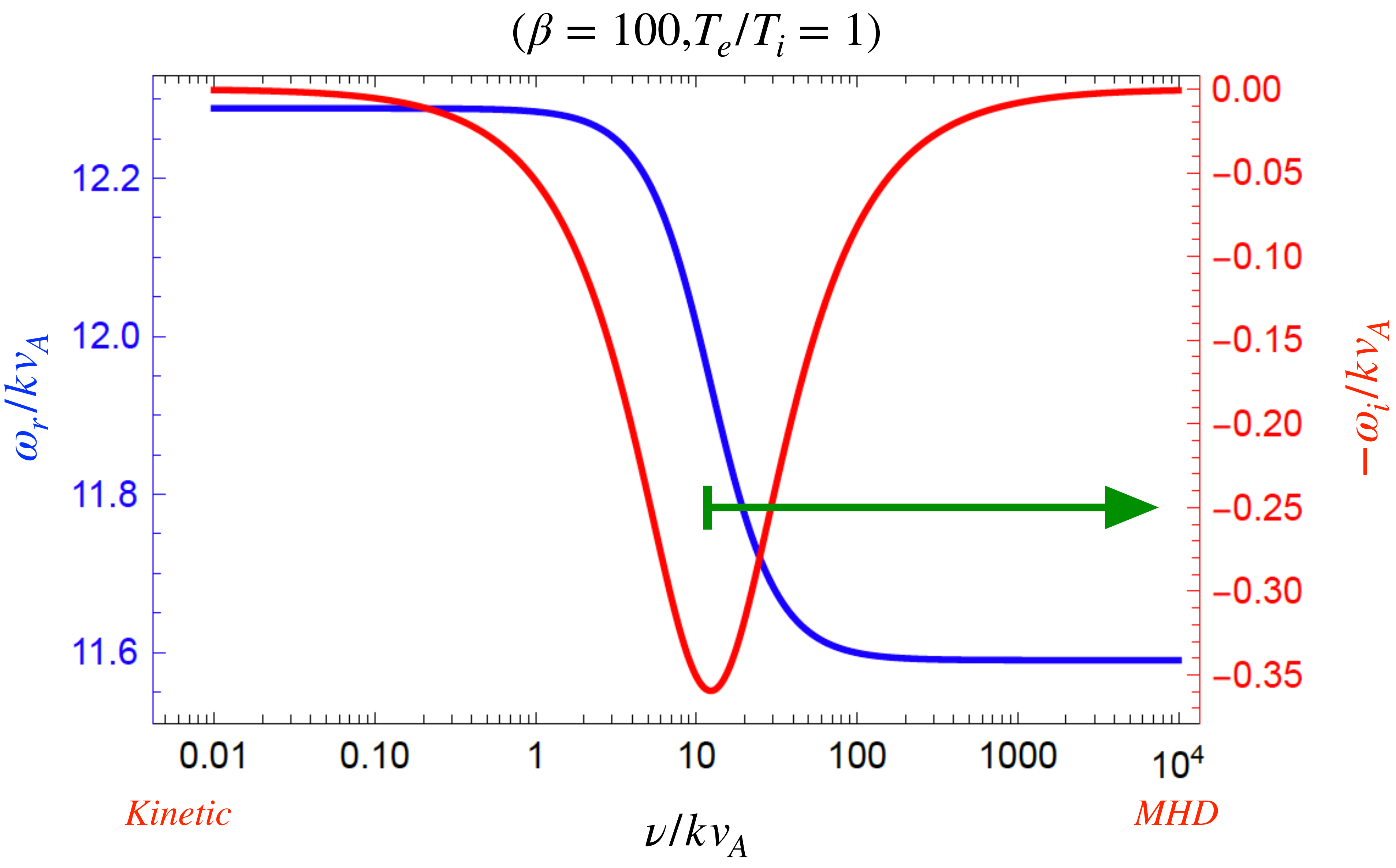


$$\nu \sim \beta k V_f \frac{\delta B_{||}}{B_0} \sim k v_{\rm th,i} \quad \textit{near threshold}$$



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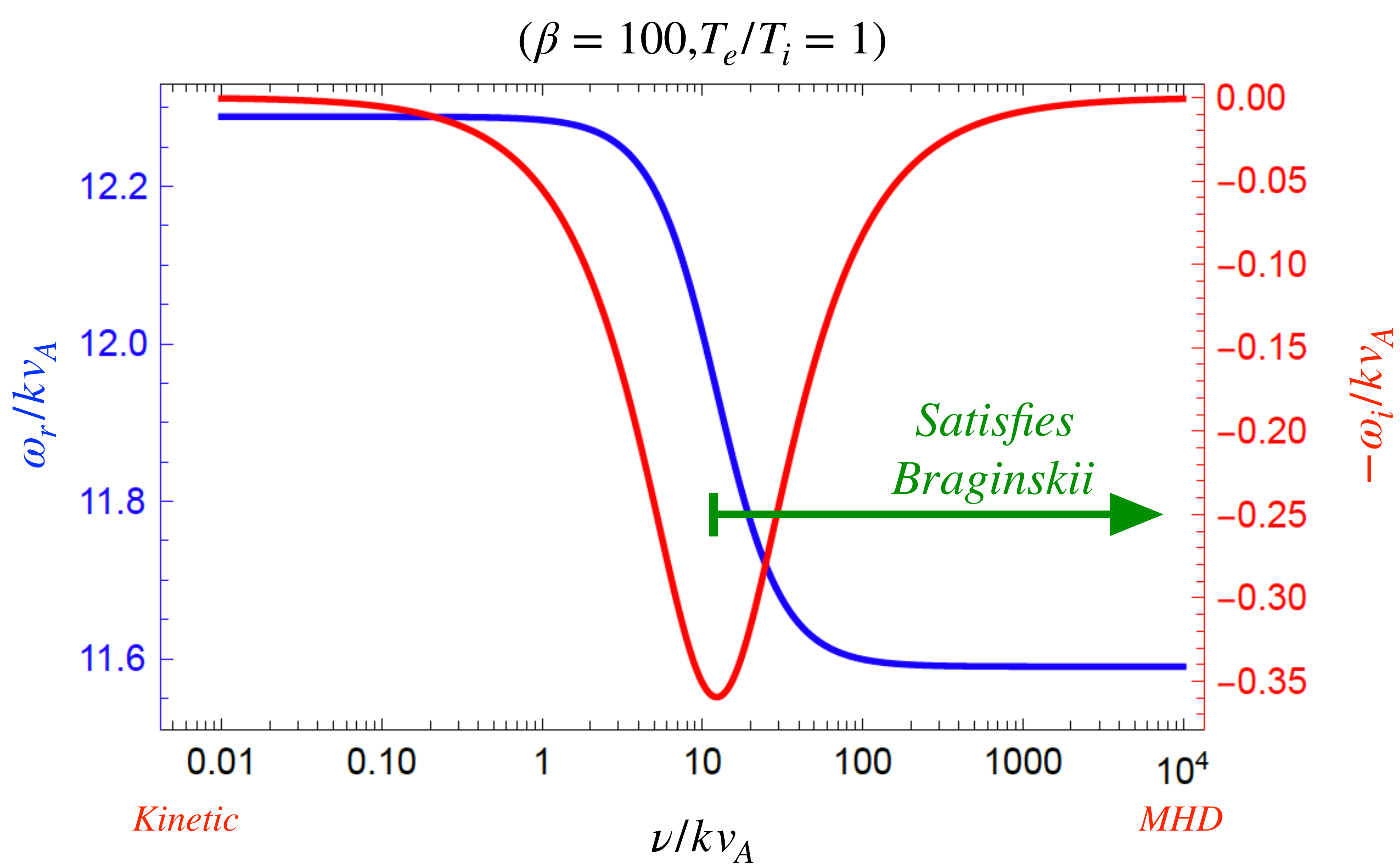
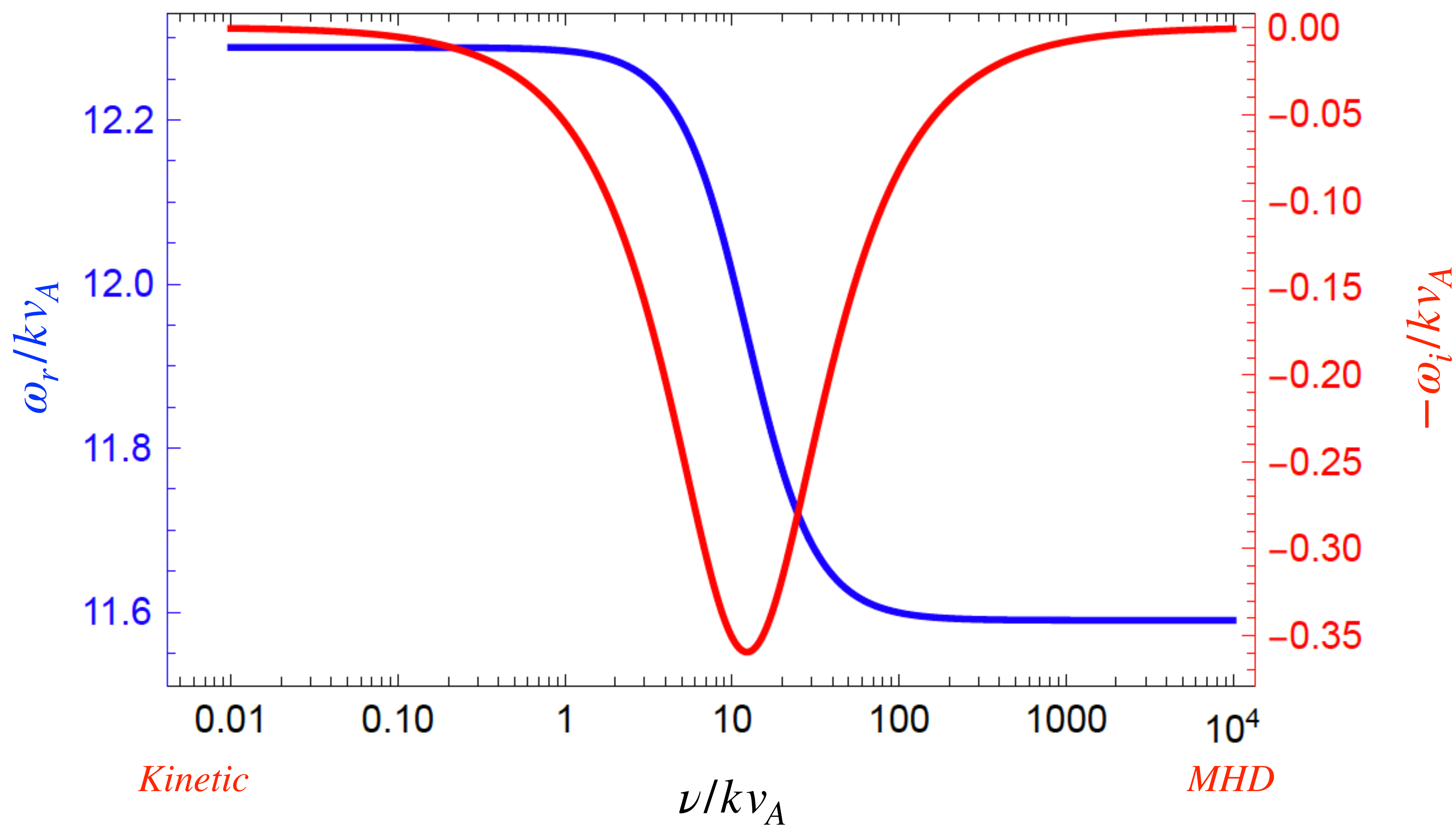


$$\nu \sim \beta k V_f \frac{\delta B_{||}}{B_0}$$

$\delta B_{||,0} \sim \beta^{-1}$ near threshold yields decay until $\delta B_{||} \leq 3B_0/2\beta$

$\delta B_{||,0} \gg \beta^{-1}$ generates strong scattering, MHD-like weaker decay

$(\beta = 100, T_e/T_i = 1)$



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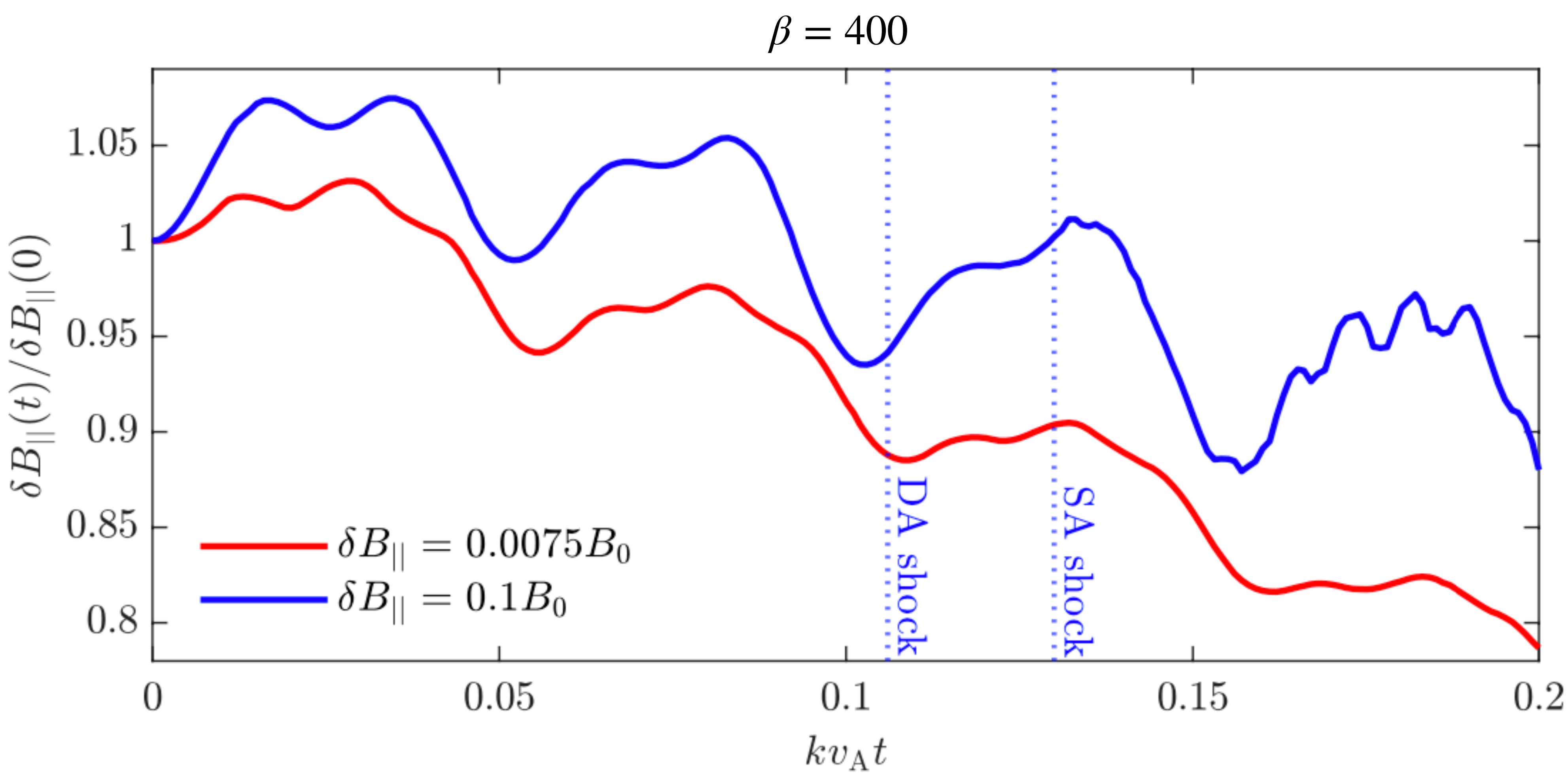

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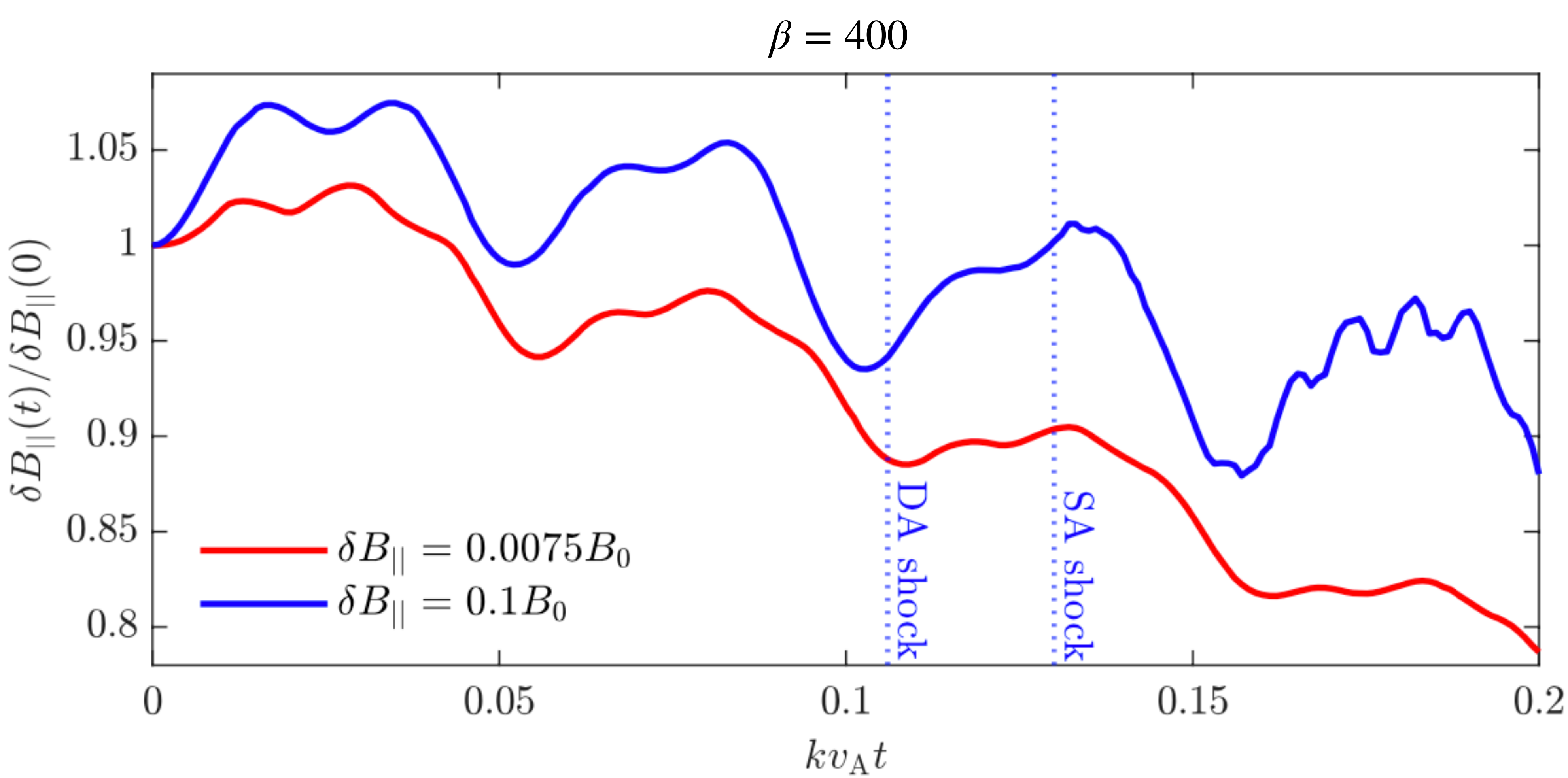
Also shocks in SA not DA time!



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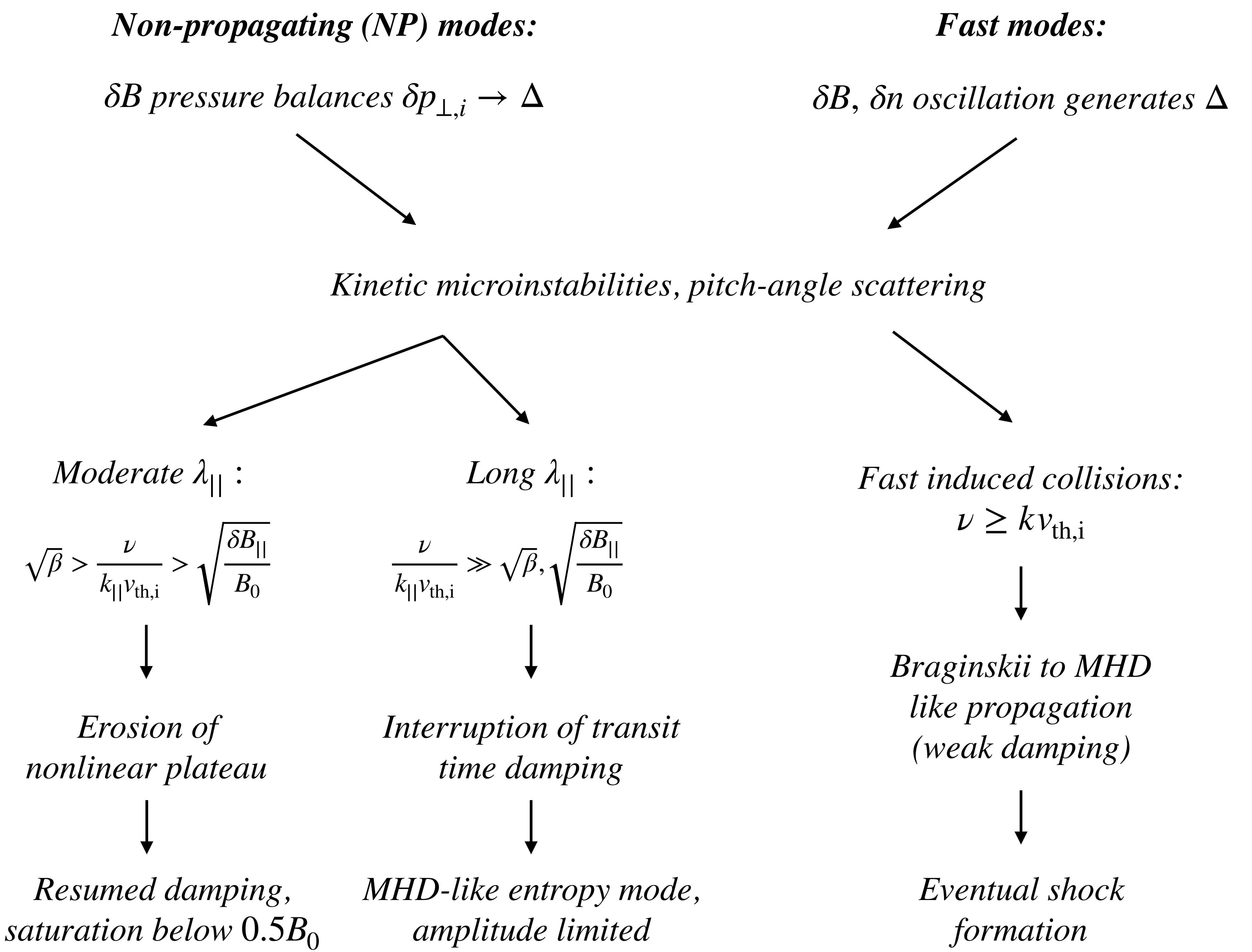
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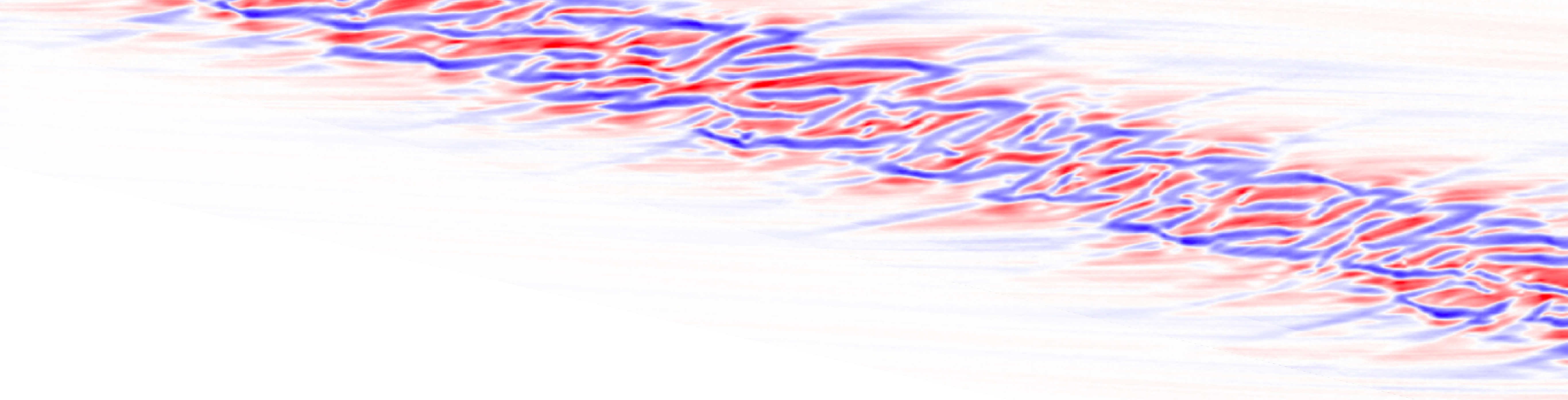
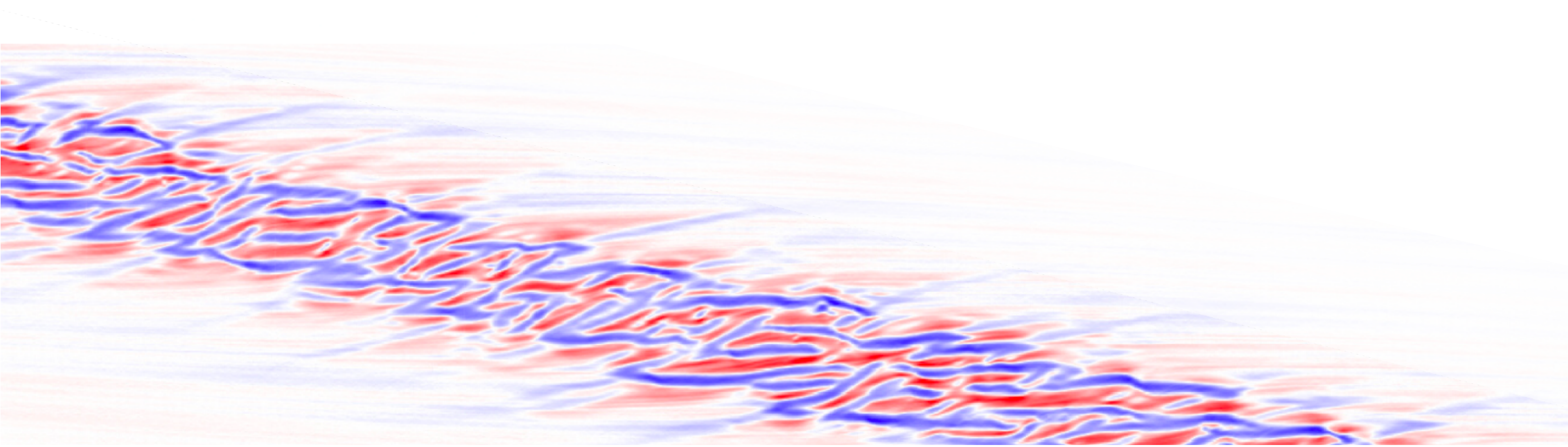
Interruption of transit time damping
MHD-like entropy mode, amplitude limited
Fast induced collisions:
$\nu \geq k v_{\mathrm{th,i}}$
Braginskii to MHD like propagation (weak damping)
Eventual shock formation



Bonus-ish: Oblique acoustic modes



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$\gamma \sim \omega \sim k_{||} v_{\mathrm{th,i}}$

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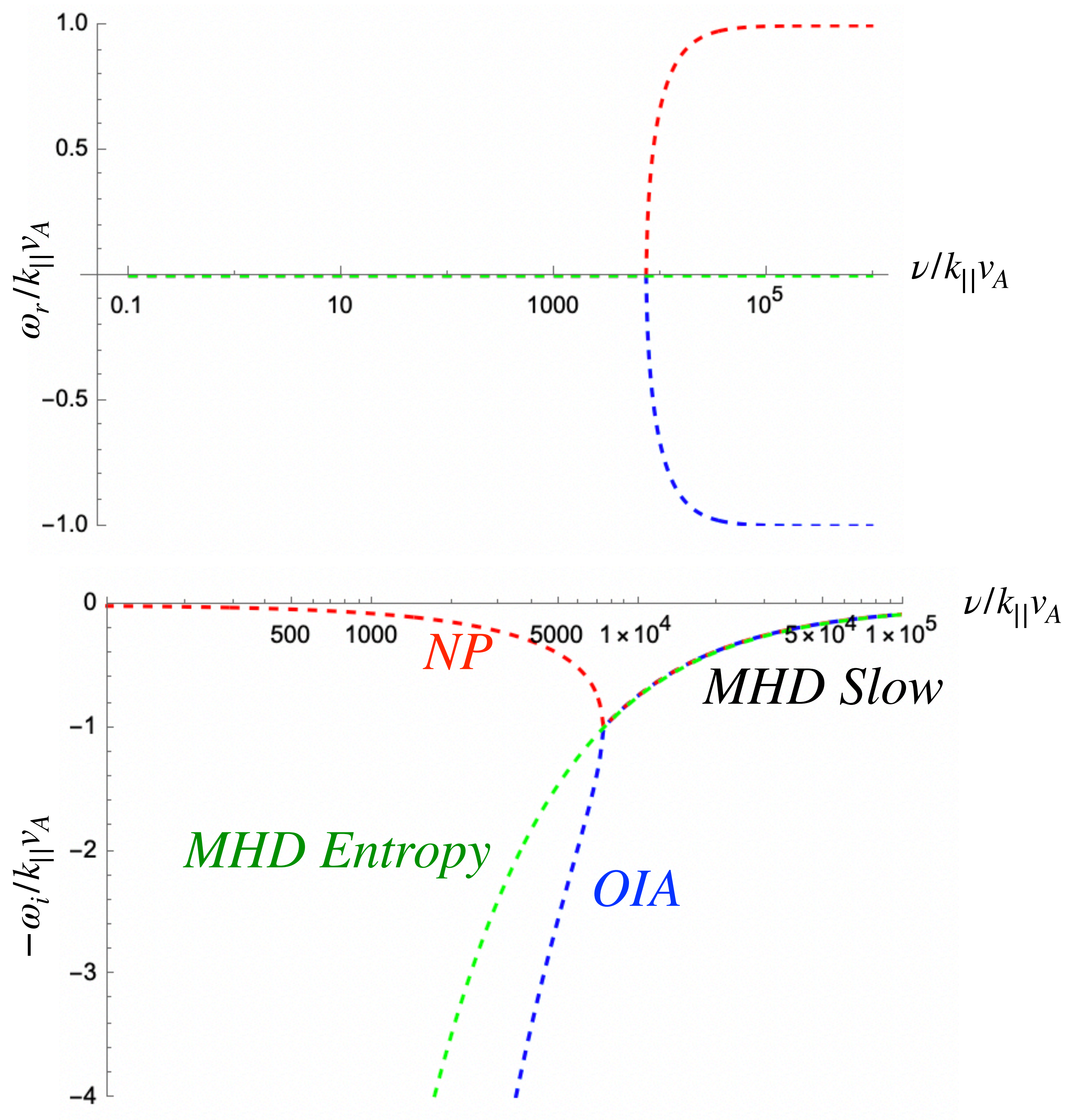
→*Meets up with NP mode to become MHD slow mode*



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2. *Behavior is expected to overlap dramatically with parallel ion acoustic modes.*

   →$\delta B_\perp$ *is quite small and plays essentially no role in the mode other than* $\vec{k} \cdot \delta\vec{B} = 0$ *(no interruption like Alfvén wave)*

   →No asymmetric anisotropy generation occurs for this mode at or near the amplitude threshold (both mirrors and firehoses occur)

Collisionless waves conserve adiabatic invariants (μ, J)

Shear Alfvén waves:

δB oscillation generates $\Delta \; (= p_\perp / p_\parallel - 1)$

Ion acoustic waves:

δn, δB oscillation generates Δ

Mirror ($\Delta\beta > 1$) and firehose ($\Delta\beta < -2$) instabilities

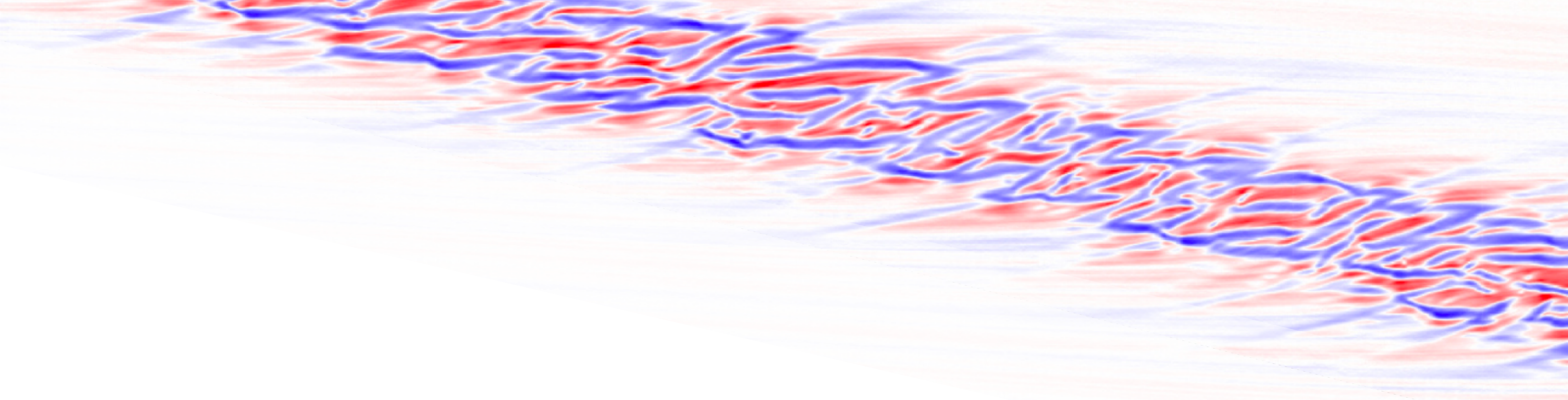
**Squire et al 2017, PRL*

Pitch-angle scattering, Braginskii-like behavior

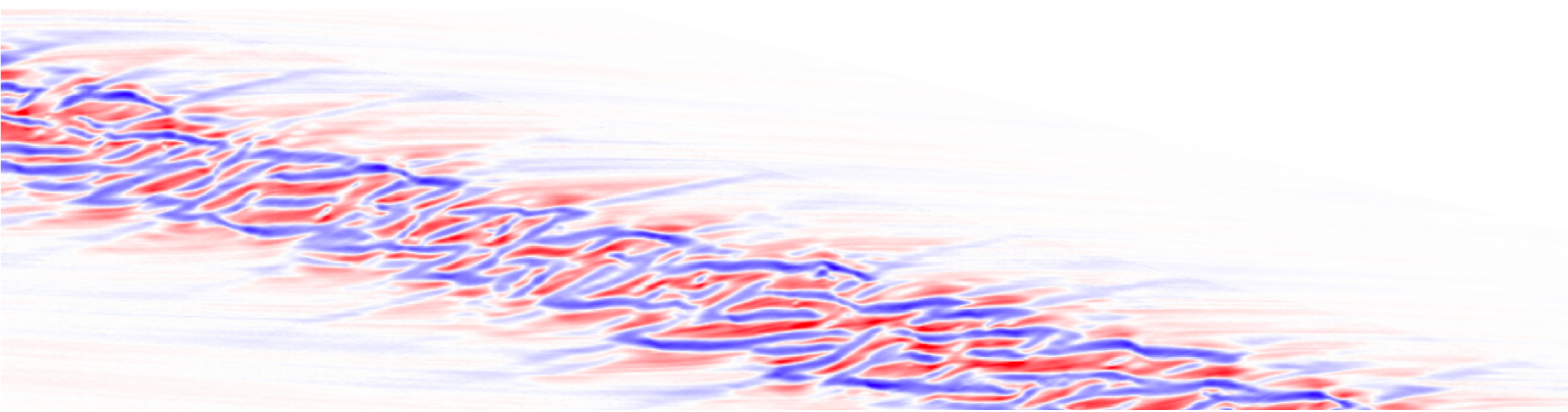
**Kunz et al 2020, JPP*

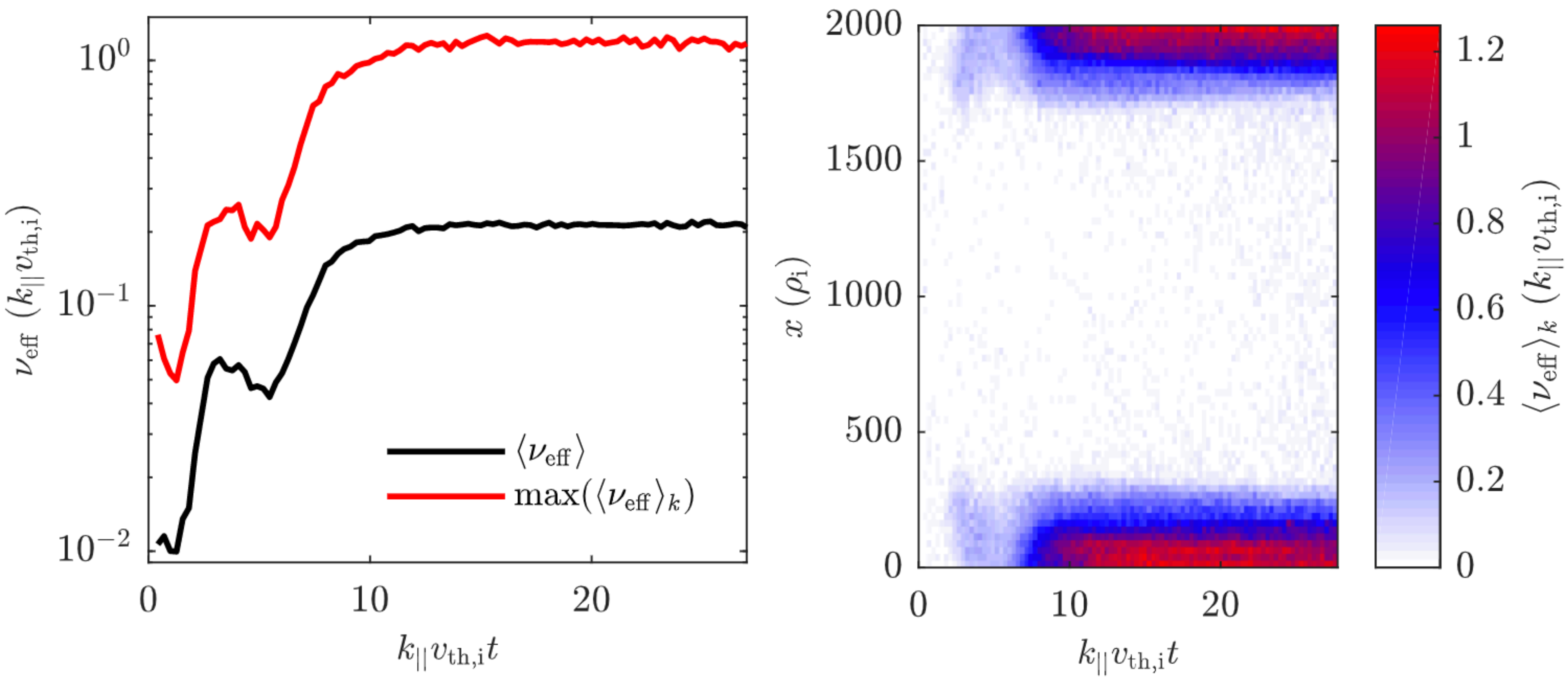
δB rapidly decays until below instability thresholds

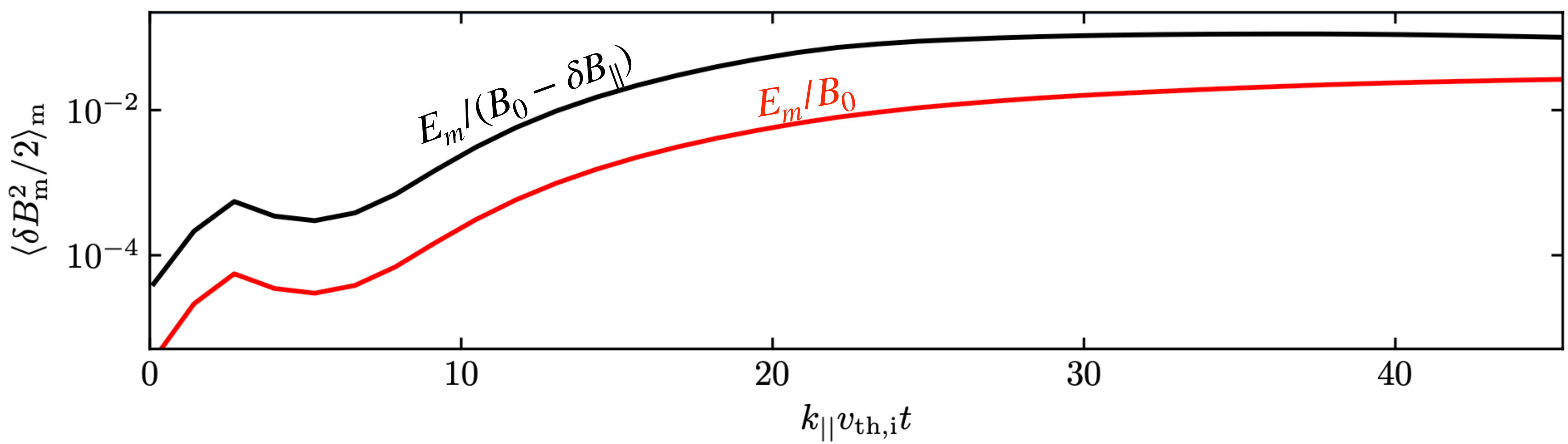
Landau damping interrupted, wave propagates undamped

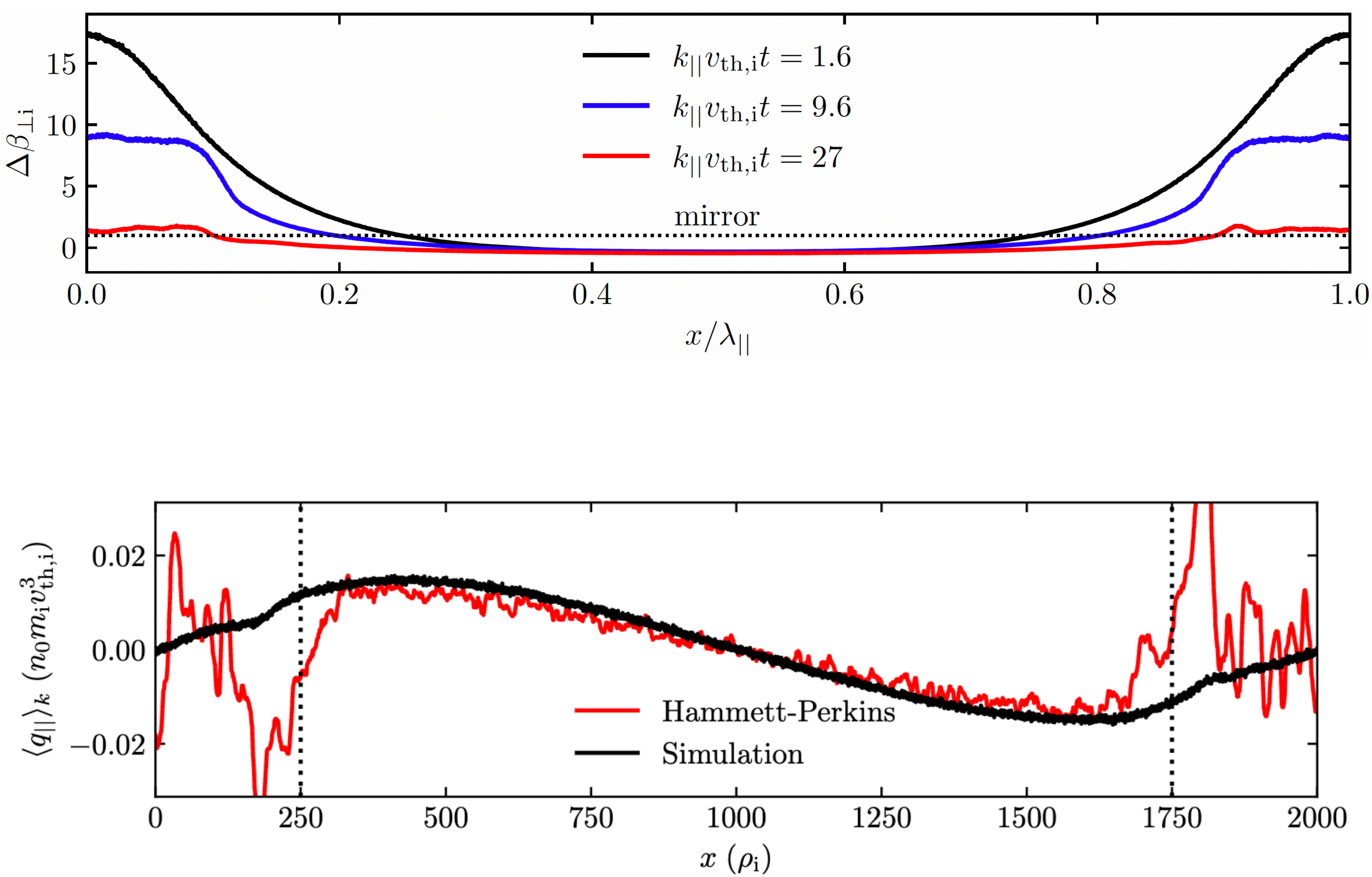


Actual Bonus



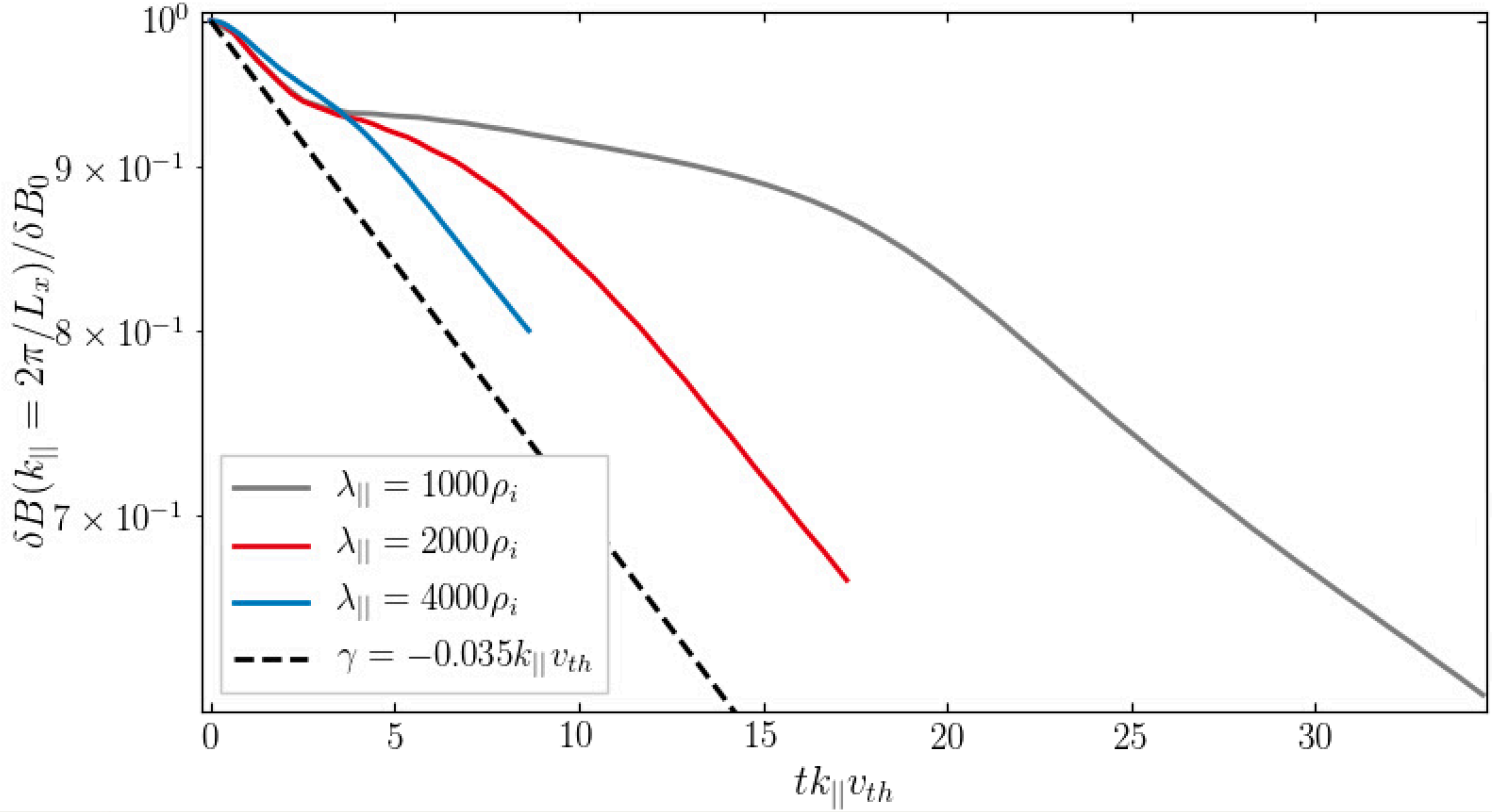

$\nu_{\mathrm{eff}}\ (k_{||}v_{\mathrm{th,i}})$
10^0
10^{-1}
10^{-2}
0
10
20
$k_{||}v_{\mathrm{th,i}}t$
$\langle\nu_{\mathrm{eff}}\rangle$
$\max(\langle\nu_{\mathrm{eff}}\rangle_k)$
$x\ (\rho_{\mathrm{i}})$
2000
1500
1000
500
0
$\langle\nu_{\mathrm{eff}}\rangle_k\ (k_{||}v_{\mathrm{th,i}})$
1.2
1
0.8
0.6
0.4
0.2


$\langle\delta B_{\mathrm{m}}^2/2\rangle_{\mathrm{m}}$
10^{-2}
10^{-4}
$E_m/(B_0-\delta B_{||})$
E_m/B_0
0
10
20
30
40
$k_{||}v_{\mathrm{th,i}}t$

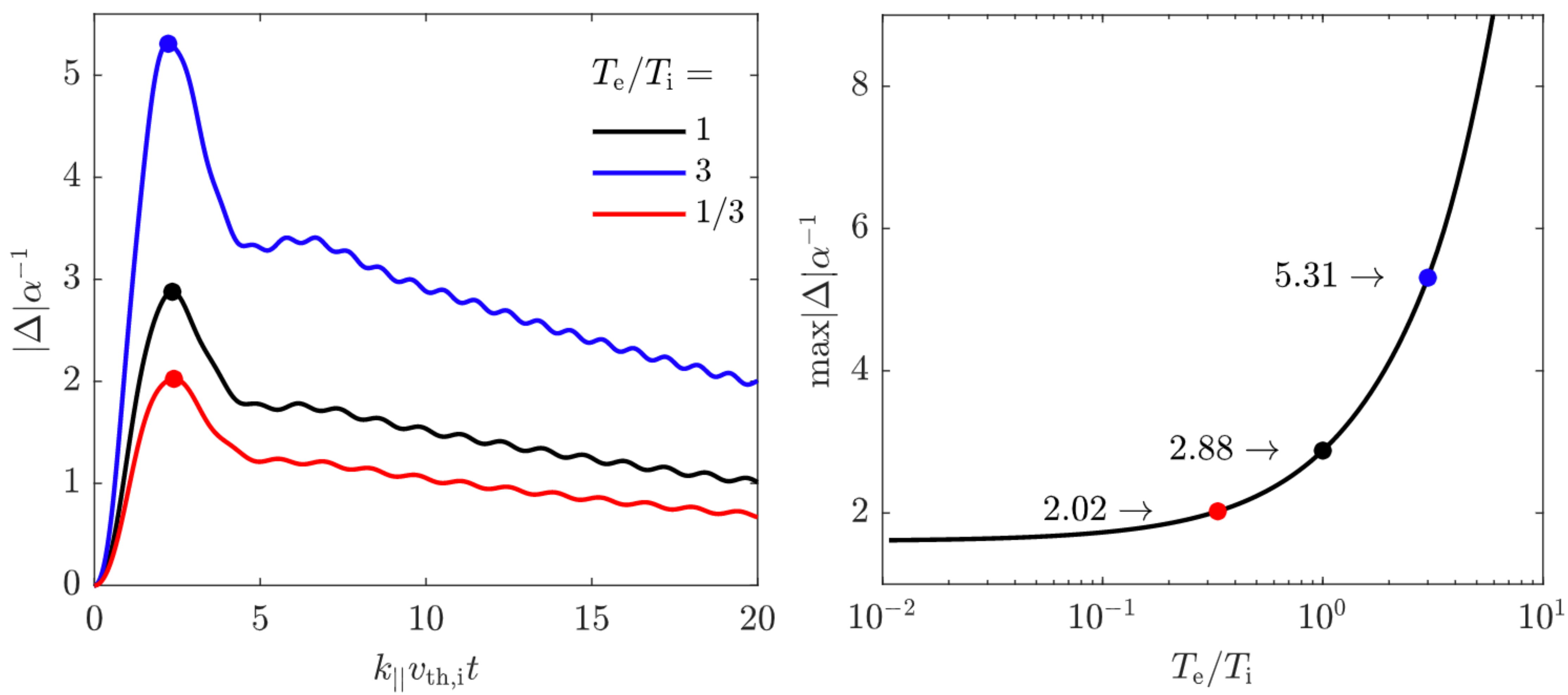


$\Delta\beta_{\perp i}$
$k_{||}v_{\rm th,i}t = 1.6$
$k_{||}v_{\rm th,i}t = 9.6$
$k_{||}v_{\rm th,i}t = 27$
mirror
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
$x/\lambda_{||}$
$\langle q_{||}\rangle_k \ (n_0 m_{\rm i} v_{\rm th,i}^3)$
Hammett-Perkins
Simulation
0.02
0.00
−0.02
0
250
500
750
1000
1250
1500
1750
2000
$x \ (\rho_{\rm i})$

Linear decay with scale separation:

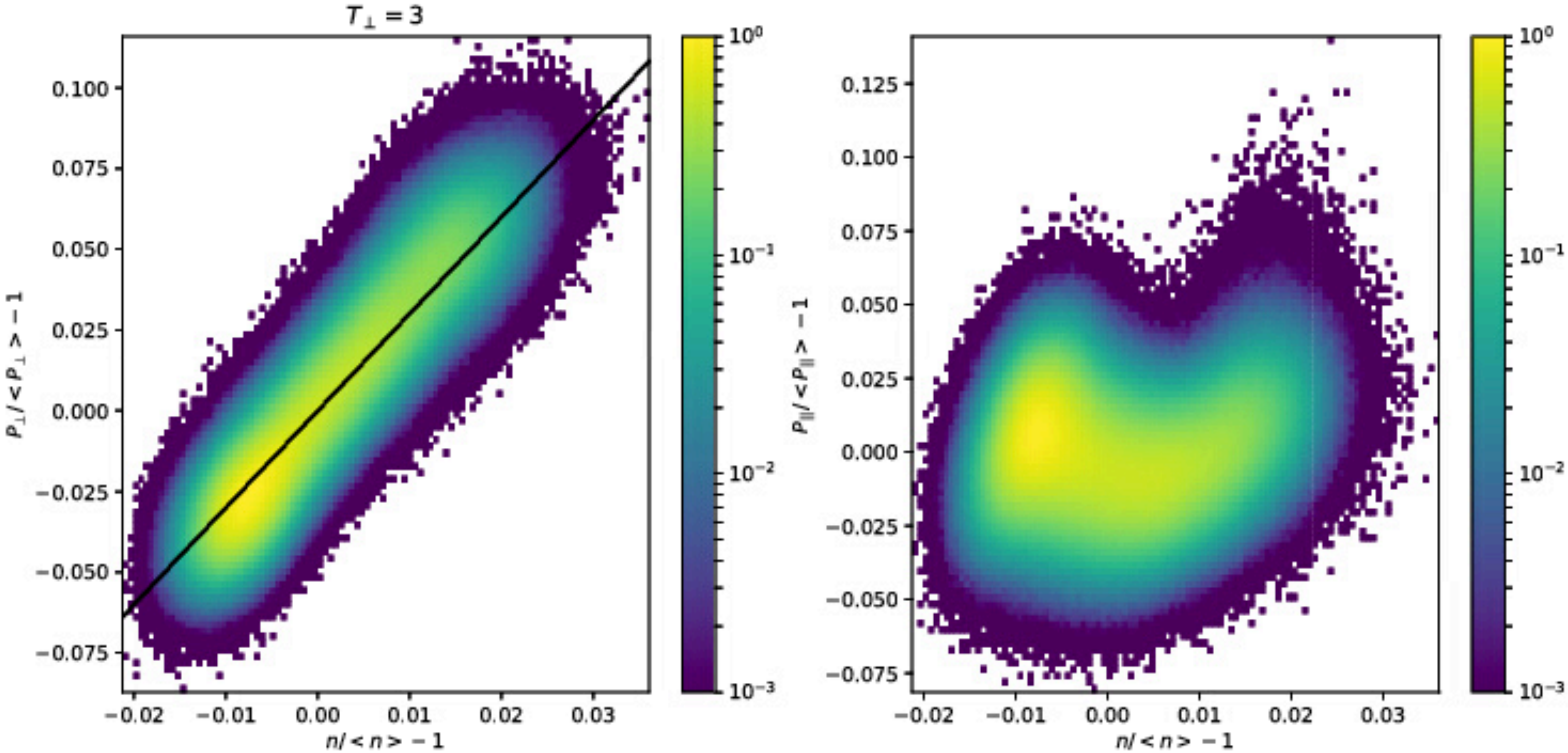


Adjustment from isothermal pressure balanced ICs



$$\Delta_{NP}(t) \simeq 2\left(1+\frac{T_e}{T_{i0}}\right)\left[\mathrm{erf}(\tau)+\left(\tau^2-\frac{\tau}{\sqrt{\pi}}\right)e^{-\tau^2}\right]\frac{\delta n(t)}{n_0}; \quad \tau \doteq \frac{k_{\|}v_{th,i}t}{2}.$$

Eigenmode relationship:



$\delta B = 0.8 B_0 \ for \ \lambda_{||} = 1000 \rho_i$

Scattering rate:

Define ν *as rate of anisotropy reduction:* $$\frac{\Delta A}{\Delta t} = -\chi N_{\rm m} \Omega_{\rm b} A \approx -\nu_{\rm eff} A,$$

where $A = \langle v_{||}^2 \rangle - \langle v_{\perp}^2 \rangle / 2$

Width of mirror region defined by where $\Delta\beta_{\perp,i} = 1$*:* $$B_0^2 + 4B_0 \delta B_{||}(x) + \delta B_{||}(x)^2 \approx 0$$

$$\rightarrow w_{\rm mirror} \approx \frac{\lambda_{||}}{\pi} \cos^{-1}\left(\frac{B_0}{\delta B}(\sqrt{3}-2)\right) \doteq \lambda_{||}\chi$$

Lastly $N_m = \chi \lambda_{||} / \lambda_{\rm mirror,||}$ *where* $k_{\rm mirror,||}\rho_{0,i} \sim \Lambda_m$

$$\rightarrow \boxed{\nu_{\rm eff} \sim \sqrt{\delta \widetilde{B}} \Omega_{\rm c,i} \Lambda_{\rm m} \chi}$$