Correlation integrals and how to use them: a discussion

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Hosking & Schekochihin (2021), *Phys. Rev. X* ***11****, 041005*

Hosking & Schekochihin (2022), arXiv:2203.03573

Hosking & Schekochihin (2022), arXiv:2202.00462

Turbulence and conservation laws

Fluid (and kinetic) systems support the conservation of certain quantities, i.e.,

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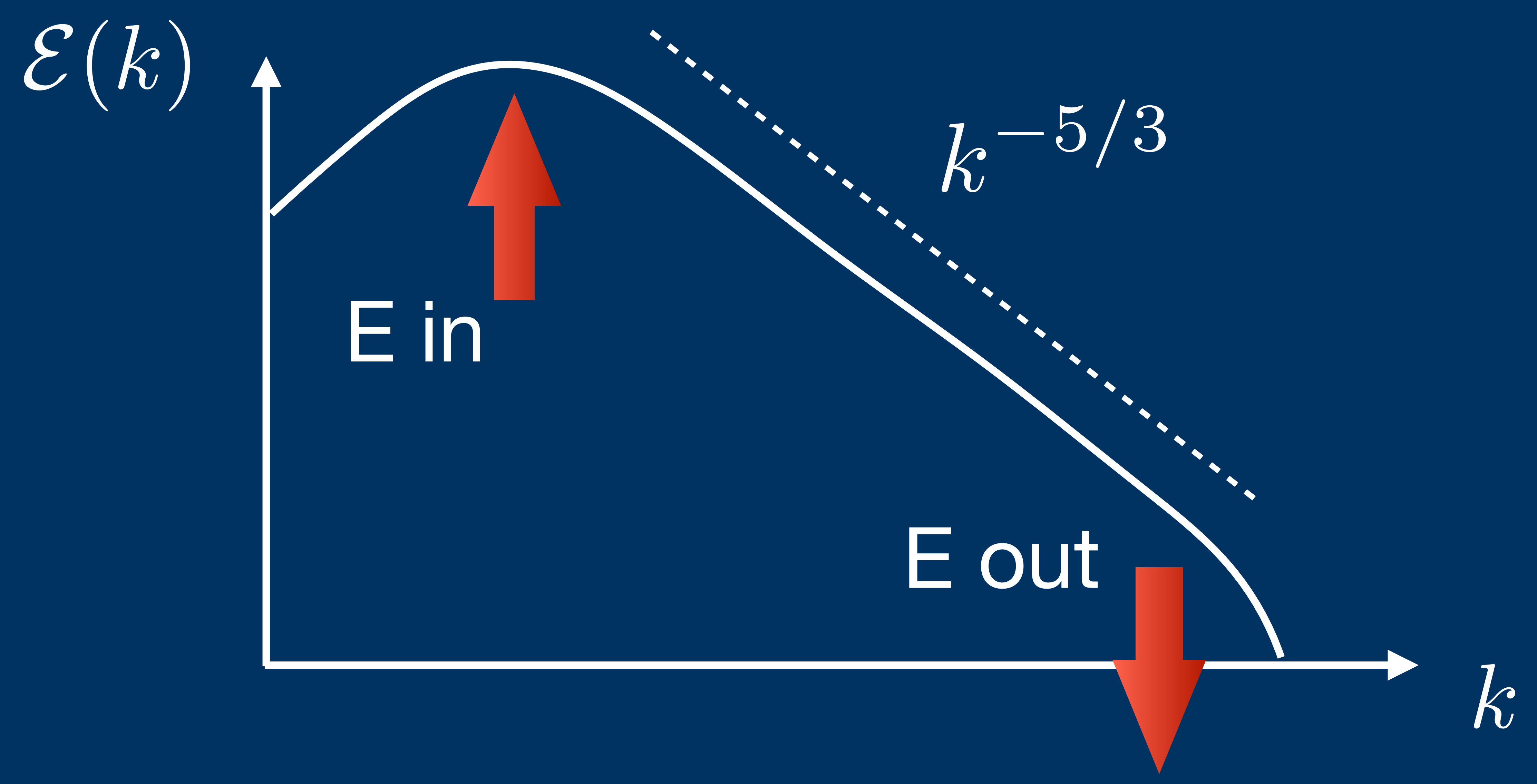
At some level, turbulence is just the nonlinear processing of these quantities between injection and dissipation scales.



Turbulence and conservation laws

The conserved quantities ψ come in two distinct types.

Type 1: ψ is a generalised energy, i.e., the norm of a dynamical field. $\langle\psi\rangle > 0$ for any possible realisation of the system. Examples are:

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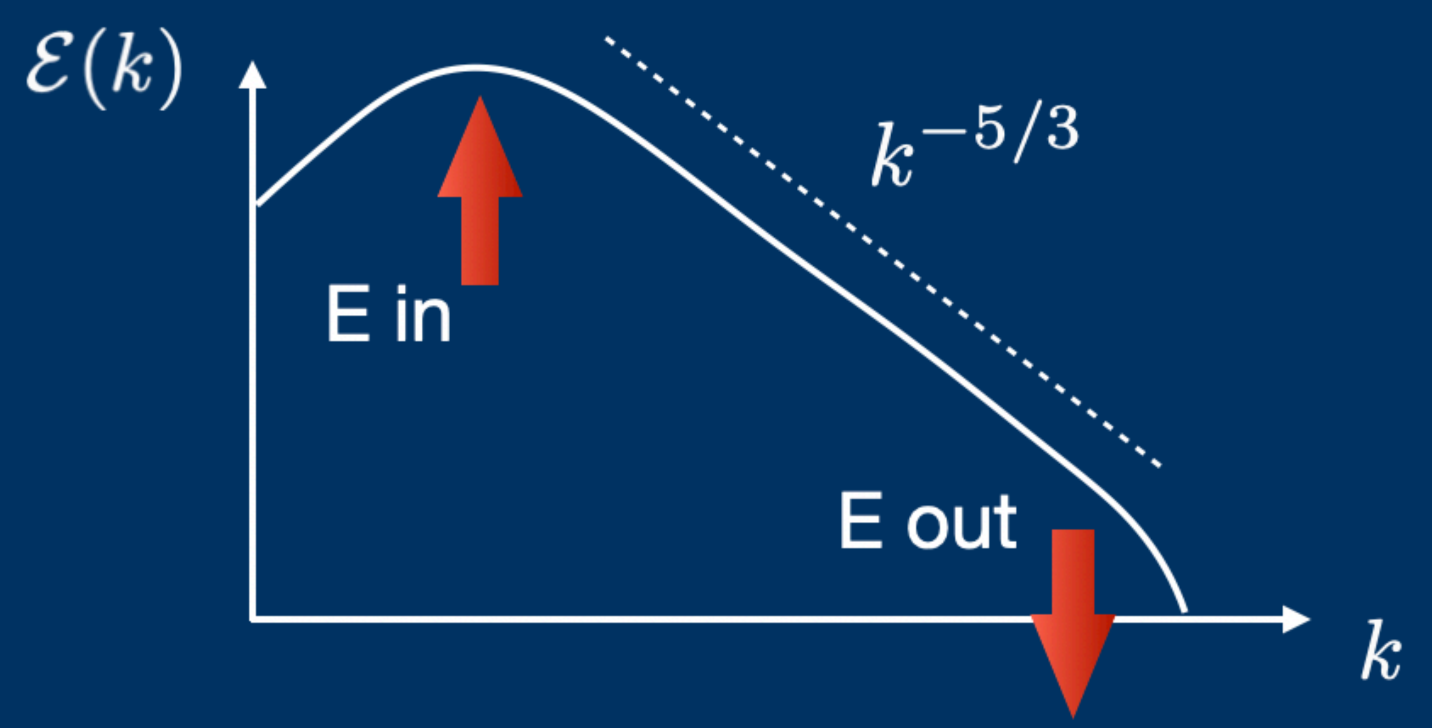
Properties of generalised energies: <u>cascades</u>

$$\varepsilon_\psi \sim \frac{\psi_l}{\tau_l} \sim \mathrm{const} \implies \mathscr{E}(k)$$

e.g., 3D hydro: (K41):

$$\varepsilon_E \sim u_l^2 \frac{u_l}{l} \implies u_l \sim (\varepsilon_E l)^{1/3} \implies \mathscr{E}(k) \propto k^{-5/3}$$



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Properties of generalised energies: inverse cascade/transfer

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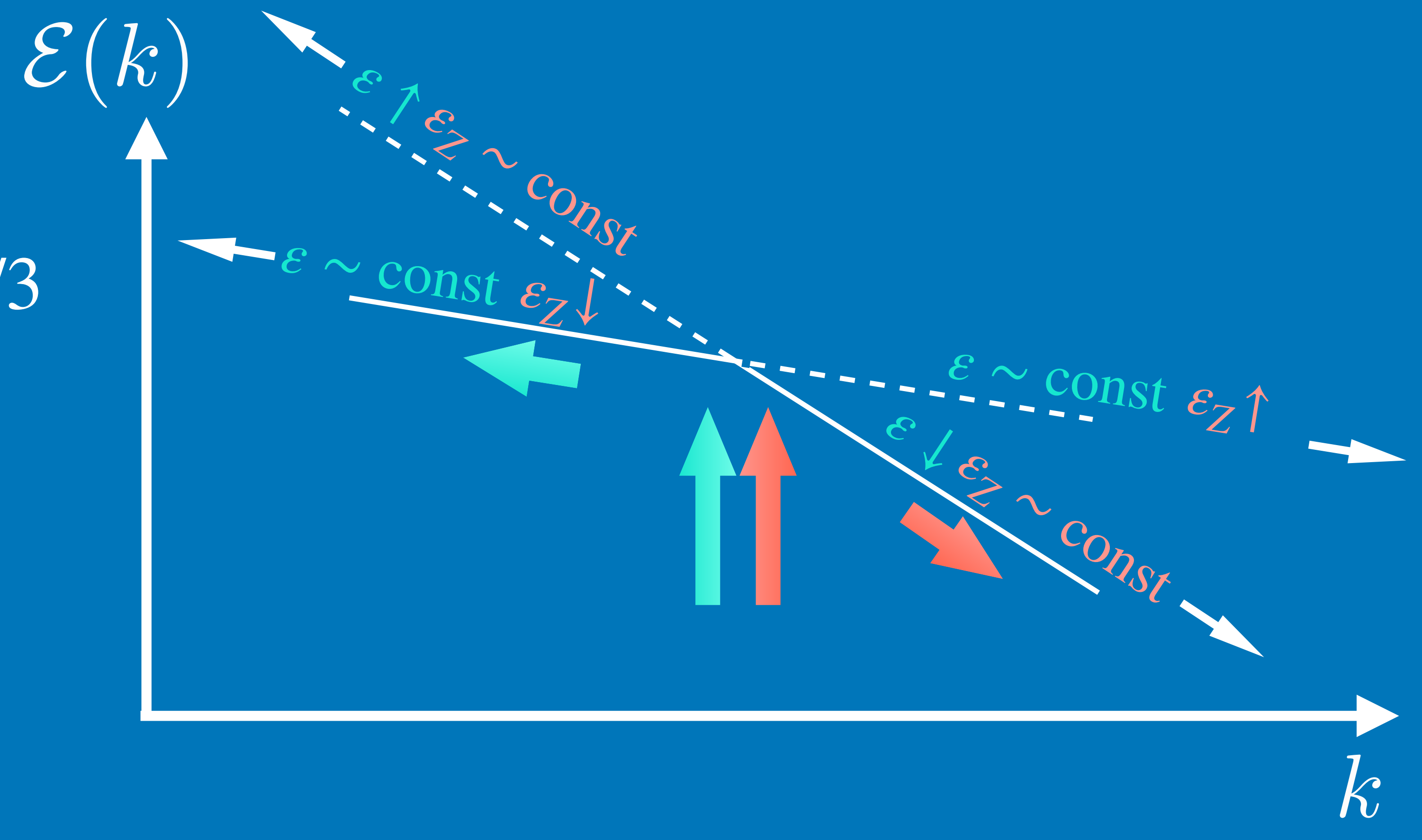

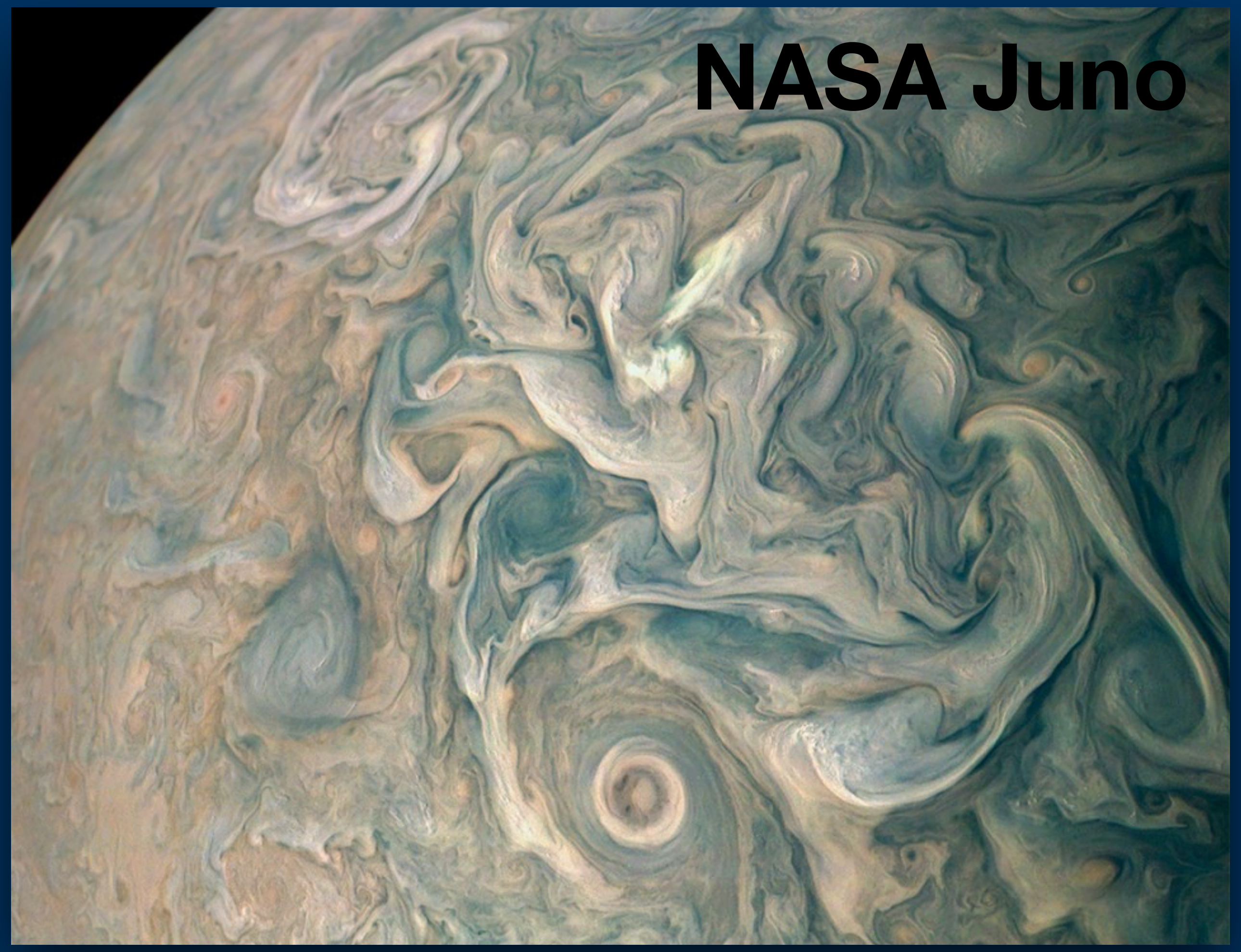

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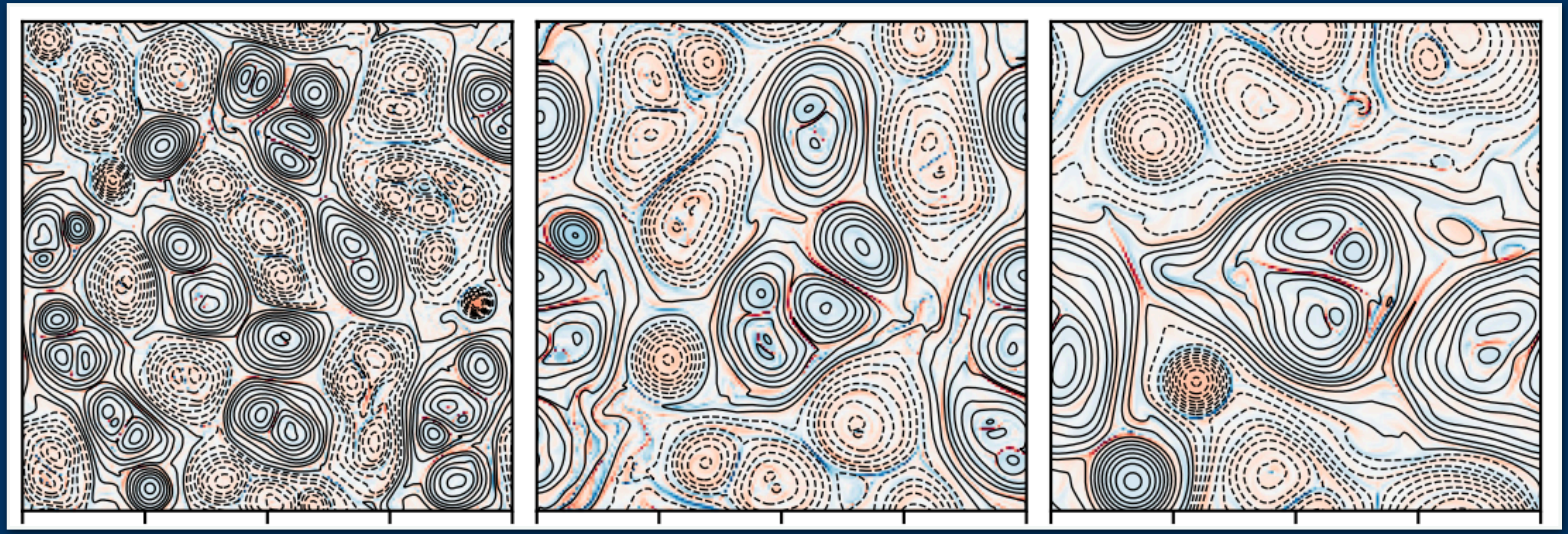
Properties of generalised energies: <u>constrain turbulent decay</u>

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\frac{E}{\tau_L}, \quad \langle\psi\rangle \sim U^\alpha L^\beta \sim \mathrm{const} \implies E \propto t^{-p}$$

e.g., 2D MHD:

$$\tau_L \sim \frac{L}{B}\epsilon_{\mathrm{rec}}^{-1}, \quad \langle A_z^2\rangle \sim B^2 L^2 \sim \mathrm{const} \implies B^2 \propto t^{-1}$$

Zhou *et. al.* 2019



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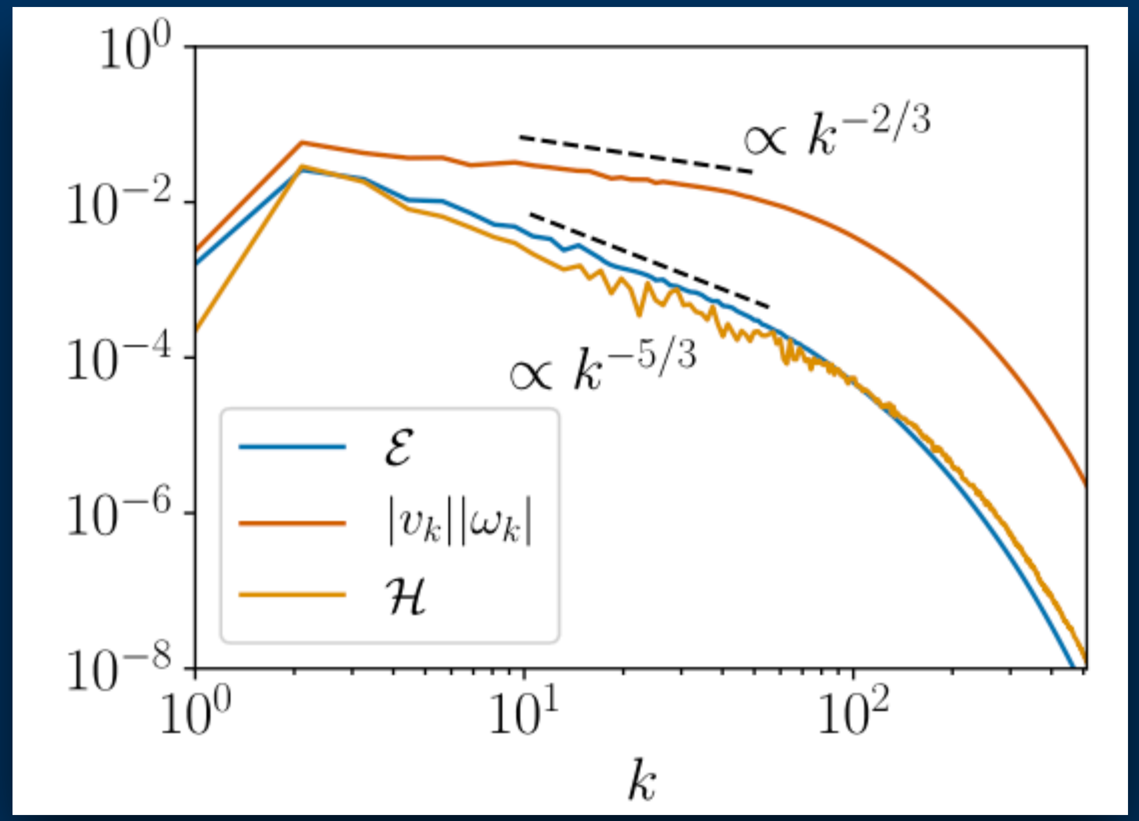
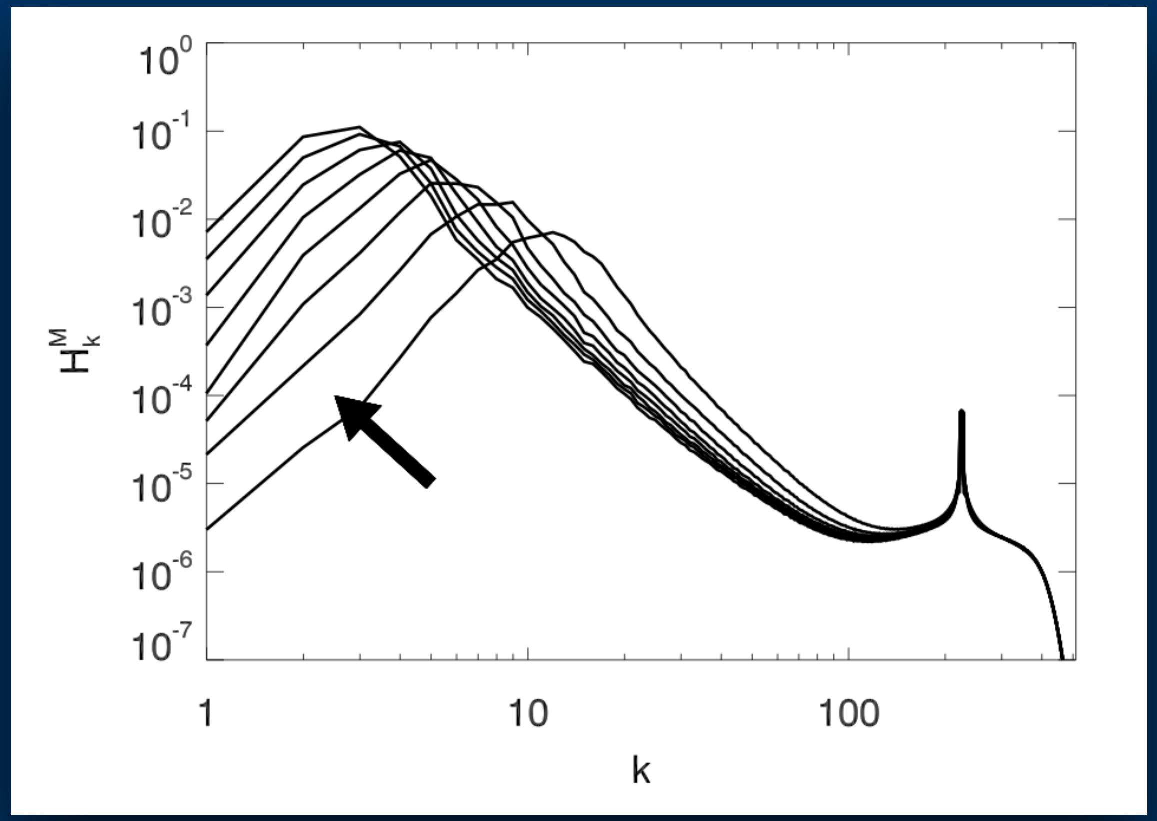
Properties of directed conserved quantities for finite $\langle\psi\rangle$:

Milanese *et. al.* 2022



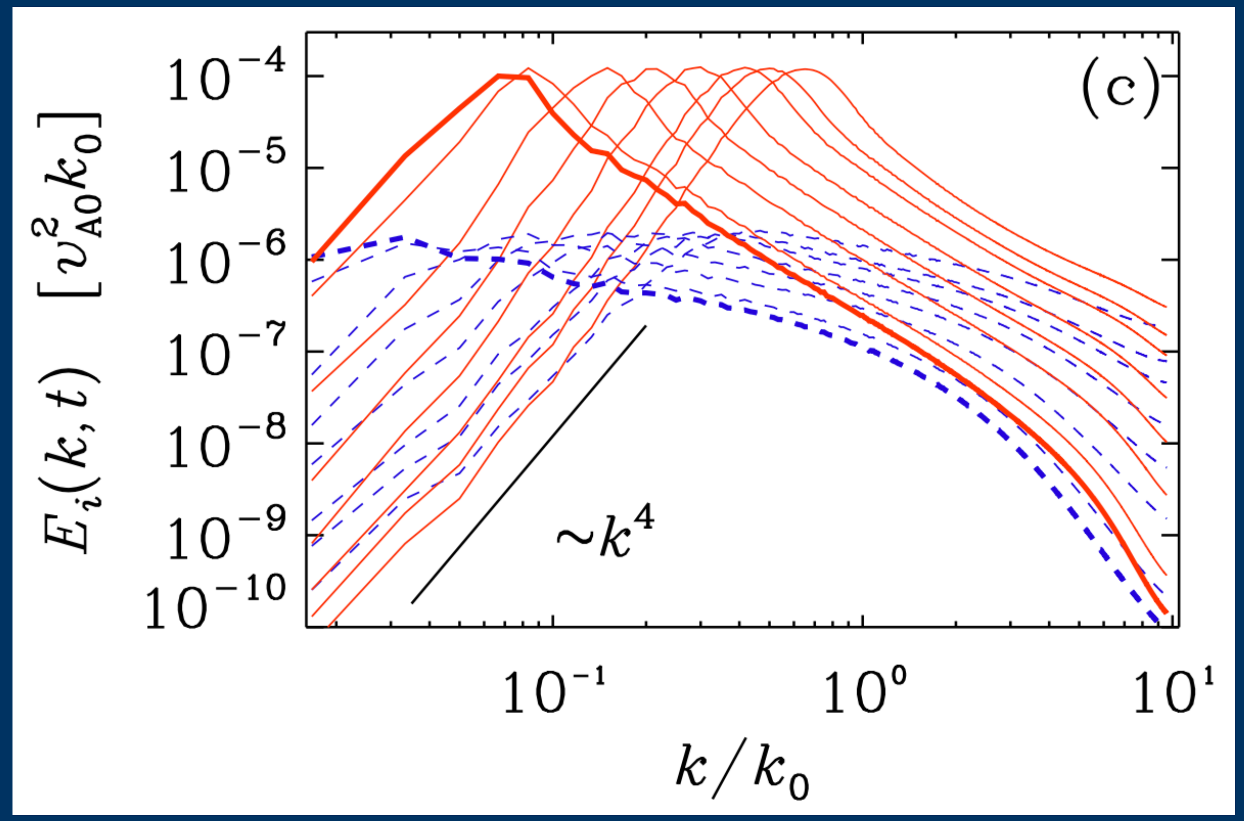
Cascades

Müller *et. al.* 2012



Dual/inverse cascade/transfer

Brandenburg *et. al.* 2017



Constrain turbulent decay

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What, if any, are then the dynamical implications of the conservation of ψ?

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- Does ψ affect decay?

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$\longrightarrow$ correlation integrals.

Correlation integrals

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \boldsymbol{F} \simeq 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\langle \psi(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{r})\rangle + 2\frac{\partial}{\partial \boldsymbol{r}} \cdot \langle \psi(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x}+\boldsymbol{r})\rangle \simeq 0$$

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Some examples:

$\psi = \boldsymbol{u}$, Saffman (1967) integral

$\psi = \boldsymbol{r} \times \boldsymbol{u}$, Loitsyanksy (1939) integral*

$\psi = \boldsymbol{u} \cdot \boldsymbol{\omega}$, “Levich-Tsinober (1983) integral”

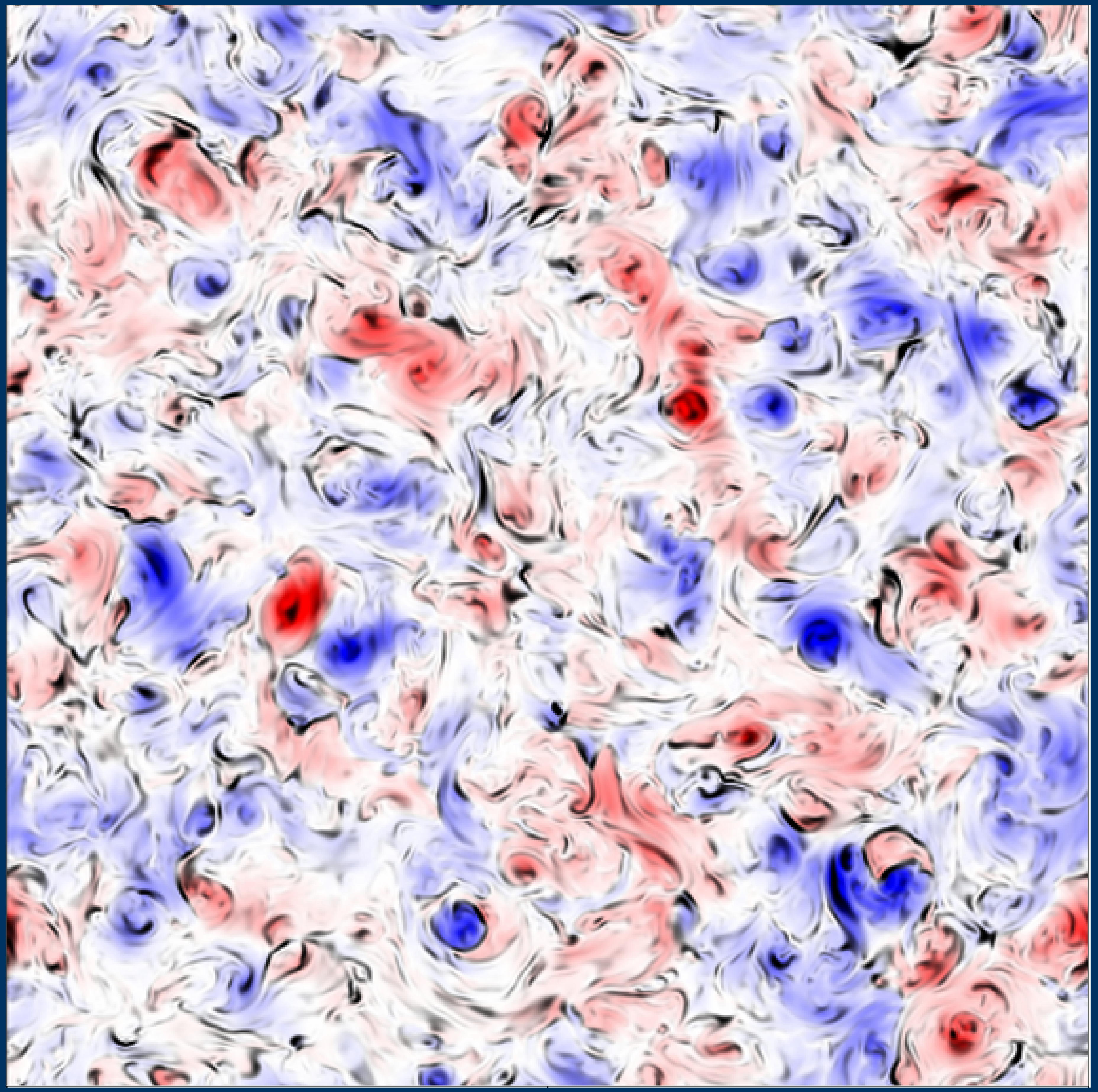
$\psi = \theta$, Corrsin (1950) integral

$\psi = \boldsymbol{B}$, “magnetic Saffman integral”

$\psi = \boldsymbol{A} \cdot \boldsymbol{B}$, “H-S (2021) integral”

$\psi = \boldsymbol{u} \cdot \boldsymbol{B}$, “Bershadskii (2019) integral ”

Correlation integrals



2D slice of $\psi = h = \boldsymbol{A} \cdot \boldsymbol{B}$ from simulation of MHD decay

$$I_\psi = \int \mathrm{d}^3\boldsymbol{r} \langle \psi(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{r})\rangle = \lim_{V\to\infty} \frac{\left\langle \left[\int_V \mathrm{d}^3\boldsymbol{x}\, \psi(\boldsymbol{x}) \right]^2 \right\rangle}{V}$$

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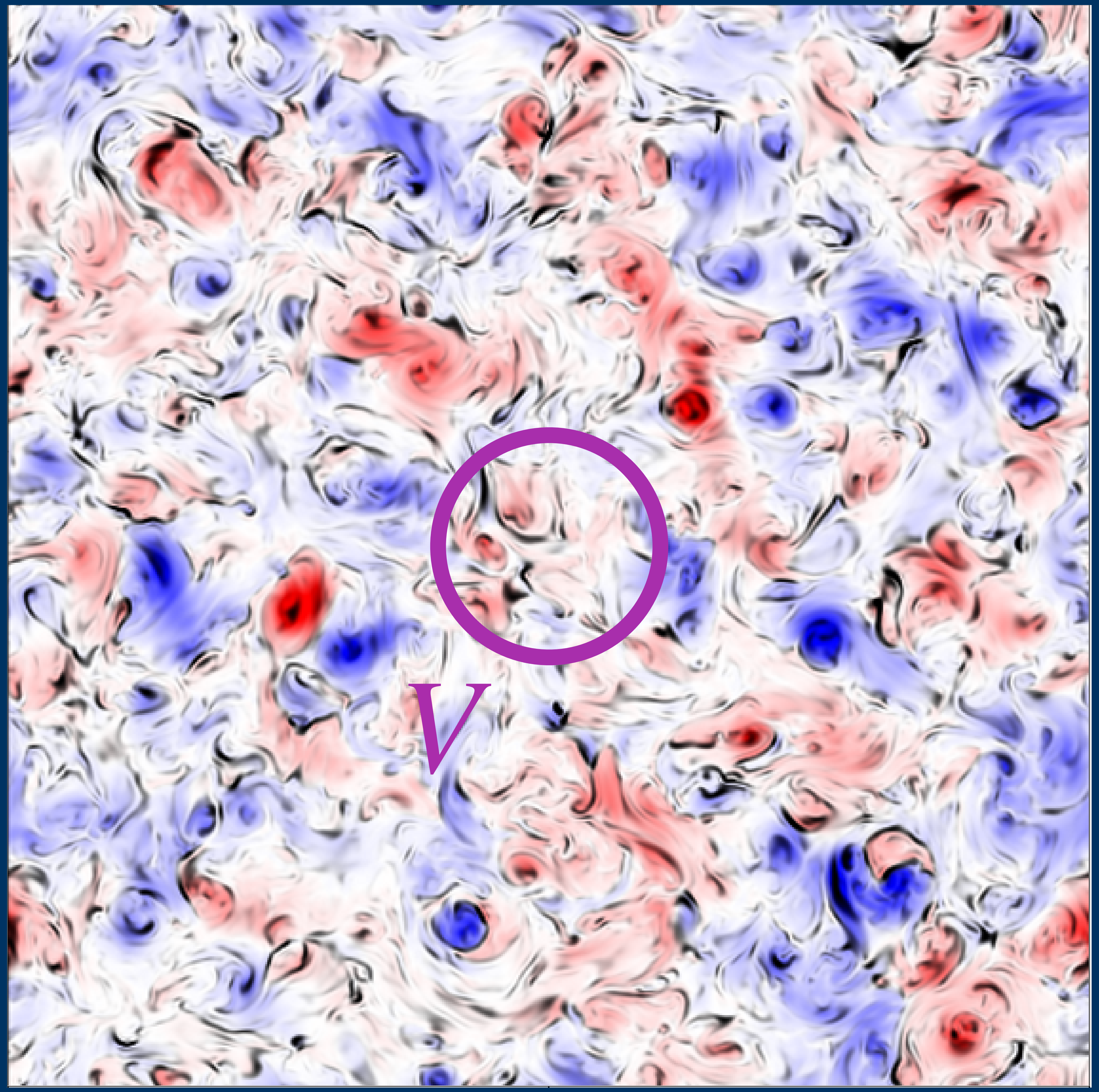

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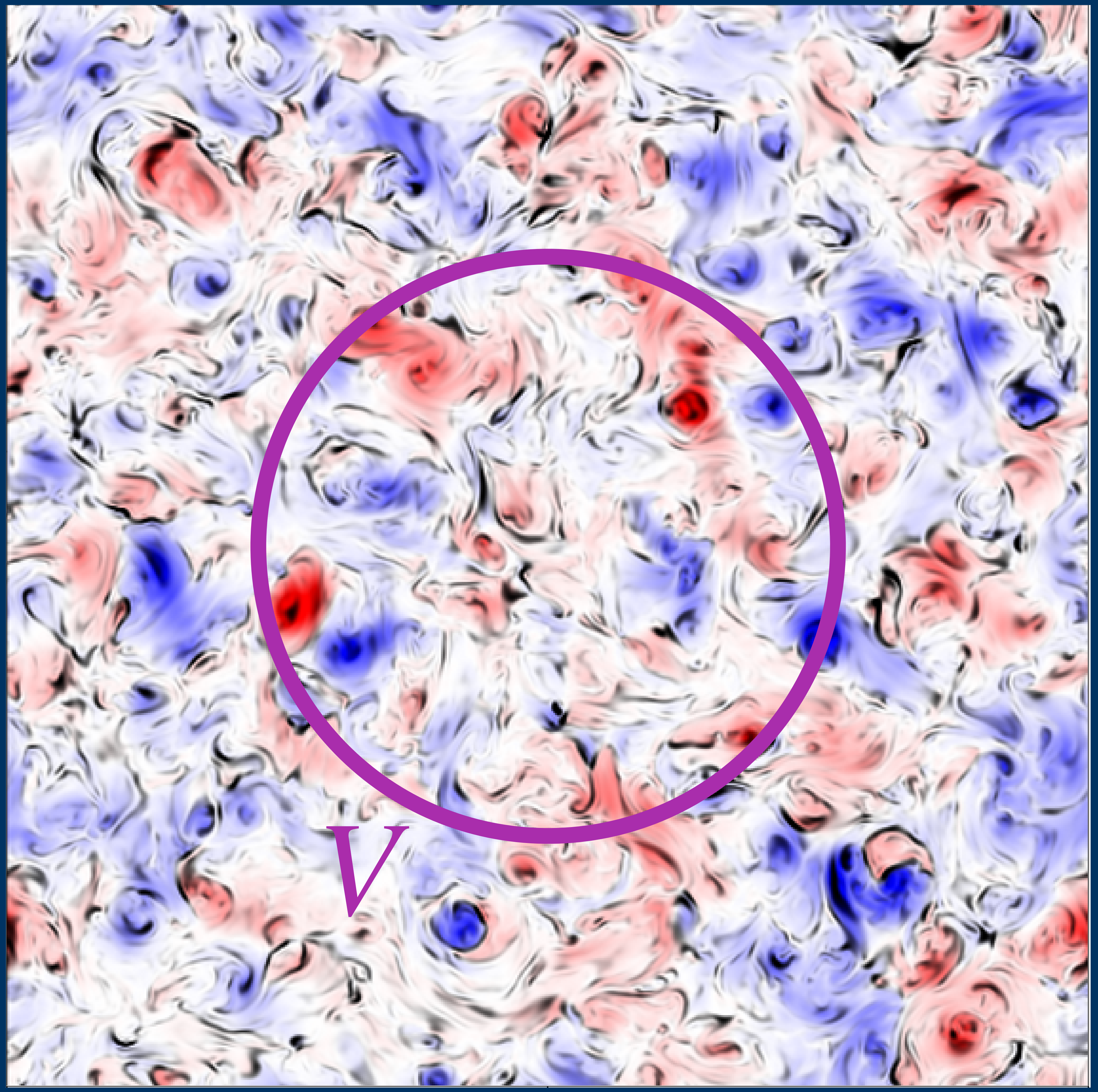

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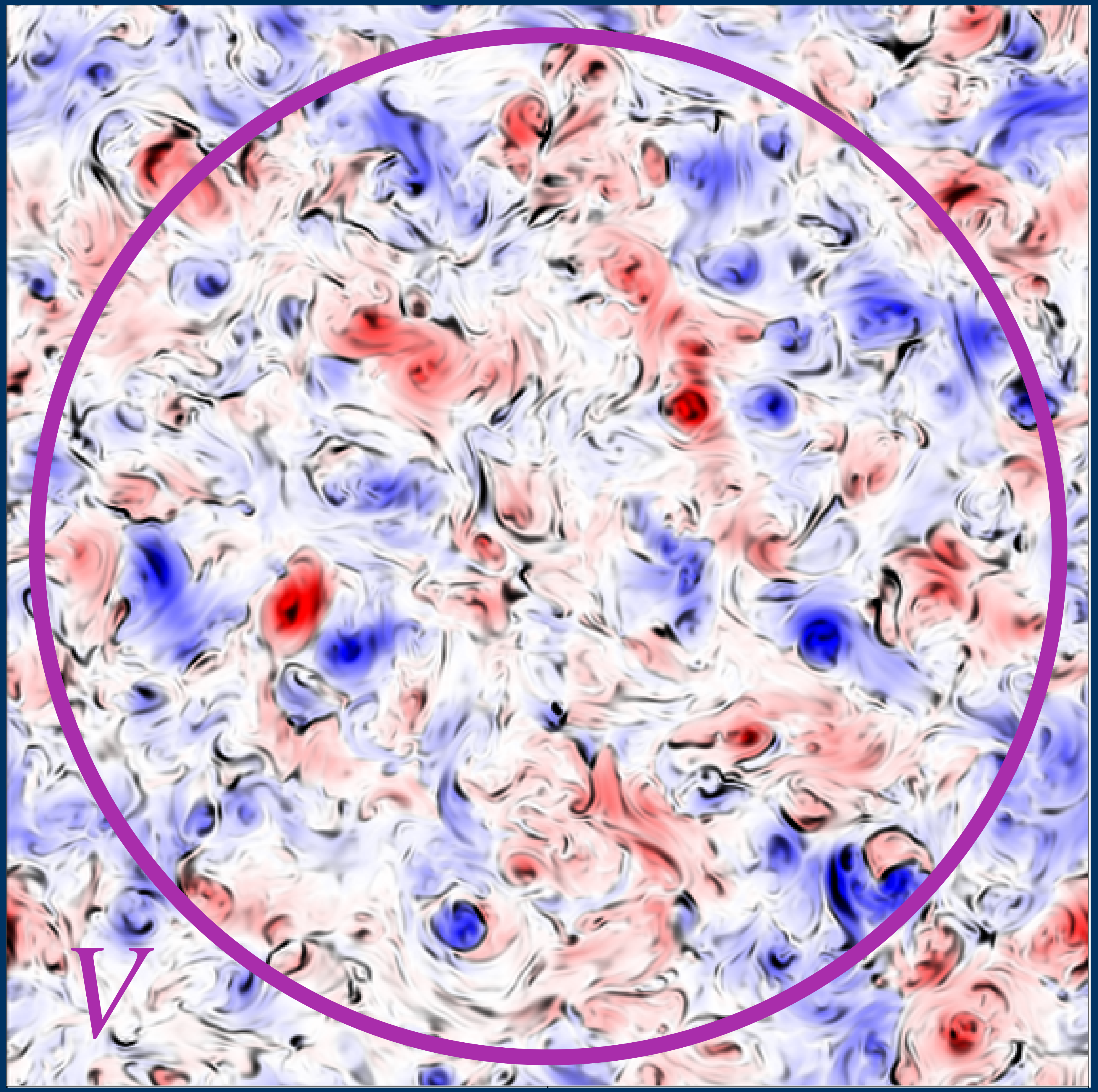

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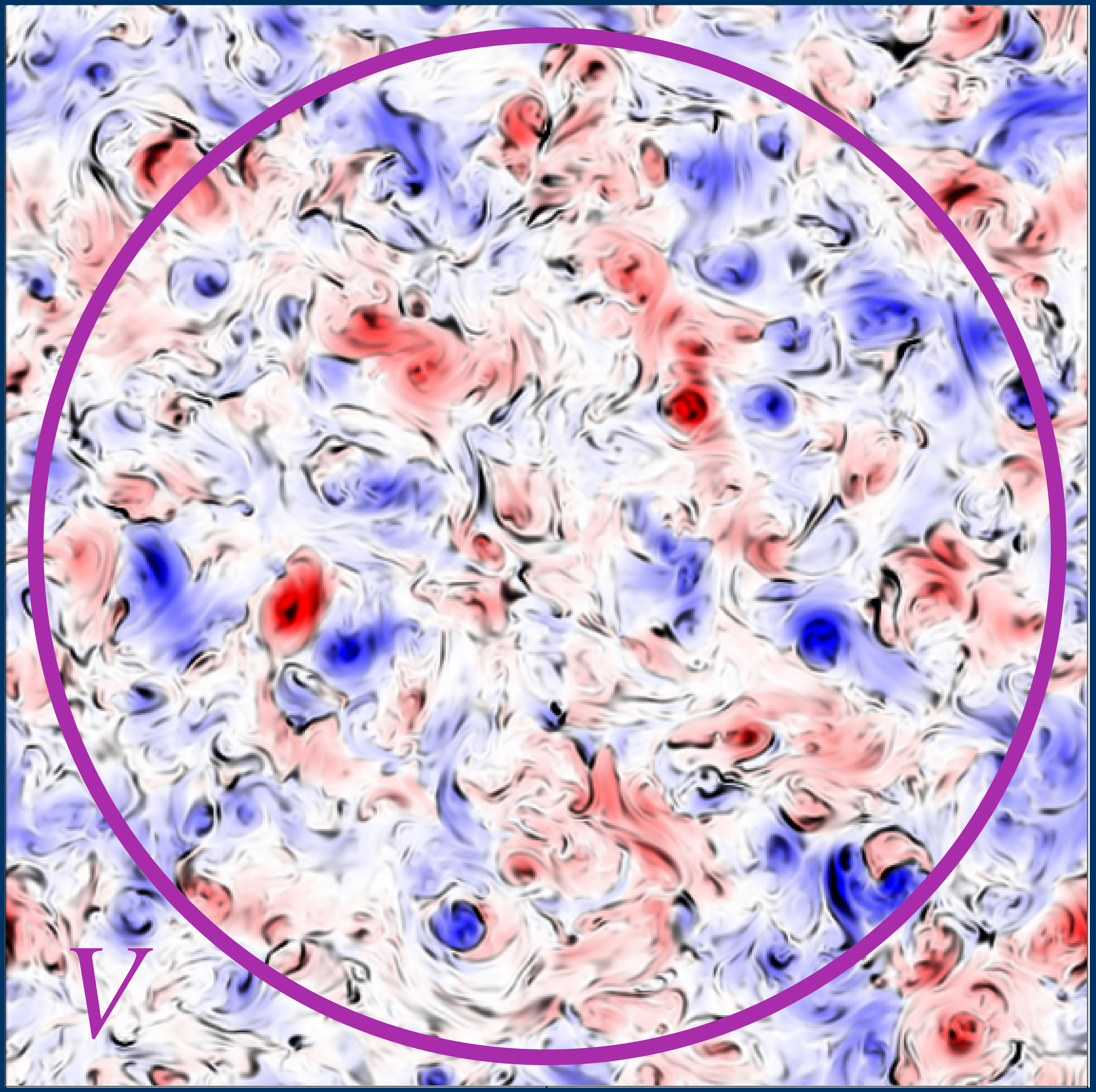

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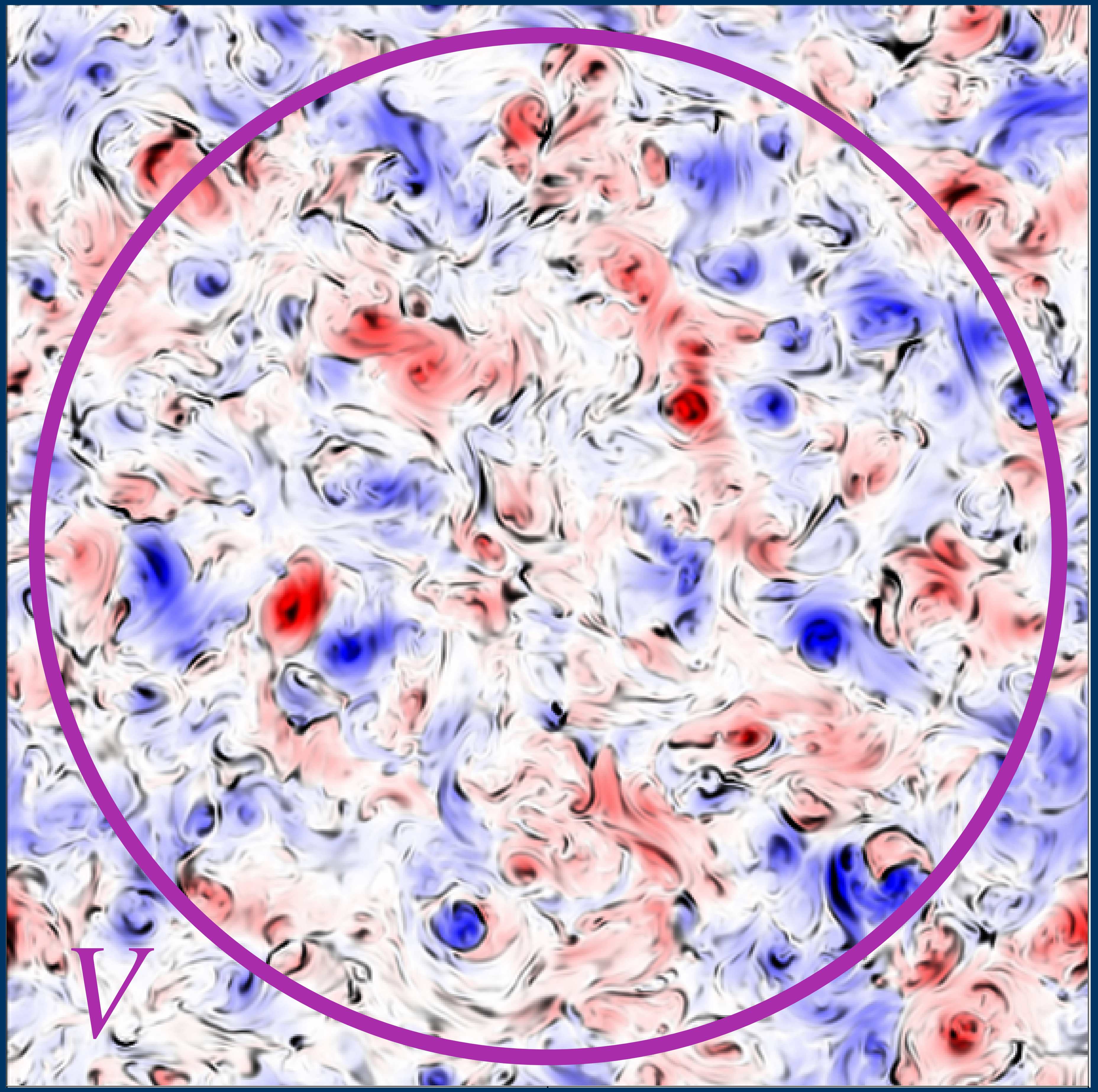

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$$\implies \frac{1}{I_\psi}\frac{\mathrm{d}I_\psi}{\mathrm{d}t} \propto \lim_{V\to\infty} V^{-2/3} = 0$$

Kinematics: correlation integrals and spectra

The power spectrum of ψ is

$$\mathscr{E}_\psi(k) = \frac{k^2}{2\pi^2} \int \mathrm{d}^3\boldsymbol{r} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \psi(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{r}) \rangle$$

If the correlation function decays sufficiently quickly with r, then

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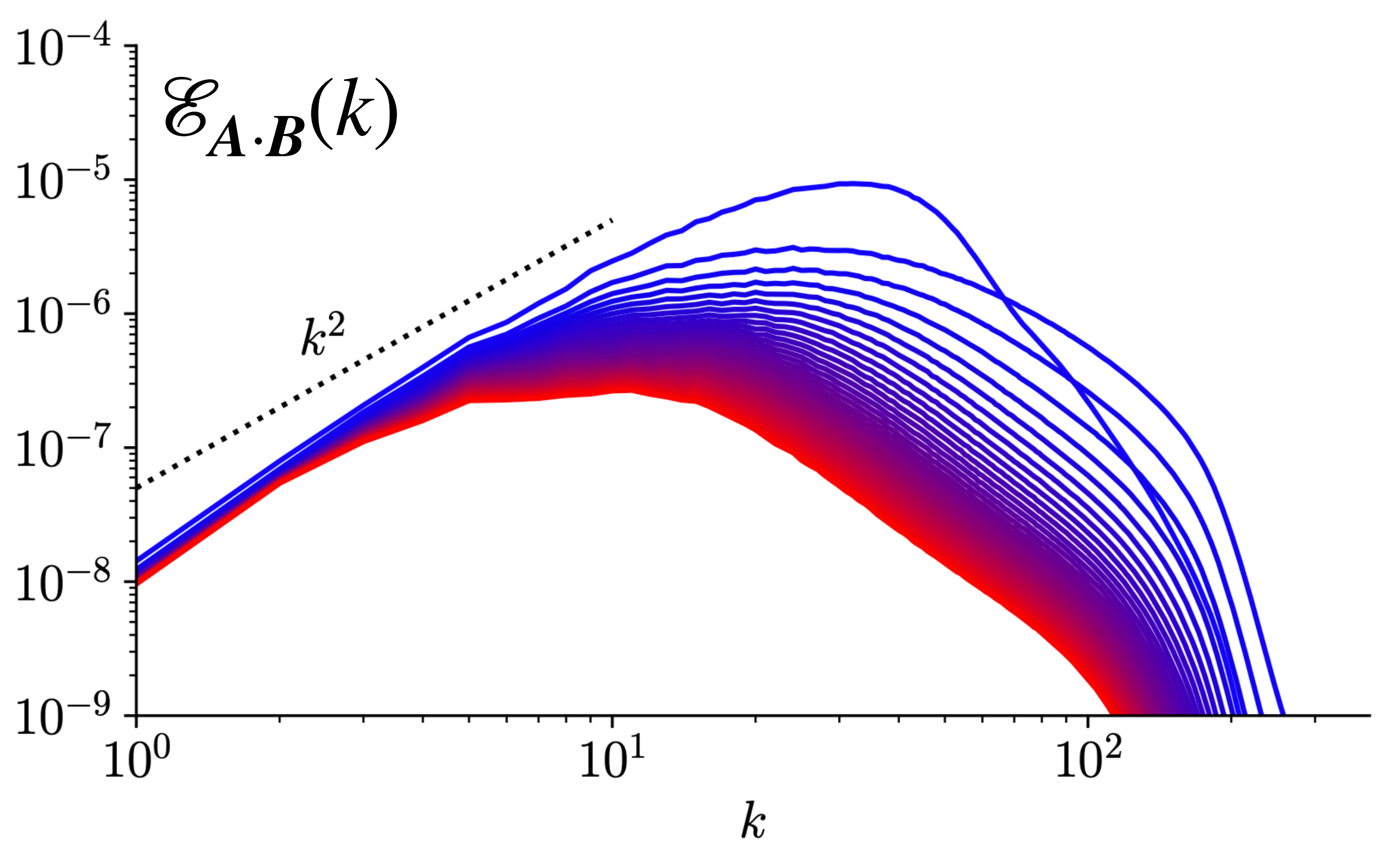

Simulation of MHD decay

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Correlation integrals and turbulent dynamics

Cascades / inertial-range phenomenology

Inverse cascades and transfers / infra-red phenomenology

Control of decaying turbulence

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$\mathscr{E}_{\boldsymbol{u}}(k \to 0)$ grows as U decays if $5\alpha > 2\beta + 3$.

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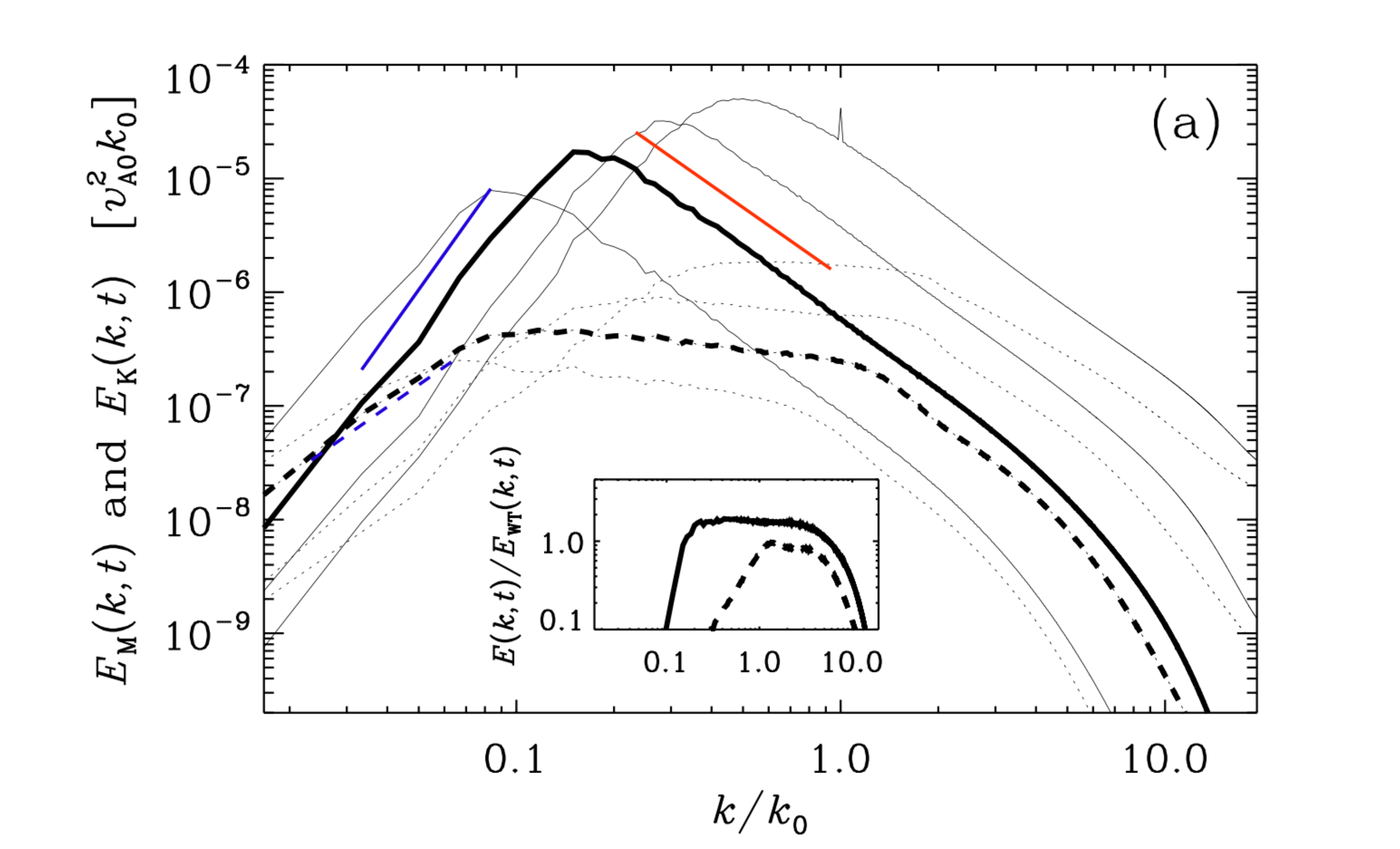
Brandenburg *et. al.* 2015



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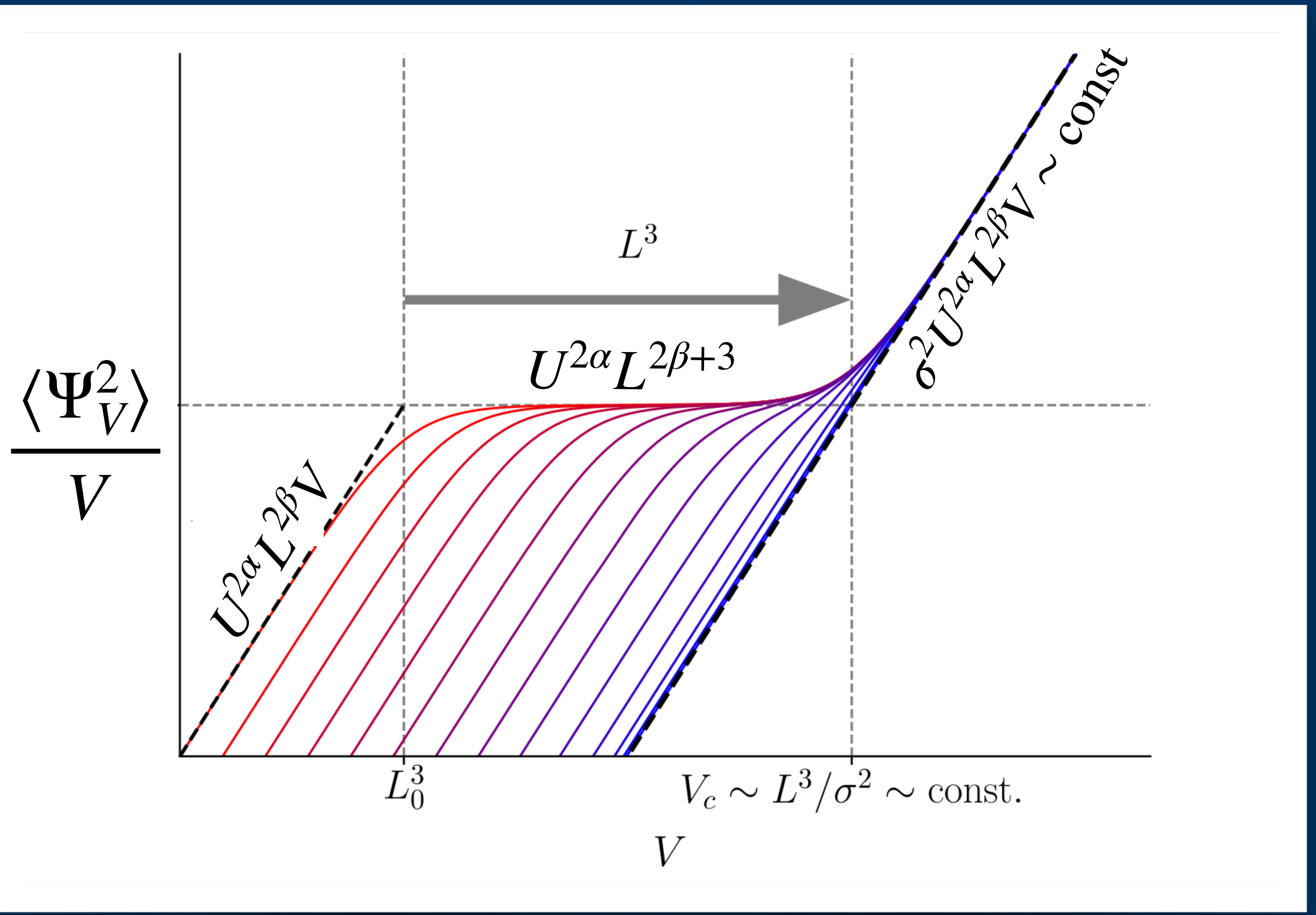

Correlation integrals and turbulent dynamics

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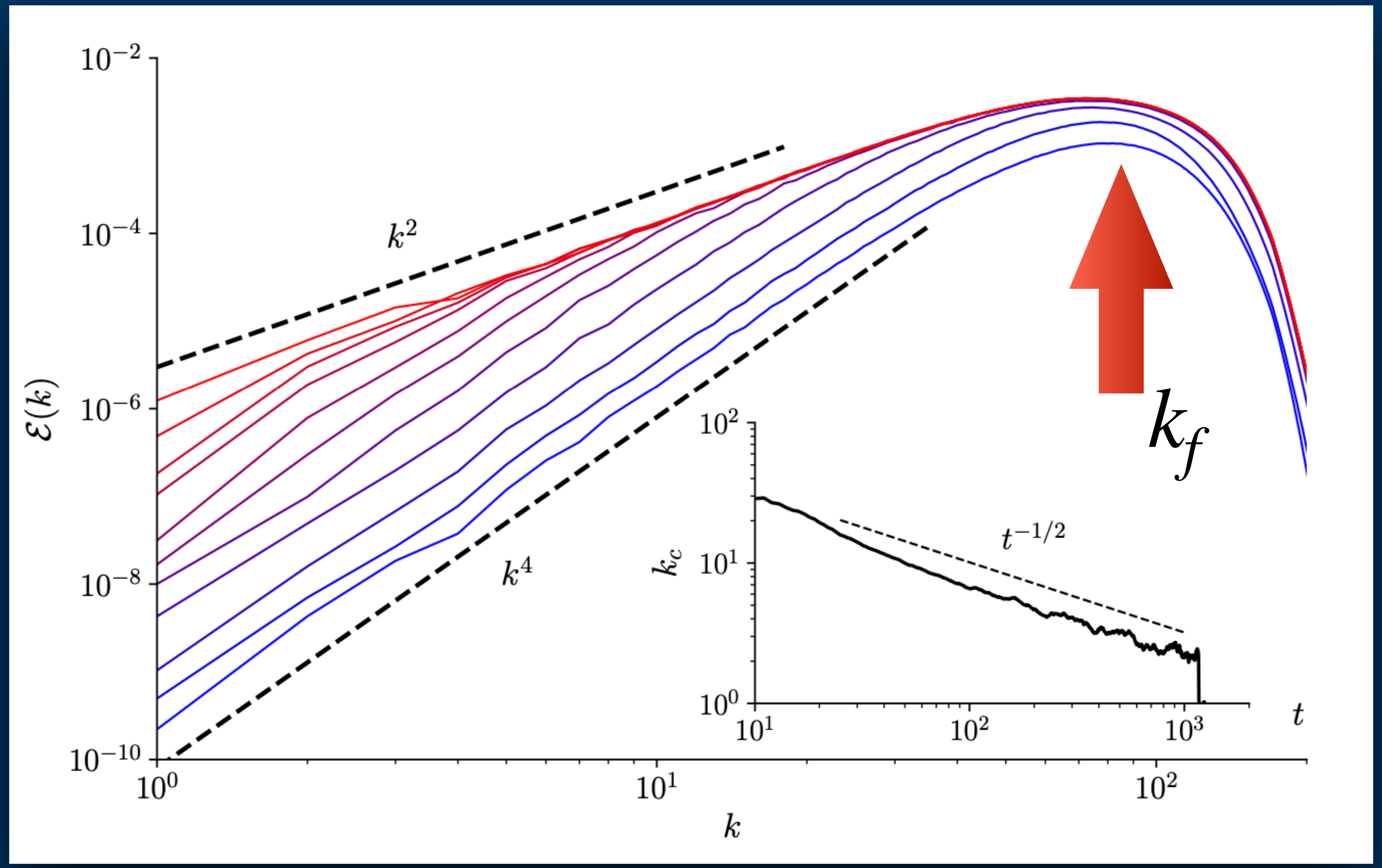
However, the connection between correlation integrals, large-scale spectra and the scaling of $\langle \Psi_V^2 \rangle$ with volume can be used to explain certain infra-red phenomena.

Correlation integrals and infra-red phenomenology

Example: “thermal equilibration” of forced hydrodynamic turbulence.



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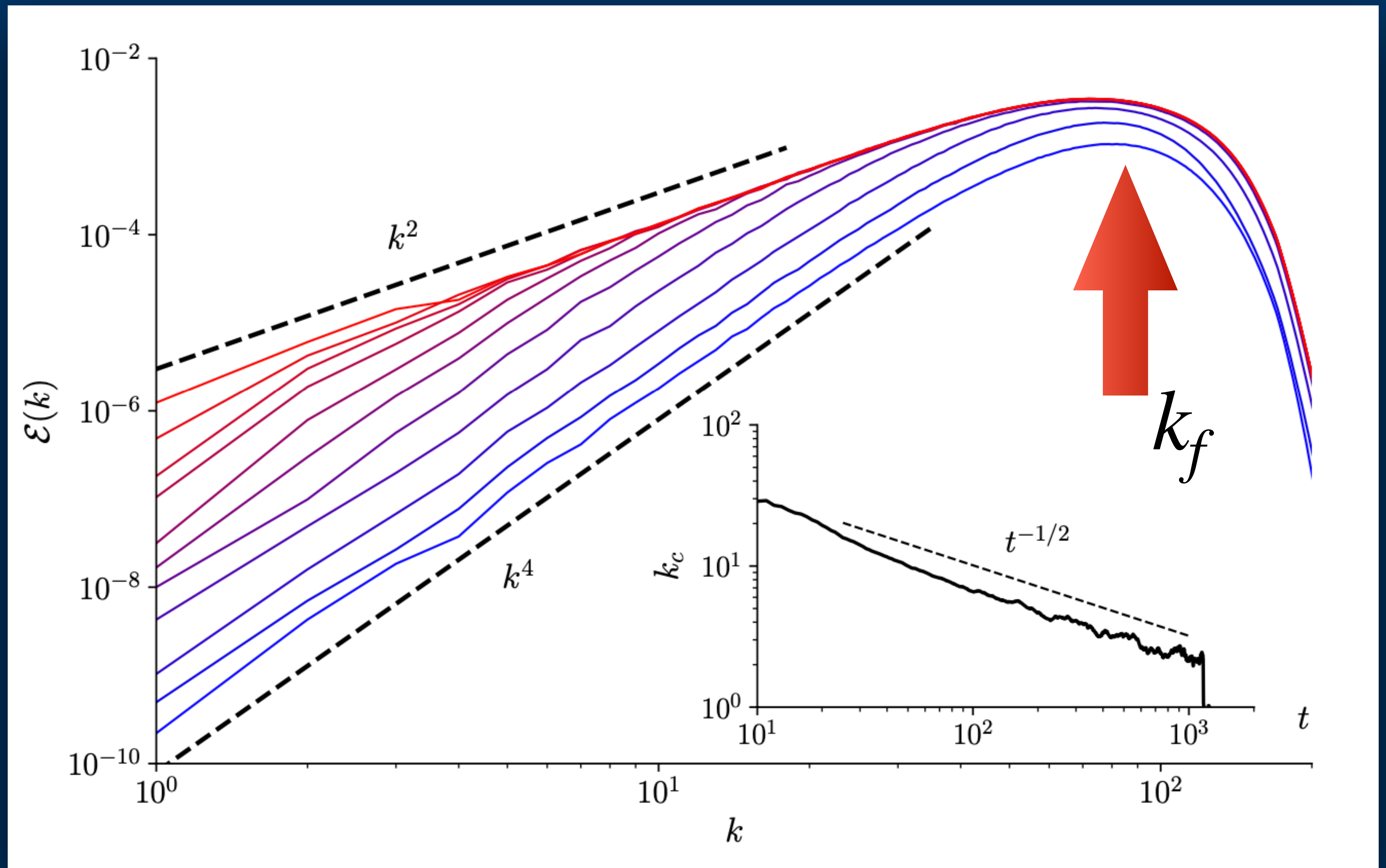

$$\mathscr{E}_{\boldsymbol{u}}(k \to 0) = \frac{I_{\boldsymbol{u}} k^2}{2\pi^2} + O(k^4)$$

$$\langle P_V^2 \rangle \equiv \left\langle \left[\int_{V=R^3} \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x}) \right]^2 \right\rangle \propto R^3, \quad L \ll k^{-1} \ll k_c^{-1}$$

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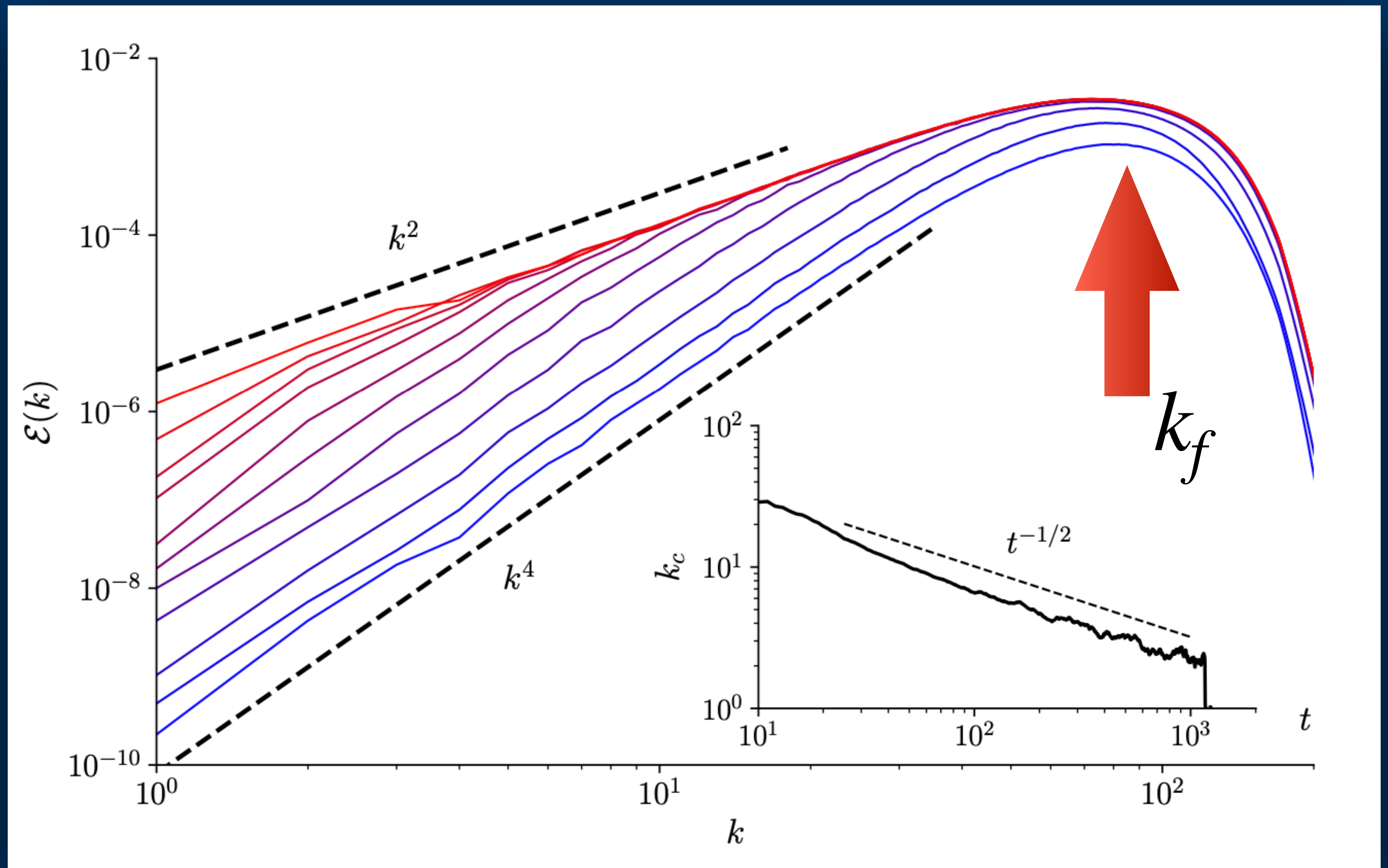

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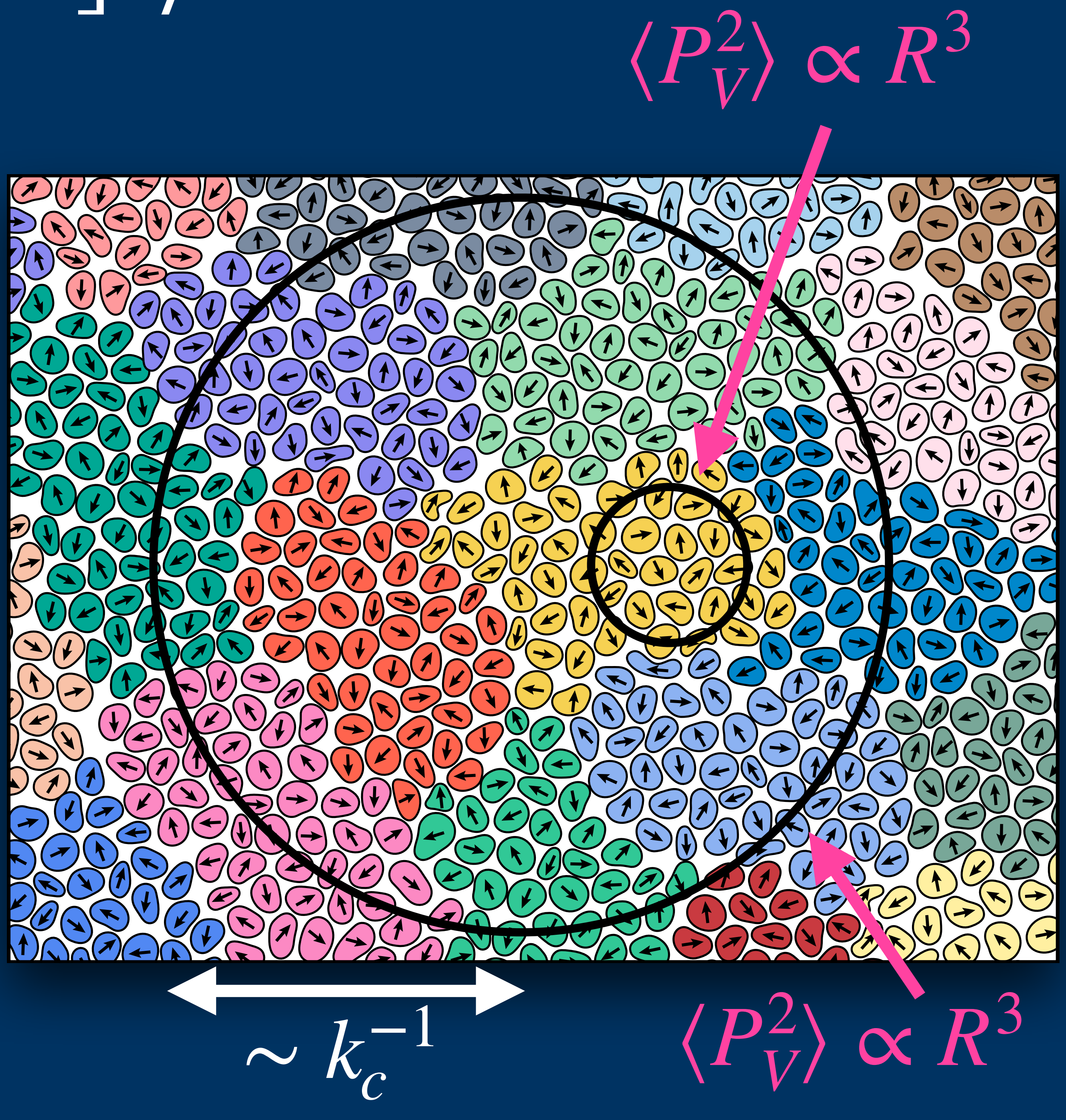

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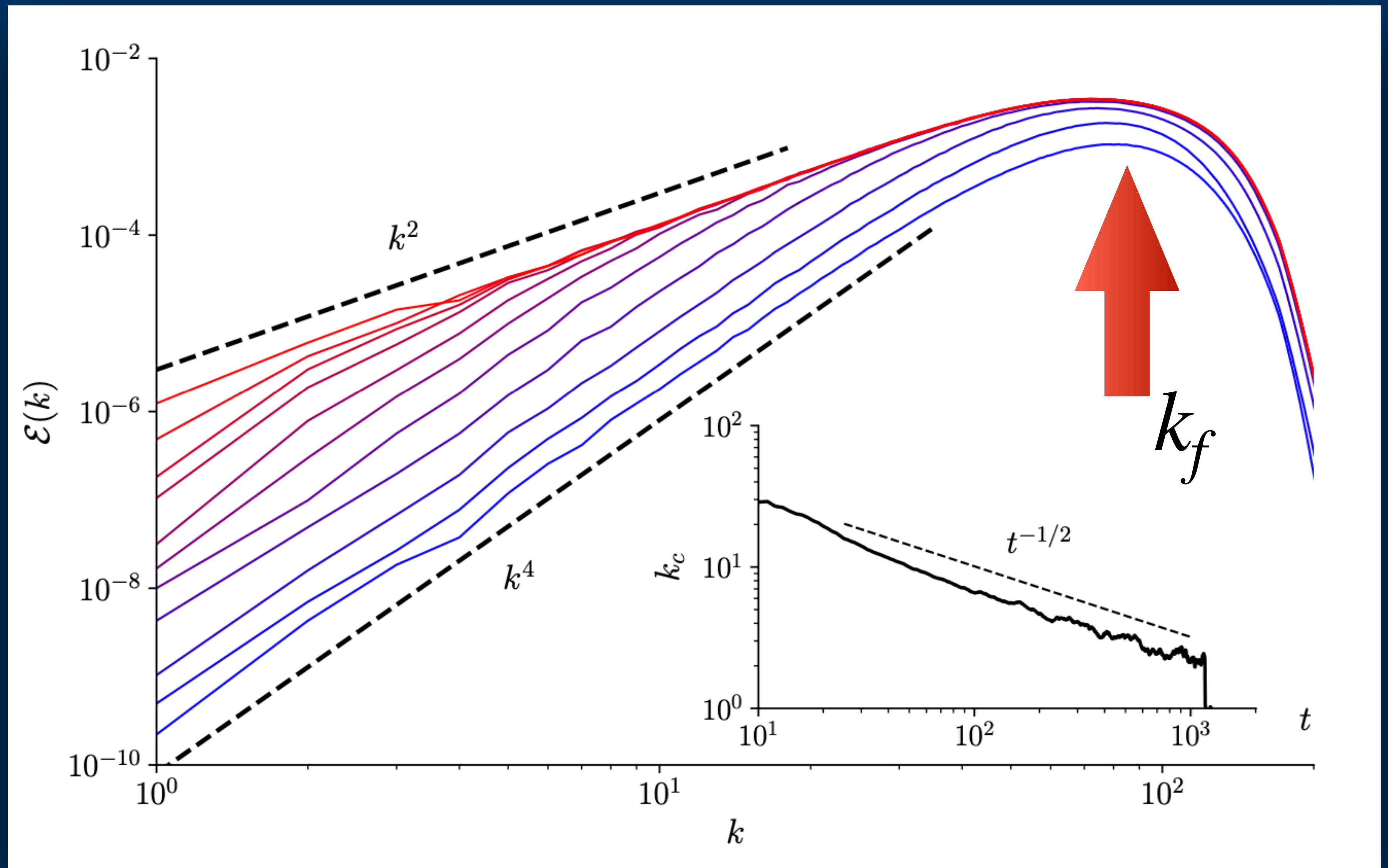

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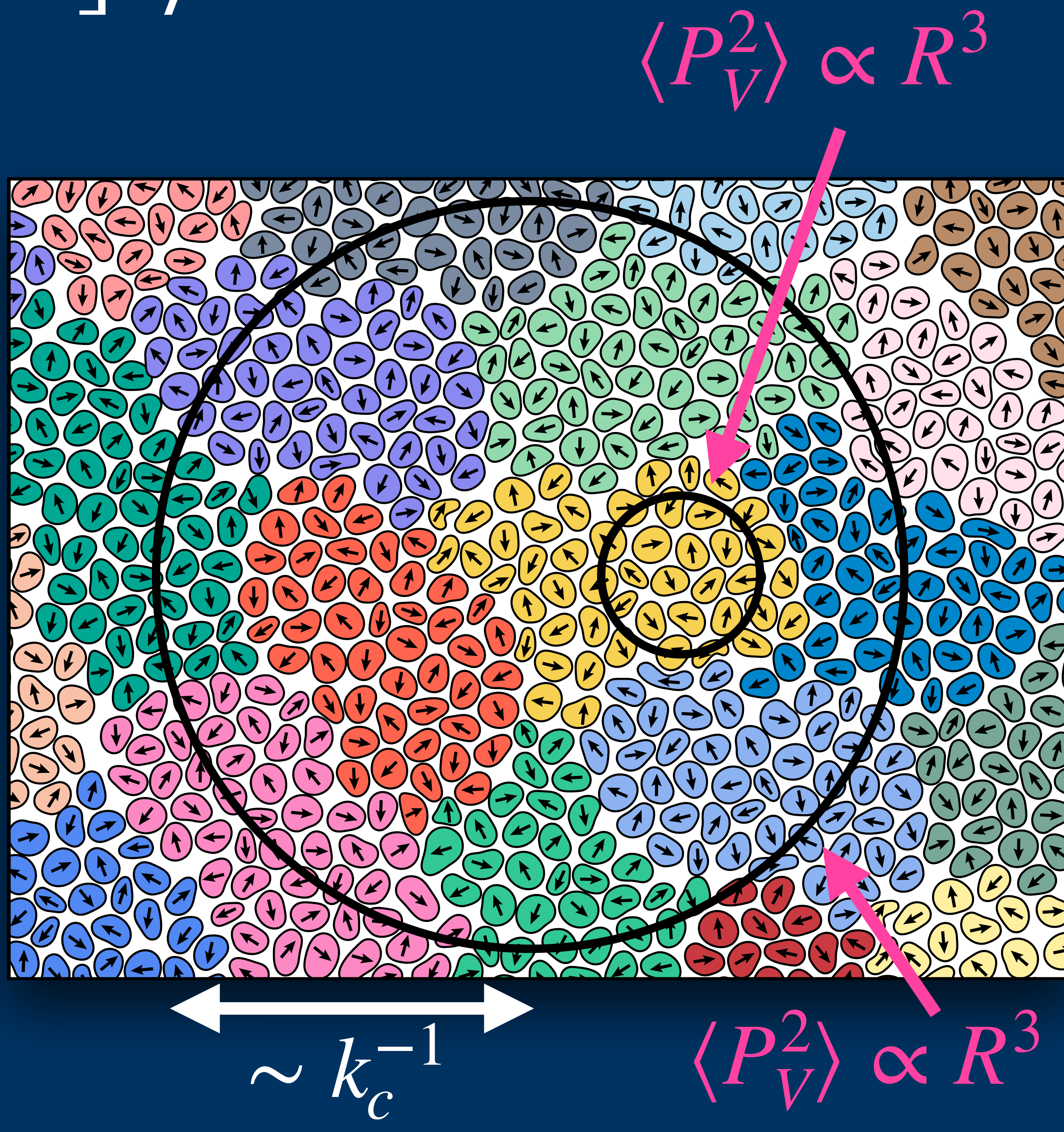

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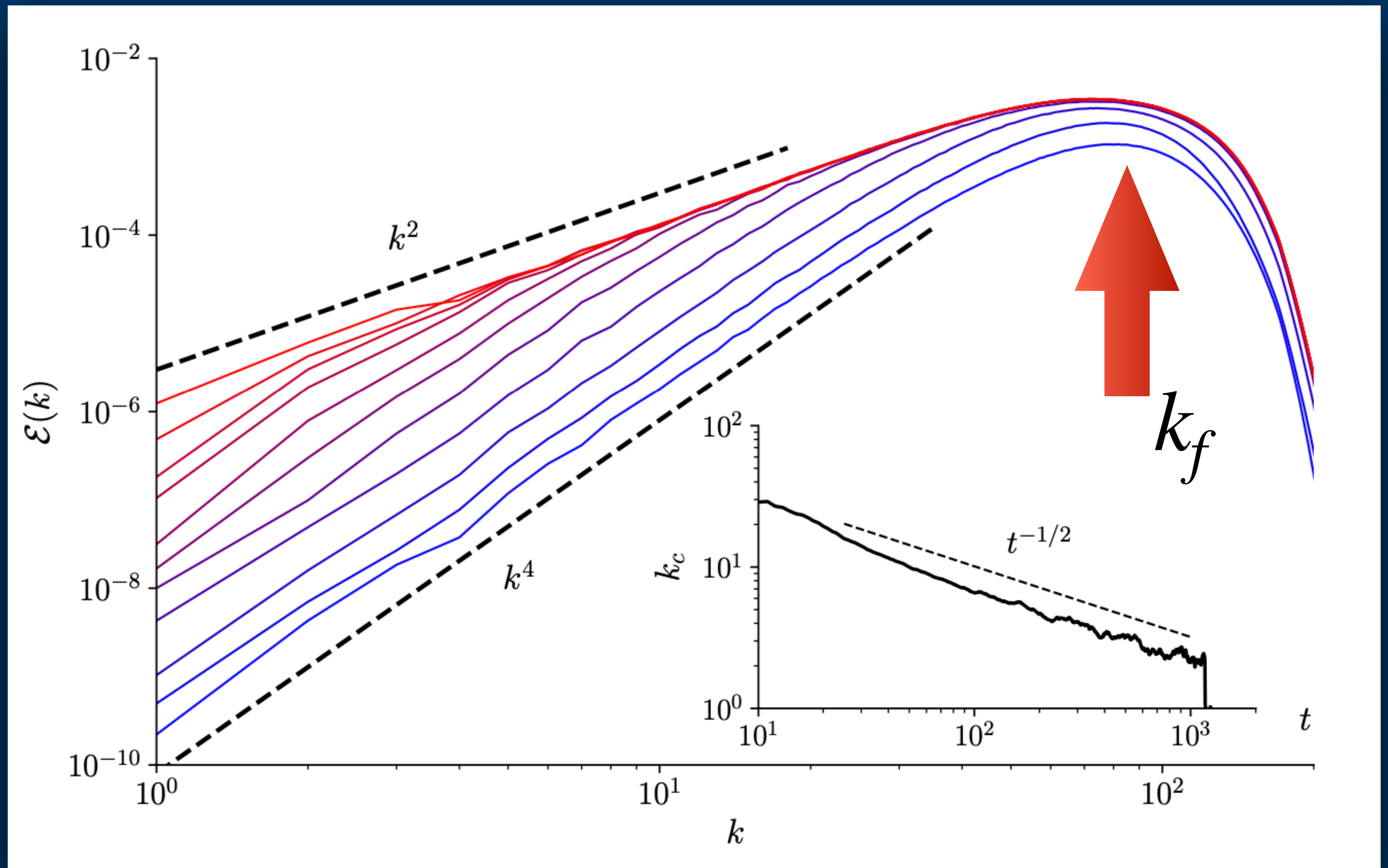

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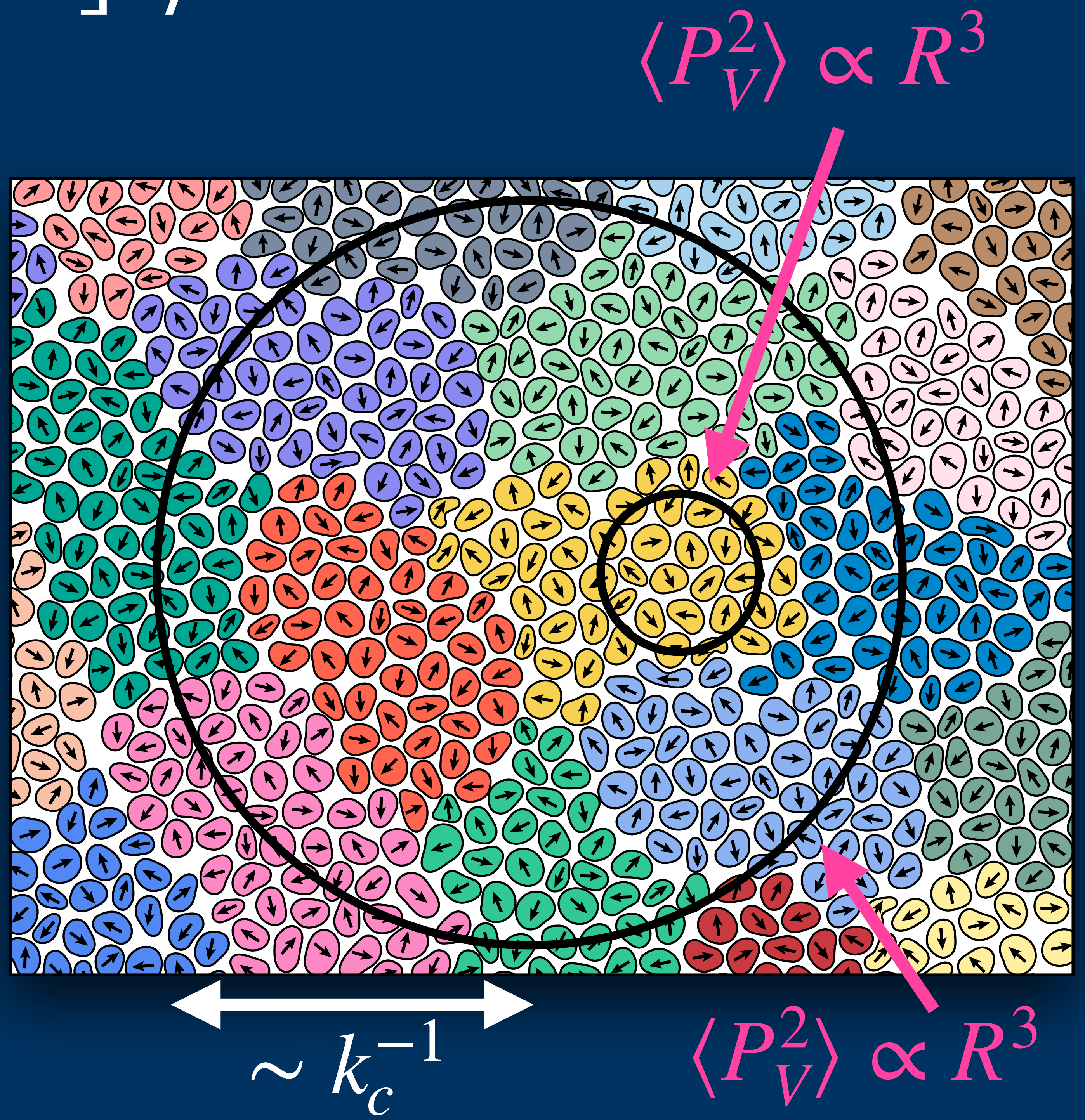

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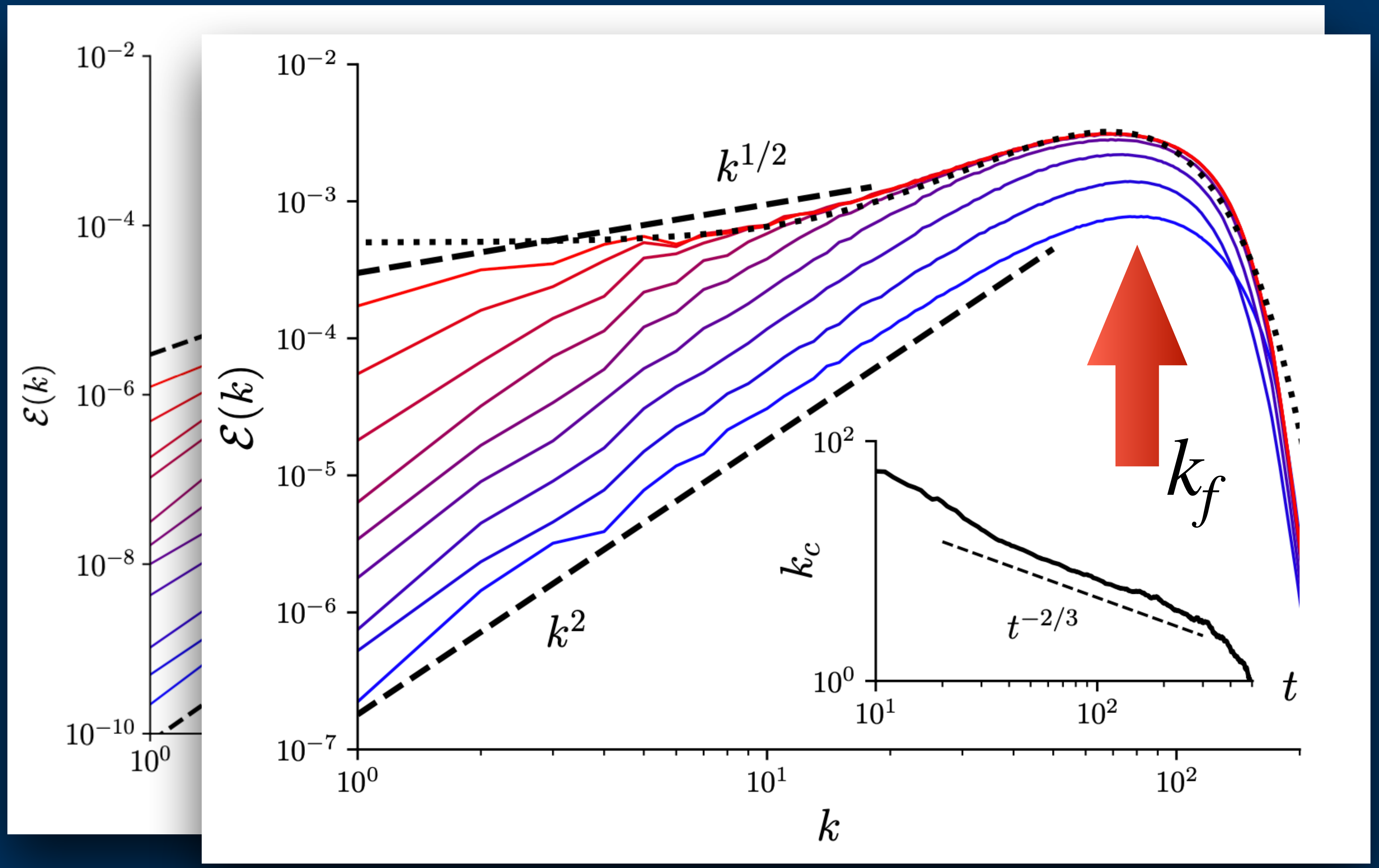

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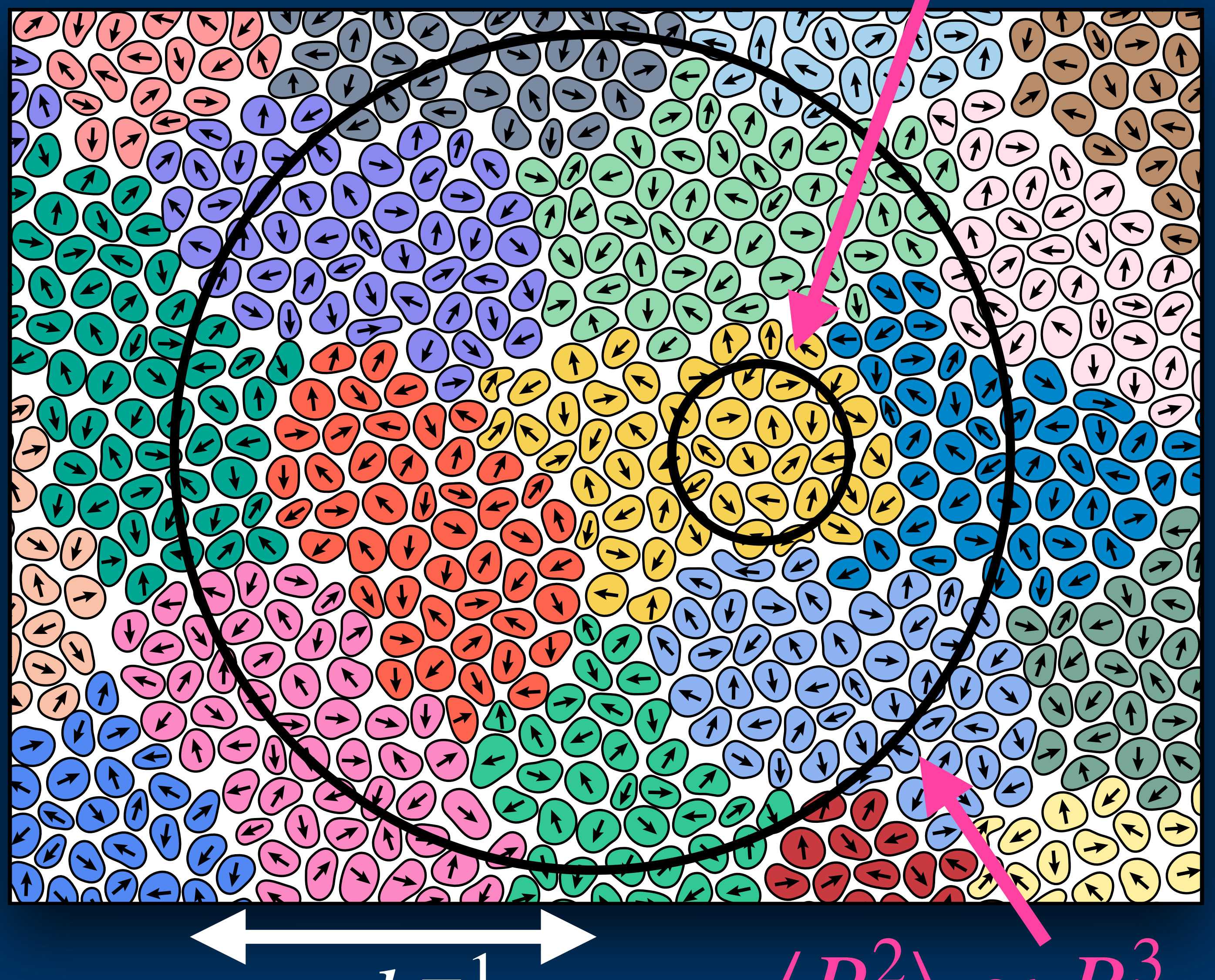

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Conclusion is that, loosely, there must be anomalously large fluctuations in kinetic helicity at small scales in hydro turbulence (cf. Milanese *et. al.* 2022).

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- Levich 2009 presents an correlation-integral-based argument for the development of anomalously strong kinetic-helicity fluctuations at the small scales of hydrodynamic turbulence. Might similar ideas be applicable to cross-helicity? I.e., to scale-dependent alignment of $\boldsymbol{u}$ and $\boldsymbol{B}$?