

Linear response theory and damped modes of stellar clusters

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Linear response theory

Klimontovich equation

Describing one **realisation** in **phase space** $\mathbf{w} = (\mathbf{x}, \mathbf{v})$

Empirical DF

$$F_d(\mathbf{w}, t) = \sum_{i=1}^N m \delta_D(\mathbf{w} - \mathbf{w}_i(t))$$

3D gravitational systems

$$U_{\text{ext}} = \frac{|\mathbf{v}|^2}{2G}$$

$$U = -\frac{G}{|\mathbf{r} - \mathbf{r}'|}$$

Empirical Hamiltonian

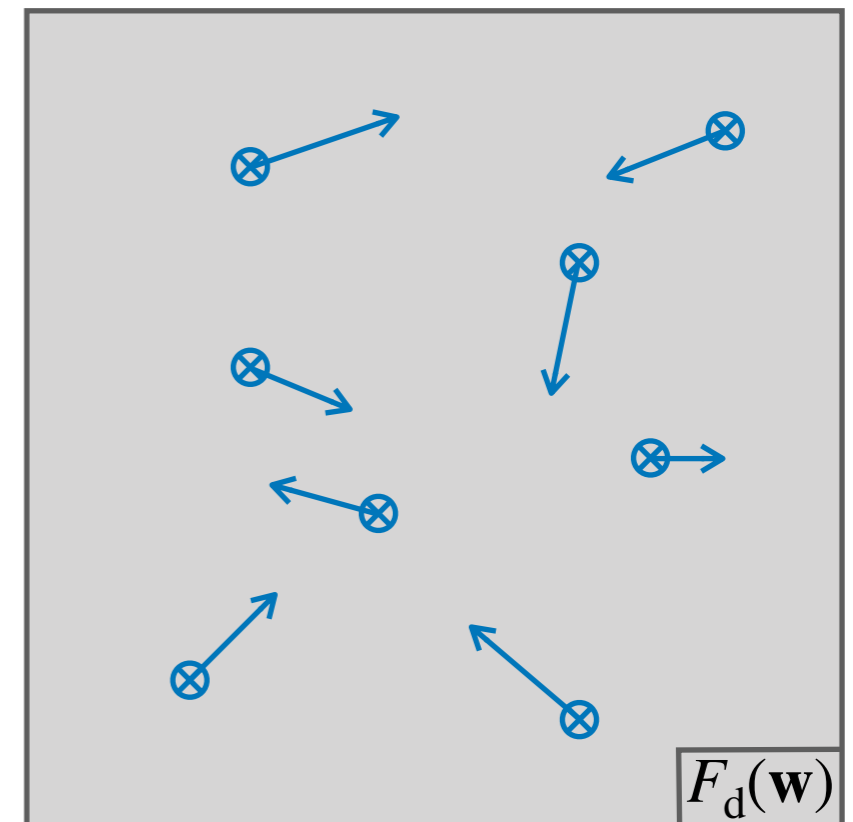
$$H_d(\mathbf{w}, t) = U_{\text{ext}}(\mathbf{w}) + \int d\mathbf{w}' F_d(\mathbf{w}', t) U(\mathbf{w}, \mathbf{w}')$$

Continuity equation in phase space

$$\frac{\partial F_d}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot \left(F_d \dot{\mathbf{w}} \right) = 0$$

Exact **Klimontovich** equation

$$\frac{\partial F_d}{\partial t} + [F_d, H_d] = 0$$



Phase space

Solving Klimontovich

Perturbative expansion

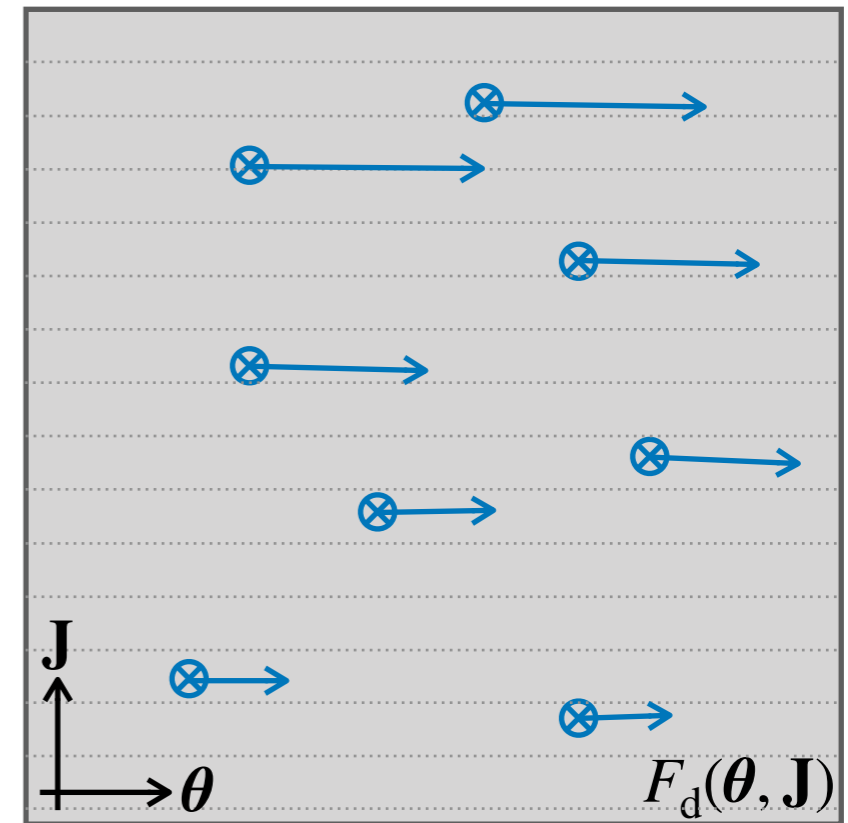
$$\begin{cases} F_d = F_0 + \delta F & \text{with } \langle \delta F \rangle = 0, \\ H_d = H_0 + \delta H & \text{with } \langle \delta H \rangle = 0. \end{cases}$$

Adiabatic approximation

$$\begin{cases} F_0 = F_0(\mathbf{J}, t), \\ H_0 = H_0(\mathbf{J}, t). \end{cases}$$

Quasi-linear evolution equations

$$\begin{aligned} \frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] &= 0 \\ \frac{\partial F_0}{\partial t} &= - \langle [\delta F, \delta H] \rangle \end{aligned}$$



Angle-Action space

Timescale separation

$$\begin{cases} T_{\delta F} \simeq T_{\text{dyn}} \\ T_{F_0} \simeq (\sqrt{N})^2 \times T_{\delta F} \end{cases}$$

Dynamics of fluctuations

Fast evolution of **perturbations** (Linearised Klimontovich equation)

$$\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] = 0$$

$$[\delta F, H_0]$$

Mean-field advection

$$[F_0, \delta H]$$

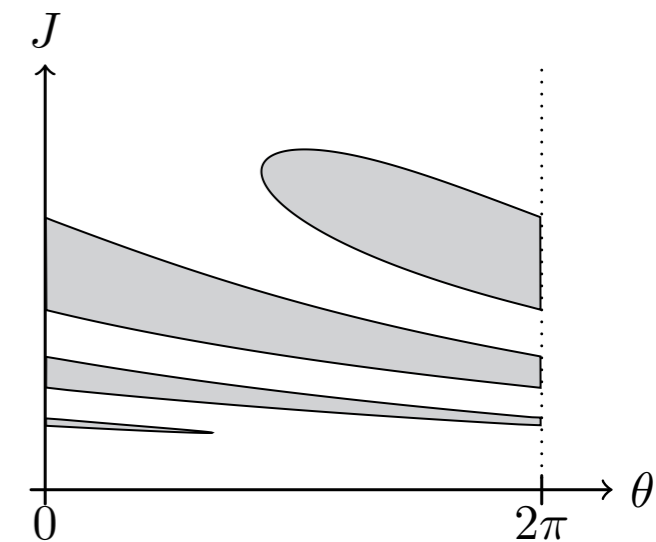
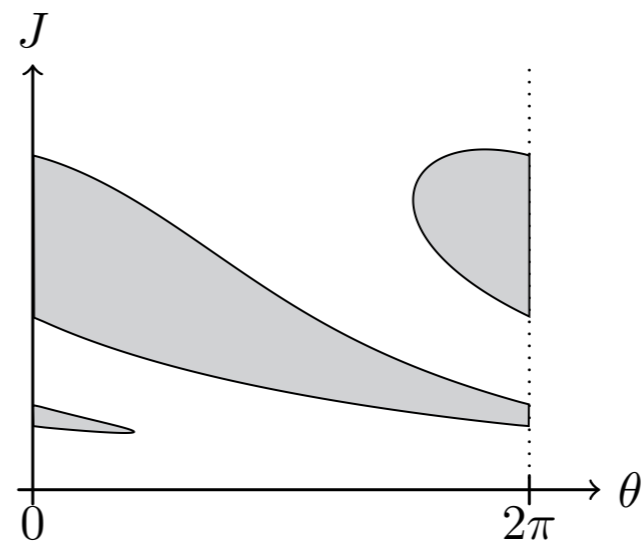
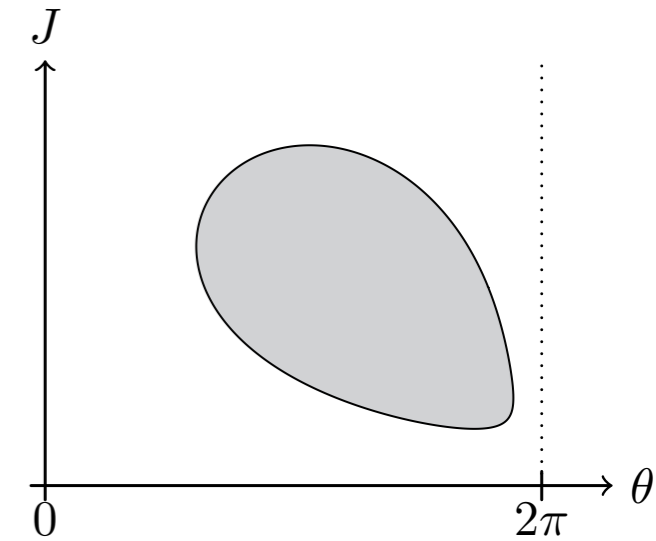
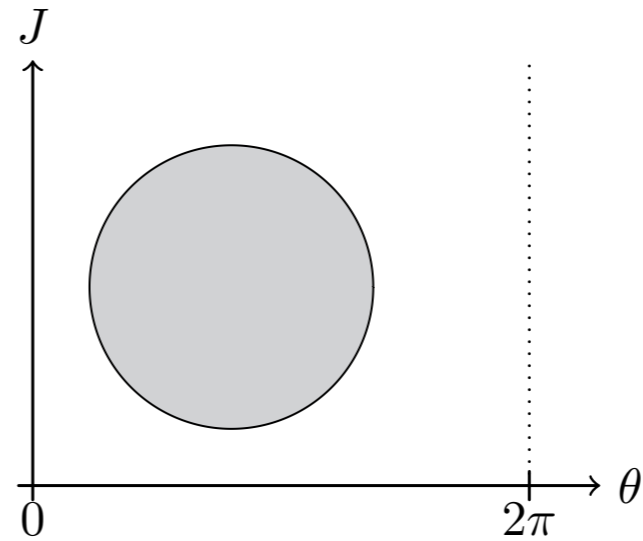
Collective effects

Self-consistent amplification

$$\delta H = \delta H [\delta F]$$

Timescale separation

$$\begin{cases} F_0(\mathbf{J}) = \text{cst} \\ H_0(\mathbf{J}) = \text{cst} \end{cases}$$



Phase Mixing

Solving for the fluctuations

Linear amplification

$$\delta\hat{F}_{\mathbf{k}}(\mathbf{J}, \omega) = - \frac{\delta F_{\mathbf{k}}(\mathbf{J}, 0)}{i(\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}))} - \frac{\mathbf{k} \cdot \partial F_0 / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \delta\hat{H}_{\mathbf{k}}(\mathbf{J}, \omega)$$

Bare noise Self-consistent amplification

with the **self-consistency**

$$\delta H(\mathbf{w}, t) = \int d\mathbf{w}' \delta F(\mathbf{w}', t) U(\mathbf{w}, \mathbf{w}') \quad U = -\frac{G}{|\mathbf{r} - \mathbf{r}'|}$$

Generic form of a **Fredholm equation**

$$[\delta H(\mathbf{J})]_{\text{dressed}} = [\delta H(\mathbf{J})]_{\text{bare}} + \int d\mathbf{J}' M(\mathbf{J}, \mathbf{J}') [\delta H(\mathbf{J}')]_{\text{dressed}}$$

Amplification kernel

Dressing of perturbations

$$[\delta H(\omega)]_{\text{dressed}} \simeq \frac{[\delta H(\omega)]_{\text{bare}}}{1 - M(\omega)} = \frac{[\delta H(\omega)]_{\text{bare}}}{|\varepsilon(\omega)|}$$

Solving for the fluctuations

Linear amplification

$$\delta\hat{F}_{\mathbf{k}}(\mathbf{J}, \omega) = - \frac{\delta F_{\mathbf{k}}(\mathbf{J}, 0)}{i(\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}))} - \frac{\mathbf{k} \cdot \partial F_0 / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \delta\hat{H}_{\mathbf{k}}(\mathbf{J}, \omega)$$

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Amplification kernel

Plasma dielectric function

$$\varepsilon_{\mathbf{k}}(\omega) = 1 + \frac{1}{k^2 \lambda_D^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega}$$

Gravitational response matrix

$$\boldsymbol{\varepsilon}_{pq}(\omega) = \mathbf{I} - \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Some properties

$$\sum_{\mathbf{k}}$$

Sum over resonances

$$\int d\mathbf{J}$$

Scan over orbital space

$$\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega$$

Resonant amplification

$$\psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Long-range interaction

Mode

$$\det[\boldsymbol{\varepsilon}(\omega)] = 0$$

Type of modes

$$\begin{cases} \text{Im}[\omega] > 0 & \text{Unstable} \\ \text{Im}[\omega] = 0 & \text{Neutral} \\ \text{Im}[\omega] < 0 & \text{Damped} \end{cases}$$

Plasmas

$$(\mathbf{x}, \mathbf{v})$$

Orbital coordinates

Galaxies

$$(\boldsymbol{\theta}, \mathbf{J})$$

Basis decomposition

$$U(\mathbf{x}, \mathbf{x}') \propto \int \frac{d\mathbf{k}}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}$$

$$U(\mathbf{w}, \mathbf{w}') = - \sum_p \psi^{(p)}(\mathbf{w}) \psi^{(p)*}(\mathbf{w}')$$

Dielectric function

$$1 + \frac{1}{k^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega}$$

$$\delta_{pq} - \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Resonance condition

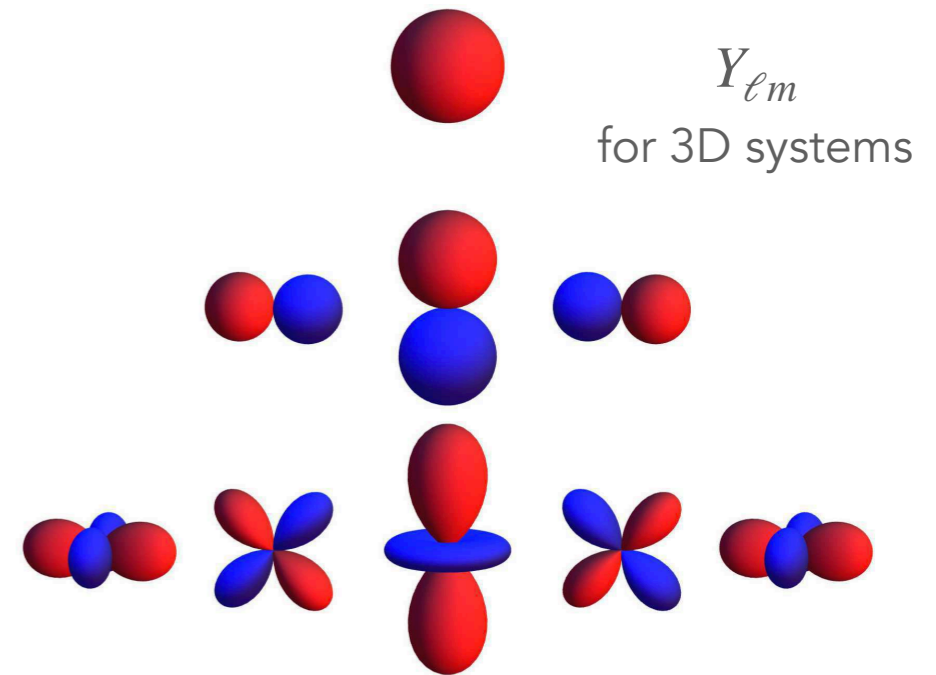
$$\delta_D(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'))$$

$$\delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \boldsymbol{\Omega}(\mathbf{J}'))$$

**How to compute the
dispersion function?**

Basis method $(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w}))$

$$\begin{cases} \psi^{(p)}(\mathbf{w}) = \int d\mathbf{w}' U(\mathbf{w}, \mathbf{w}') \rho^{(p)}(\mathbf{w}'), \\ \int d\mathbf{w} \psi^{(p)}(\mathbf{w}) \rho^{(q)*}(\mathbf{w}) = -\delta_{pq}. \end{cases}$$



“Separable” pairwise interaction

$$U(\mathbf{w}, \mathbf{w}') = - \sum_p \psi^{(p)}(\mathbf{w}) \psi^{(p)*}(\mathbf{w}')$$

Plasmas

$$U(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \int \frac{d\mathbf{k}}{|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

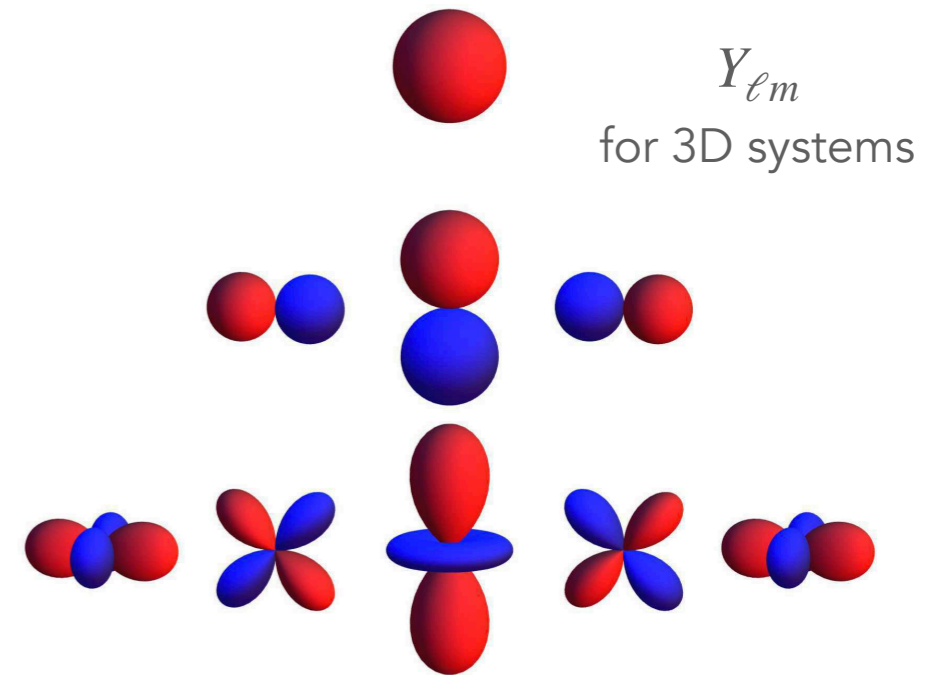
Galaxies

$$\Delta\Phi = 4\pi G\rho$$

Poisson equation

Basis method $(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w}))$

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Newtonian interaction

$$\begin{aligned} U(\mathbf{r}, \mathbf{r}') &= -\frac{G}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\int \frac{d\mathbf{k}}{\mathbf{k}^2} e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \\ &= -\sum_{\ell, m} Y_{\ell m}(\hat{\mathbf{r}}) Y_{\ell m}(\hat{\mathbf{r}}') \frac{\text{Min}[r, r']^\ell}{\text{Max}[r, r']^{\ell+1}} \end{aligned}$$

Scale invariance

Translation invariance

Rotation invariance

Biorthogonal basis

What matters is the **mean potential**

$$\begin{cases} \rho_{\ell=0,n=1}(r) = \rho_0(r), \\ \rho_{\ell=1,n=1}(r) = d\rho_0/dr, \\ \dots \end{cases}$$

cf. Self-consistent field simulations

What matters are the **perturbations**

$$\delta\rho(\mathbf{r}, t) = \sum_p A_p(t) \rho^{(p)}(\mathbf{r})$$

cf. Linear response in time domain

What matters is the **pairwise interaction**

$$U(\mathbf{r}, \mathbf{r}') = - \sum_p \psi^{(p)}(\mathbf{r}) \psi^{(p)*}(\mathbf{r}')$$

cf. Kinetic theory

How to chose the basis?

Self-consistent amplification

Linear response

$$\left[\delta H(\mathbf{J}) \right]_{\text{dressed}} = \left[\delta H(\mathbf{J}) \right]_{\text{bare}} + \int d\mathbf{J}' M(\mathbf{J}, \mathbf{J}') \left[\delta H(\mathbf{J}') \right]_{\text{dressed}}$$

Amplification kernel

In terms of **coupling coefficients**

$$\psi_{\mathbf{k}\mathbf{k}'}^{\text{d}}(\mathbf{J}, \mathbf{J}', \omega) = \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}') + (2\pi)^d \sum_{\mathbf{k}''} \int d\mathbf{J}'' \frac{\mathbf{k}'' \cdot \partial F / \partial \mathbf{J}''}{\mathbf{k}'' \cdot \boldsymbol{\Omega}(\mathbf{J}'') - \omega} \psi_{\mathbf{k}\mathbf{k}''}(\mathbf{J}, \mathbf{J}'') \psi_{\mathbf{k}''\mathbf{k}}^{\text{d}}(\mathbf{J}'', \mathbf{J}', \omega)$$

$\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J}, \mathbf{J}')$ Bare coefficient, Landau

$\psi_{\mathbf{k}\mathbf{k}'}^{\text{d}}(\mathbf{J}, \mathbf{J}', \omega)$ Dressed coefficient, Balescu-Lenard

Can one compute the dressed coefficients without any basis?

Gravitational response matrix

$$\boldsymbol{\varepsilon}_{pq}(\omega) = \mathbf{I} - \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Some properties

$$\sum_{\mathbf{k}}$$

Sum over resonances

$$\int d\mathbf{J}$$

Scan over orbital space

$$\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega$$

Resonant amplification

$$\psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Long-range interaction

Mode

$$\det[\boldsymbol{\varepsilon}(\omega)] = 0$$

Type of modes

$$\begin{cases} \text{Im}[\omega] > 0 & \text{Unstable} \\ \text{Im}[\omega] = 0 & \text{Neutral} \\ \text{Im}[\omega] < 0 & \text{Damped} \end{cases}$$

Landau's prescription

$$\int_{\mathcal{L}}^{\infty} du \frac{G(u)}{u - \omega} = \begin{cases} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} & \text{if } \text{Im}[\omega] > 0 \\ \mathcal{P} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + i\pi G(\omega) & \text{if } \text{Im}[\omega] = 0 \\ \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + 2i\pi G(\omega) & \text{if } \text{Im}[\omega] < 0 \end{cases}$$

Unstable, e.g., ROI

Neutral, e.g., BL

Damped, e.g., sloshing

Some remarks

“Causality breaking”

$$+ \text{Im}[\omega] \quad \text{vs} \quad - \text{Im}[\omega]$$

“Aligned” resonant denominator

$$\frac{1}{u - \omega} \quad \text{vs} \quad \frac{1}{f(u) - \omega}$$

Analytic integrand

$$G(\omega) \quad \text{for } \omega \in \mathbb{C}$$

Infinite frequency support

$$\int_{-\infty}^{+\infty} du \quad \text{vs} \quad \int_{-1}^1 du$$

Landau's prescription

$$\int_{\mathcal{L}}^{\pm\infty} du \frac{G(u)}{u - \omega} = \begin{cases} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} & \text{if } \text{Im}[\omega] > 0 & \textit{Unstable, e.g., ROI} \\ \mathcal{P} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + i\pi G(\omega) & \text{if } \text{Im}[\omega] = 0 & \textit{Neutral, e.g., BL} \\ \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + 2i\pi G(\omega) & \text{if } \text{Im}[\omega] < 0 & \textit{Damped, e.g., sloshing} \end{cases}$$

Plasmas

$$\varepsilon_{\mathbf{k}}(\omega) = 1 + \frac{1}{k^2 \lambda_D^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega}$$

Resonant denominator is aligned $u = \mathbf{k} \cdot \mathbf{v}$ Integrand is typically **analytic** $F(v) \propto e^{-v^2/2}$ Frequency support is typically **infinite** $\int_{-\infty}^{+\infty} dv$ "Vanilla" Maxwellian case $Z[\text{zeta}_-] := \text{I Sqrt}[\text{Pi}] \text{Exp}[-\text{zeta}^2] (1 + \text{I Erfi}[\text{zeta}])$

Aligning the denominator

One can equivalently label **orbits** with their **frequencies**

$$M(\omega) = \int d\mathbf{J} \frac{G(\mathbf{J})}{\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega}$$

To resonant frequency

$$u \propto \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})$$

$$M(\omega) = \int_{\mathcal{L}}^1 du \frac{G(u)}{u - \omega}$$

Analytic continuation

Initial expression

$$M(\omega) = \int_{\mathcal{L}}^{-1}^1 du \frac{G(u)}{u - \omega}$$

On $[-1, 1]$ with unit weight: **Legendre projection**

$$G(u) = \sum_k a_k P_k(u)$$

Polynomial, therefore analytic

Hence the **separable** writing

$$M(\omega) = \sum_k a_k D_k(\omega)$$

with

$$D_k(\omega) = \int_{\mathcal{L}}^{-1}^1 du \frac{P_k(u)}{u - \omega}$$

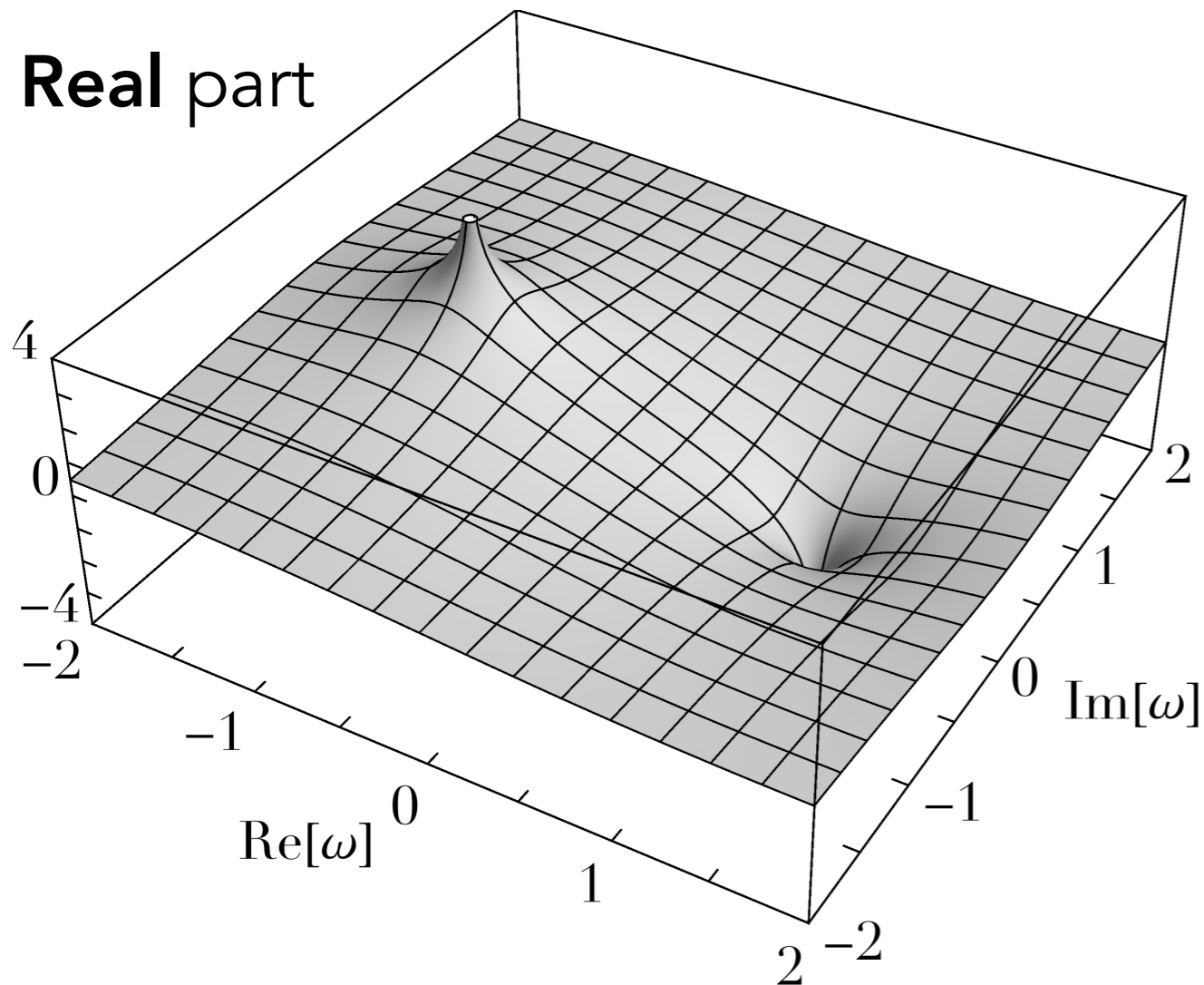
$\{F(\mathbf{J}), \Omega(\mathbf{J}), \psi^{(p)}(\mathbf{r})\}$

The resonant integral

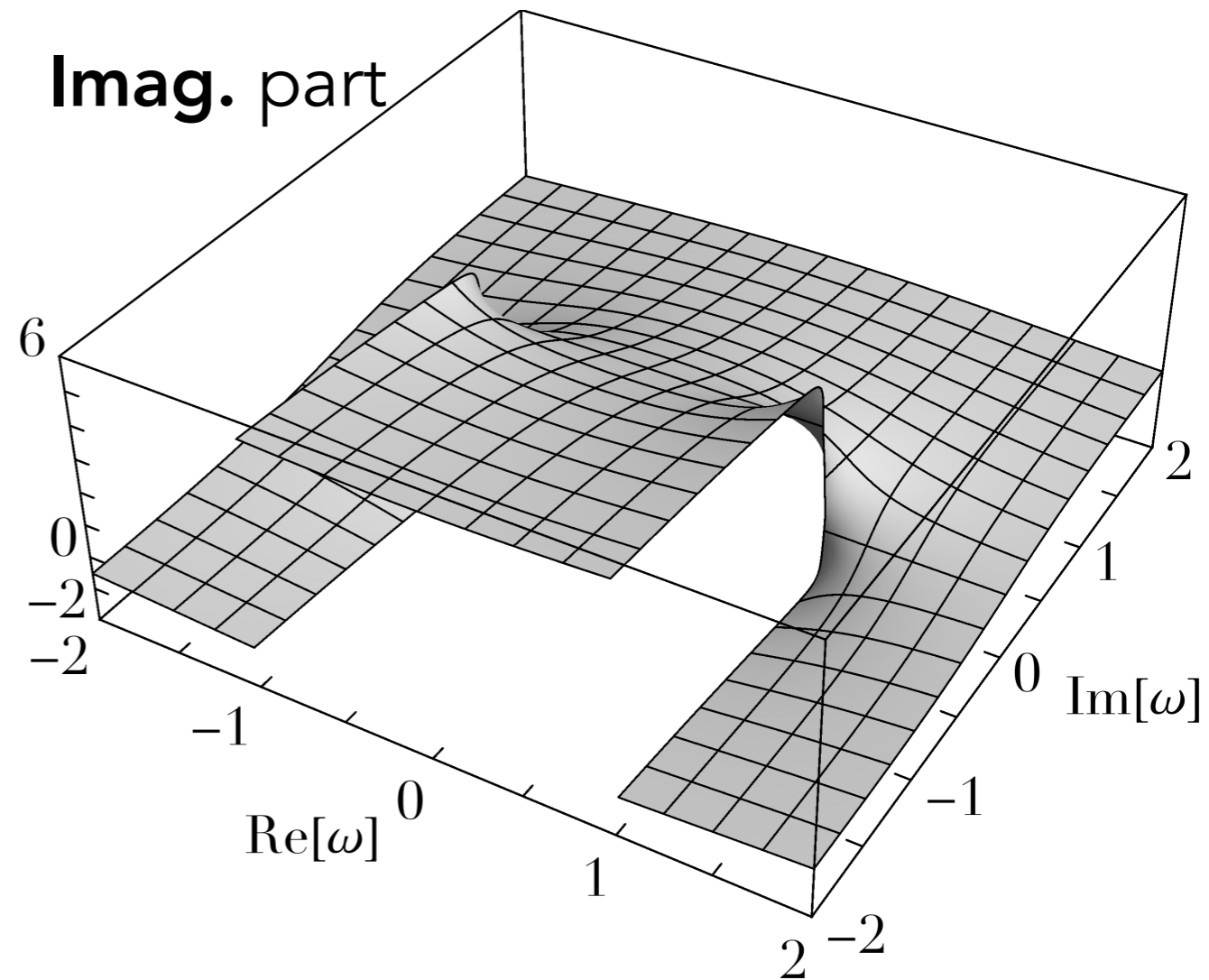
Finite frequency-domain

$$D_0(\omega) = \int_{\mathcal{L}}^{-1} du \frac{1}{u - \omega}$$

Real part



Imag. part



Only one difficult integral

We know the one integral

$$D_0(\omega) = \int_{\mathcal{L}}^1 du \frac{1}{u - \omega}$$

“Pain de sucre”

$$\begin{aligned} D_1(\omega) &= \int_{\mathcal{L}}^1 du \frac{u}{u - \omega} = \int_{\mathcal{L}}^1 du \frac{u - (\omega - \omega)}{u - \omega} \\ &= 2 + \omega D_0(\omega) \end{aligned}$$

Legendre recurrence gives $P_{k+2}(\omega) = \text{Linear}[P_k(\omega), P_{k+1}(\omega)]$

Hence, we know all

$$D_k(\omega) = \int_{\mathcal{L}}^1 du \frac{P_k(u)}{u - \omega}$$

Ready to compute

Generic expression

$$M_{pq}(\omega) = \sum_{\mathbf{k}} \sum_k a_k[p, q, \mathbf{k}] D_k(\varpi_{\mathbf{k}})$$

Projection to get $\{a_k\}$

$$\mathcal{O} \left[K \times N_{\text{radial}}^2 \times k_1^{\text{max}} \times \ell_{\text{max}} \times K_u \times K_v \right]$$

Evaluation to get $\mathbf{M}(\omega)$

$$\mathcal{O} \left[N_{\text{radial}}^2 \times k_1^{\text{max}} \times \ell_{\text{max}} \times K_u \right]$$

 K Sampling of the orbit-average N_{radial} Number of basis elements k_1^{max} Number of radial resonances ℓ_{max} Considered harmonics K_u Number of Legendre functions K_v Number of sampling 2nd dim.

Damped modes in globular clusters

Globular clusters

Dense, spherical stellar systems

Radii \sim a few parsecs

Contains $N \sim 10^5$ stars

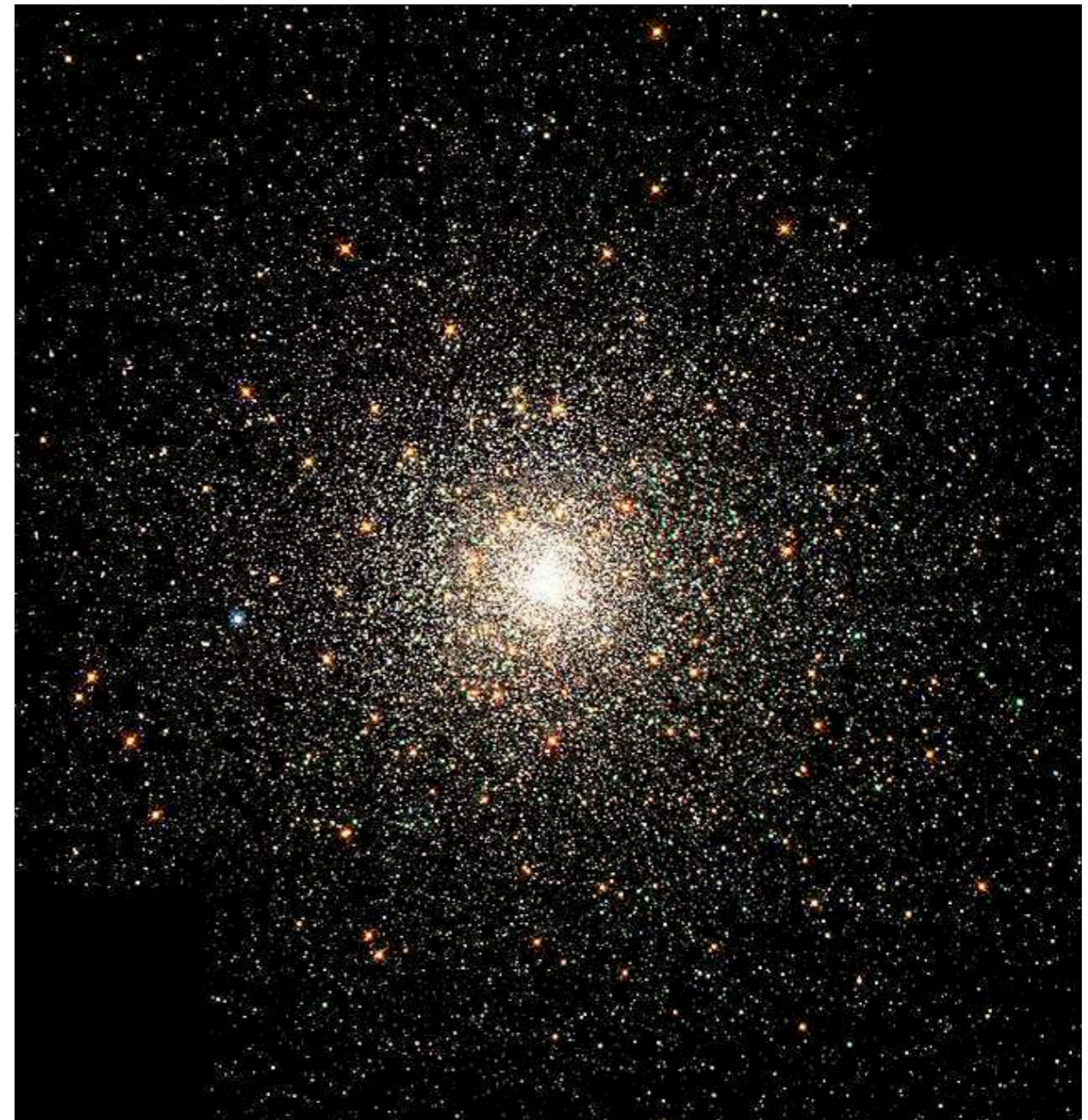
Very old $\sim 10^{10}$ yr

Crossing time $\sim 10^5$ yr

Relaxation time $\sim 10^{10}$ yr

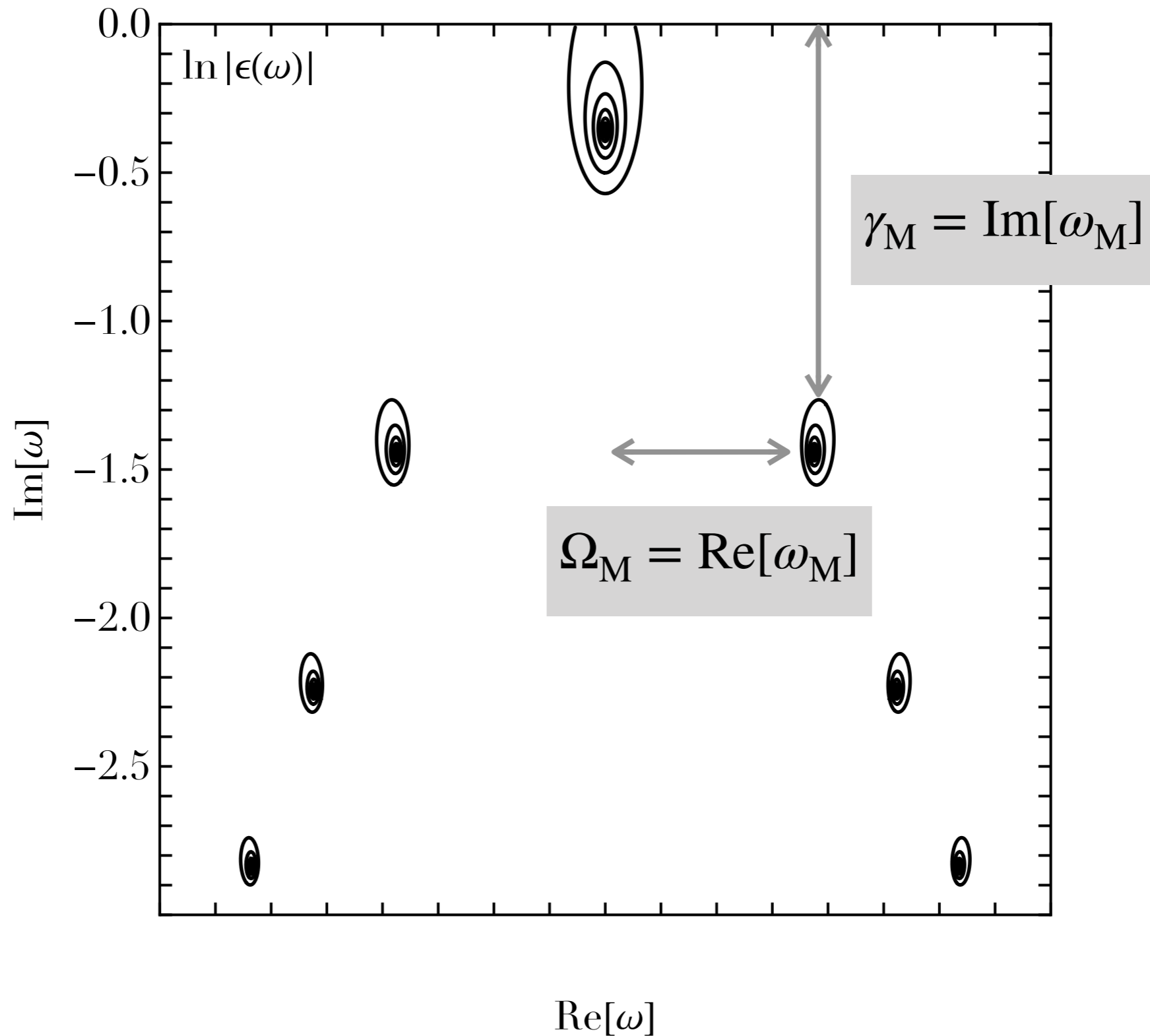
Expected to be **linearly stable**

No maximum entropy, i.e. no Maxwellian



Dispersion function

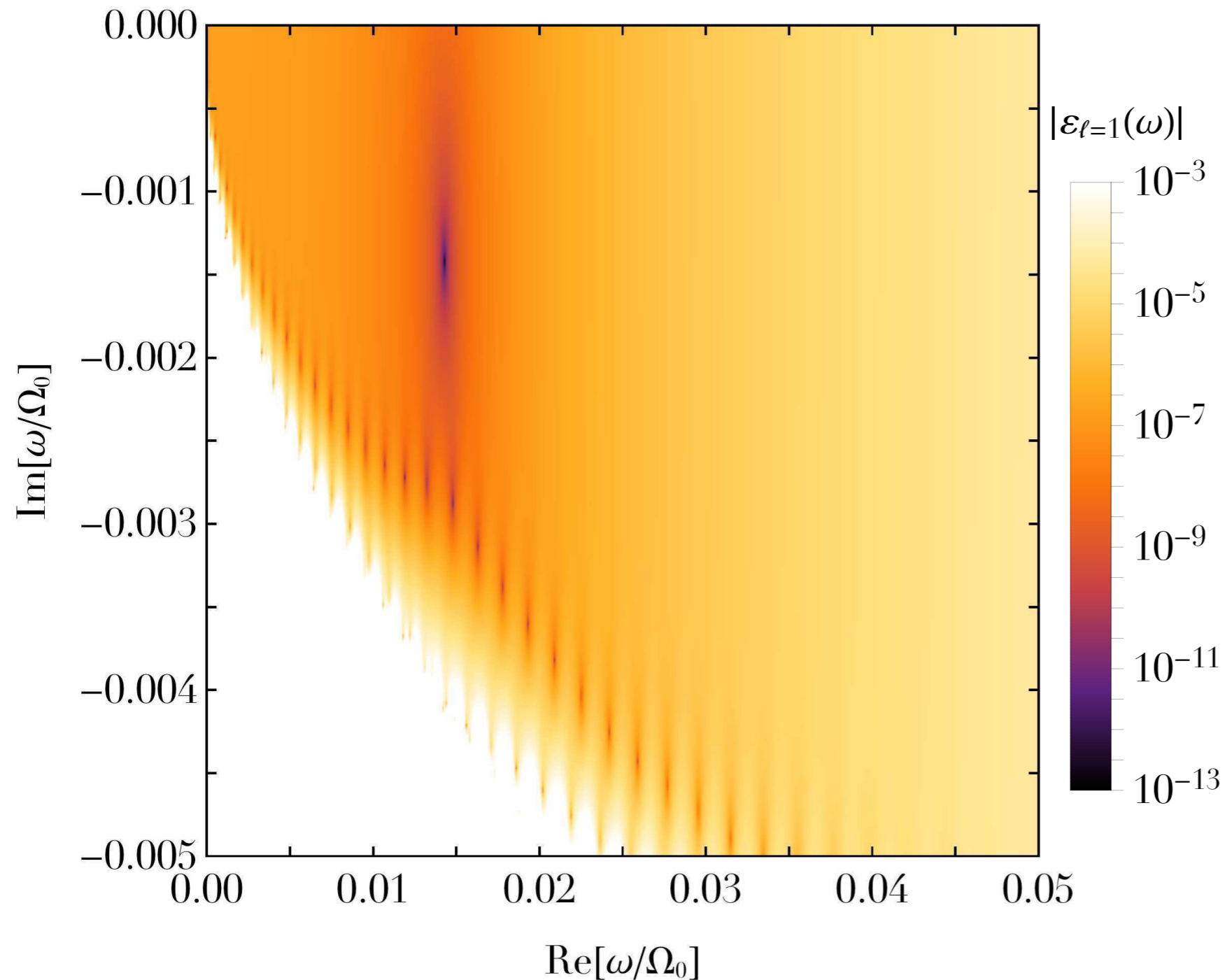
(Landau) damped modes in a (periodic) **plasma**



Dispersion function

Isotropic isochrone cluster

$$\det[\boldsymbol{\varepsilon}_\ell(\omega)]$$



$\ell = 1$ damped mode

$$\omega_{\text{M}}/\Omega_0 = 0.0143$$

$$-0.00142 i$$

Slow mode

$$\text{Re}[\omega_{\text{M}}]/\Omega_0 \ll 1$$

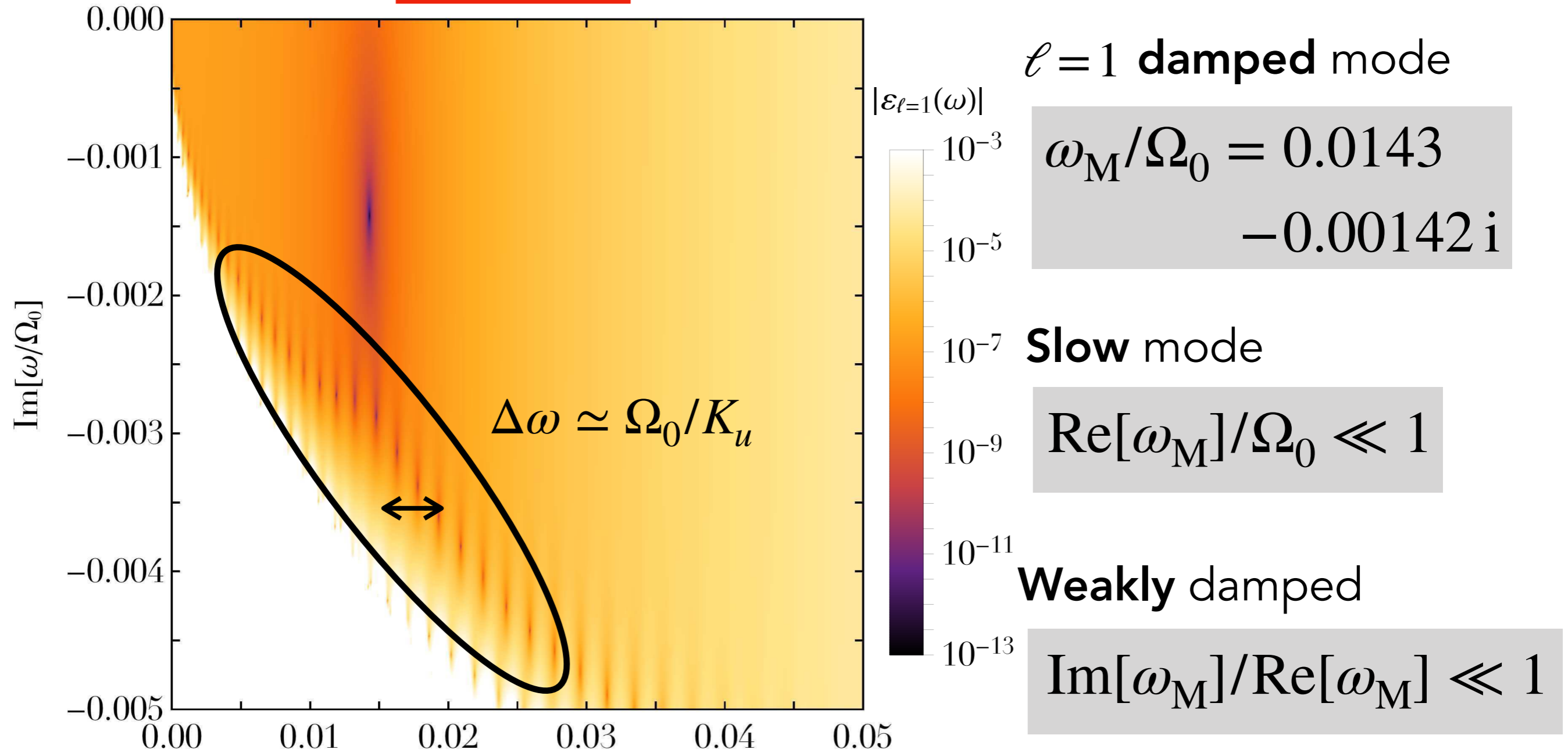
Weakly damped

$$\text{Im}[\omega_{\text{M}}]/\text{Re}[\omega_{\text{M}}] \ll 1$$

Dispersion function

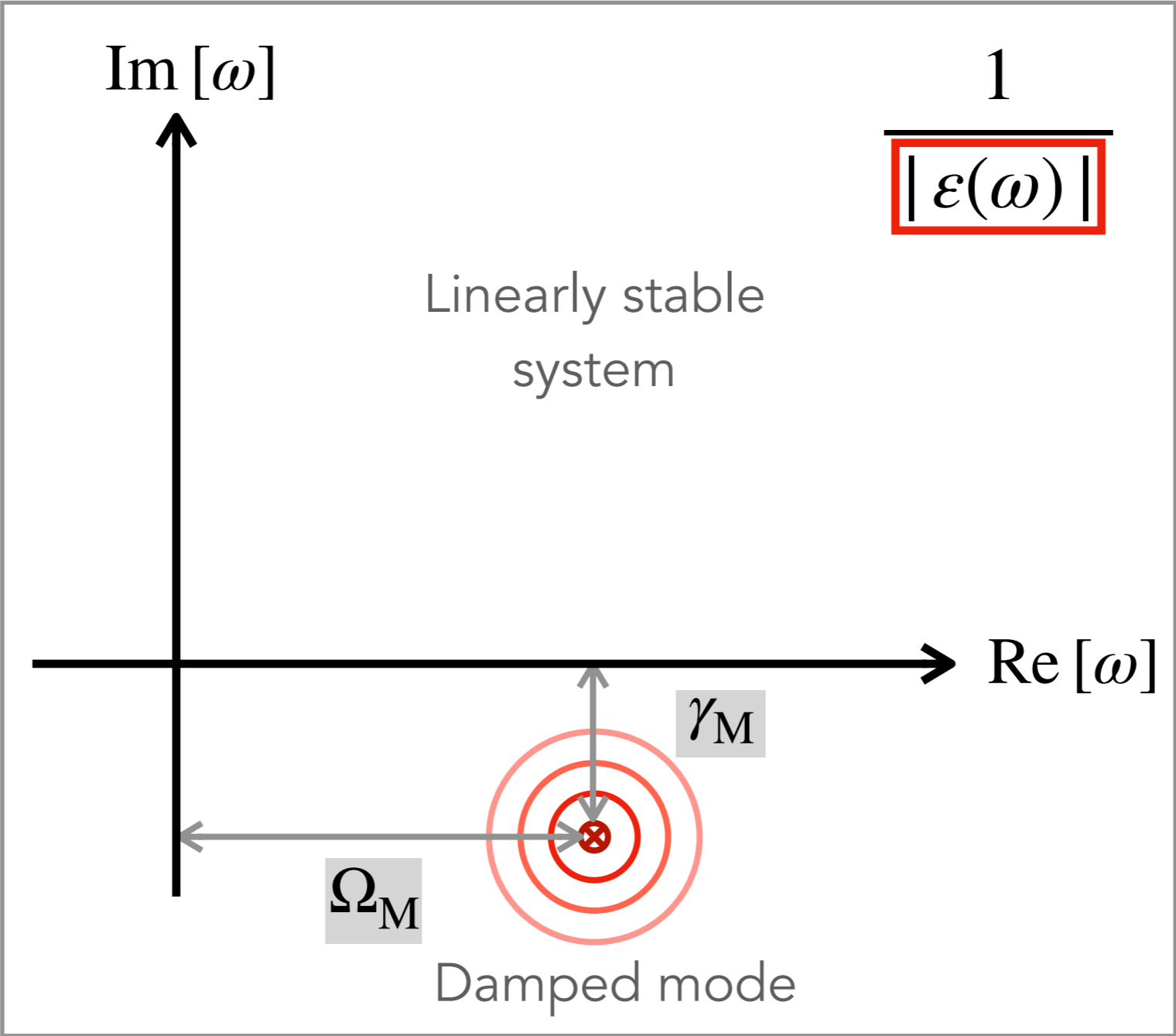
Isotropic isochrone cluster

$$\det[\boldsymbol{\varepsilon}_\ell(\omega)]$$



How to reduce the spurious oscillations stemming from Legendre?

Amplification



Susceptibility

$$\frac{1}{|\varepsilon(\Omega_M)|} \gg 1$$

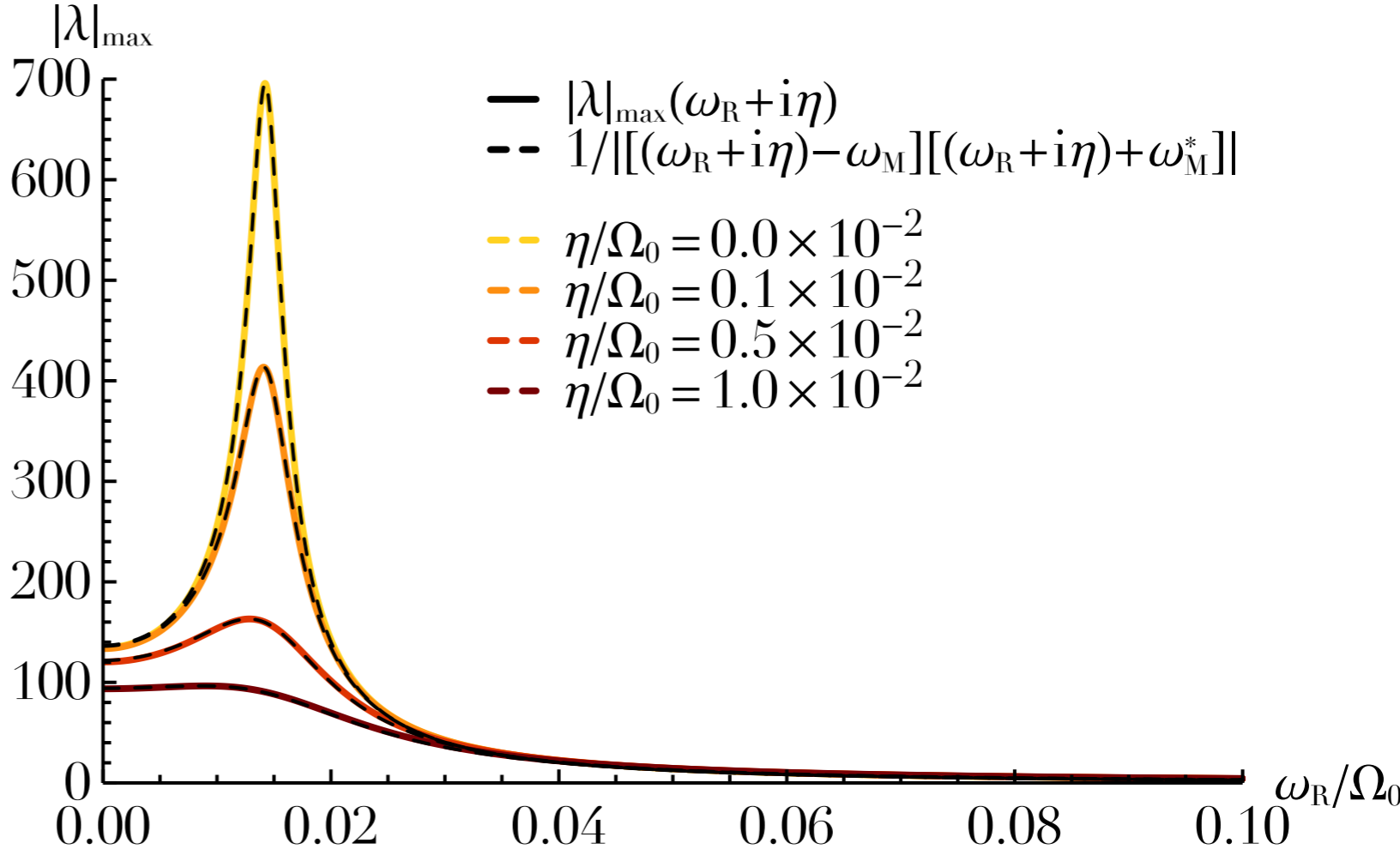
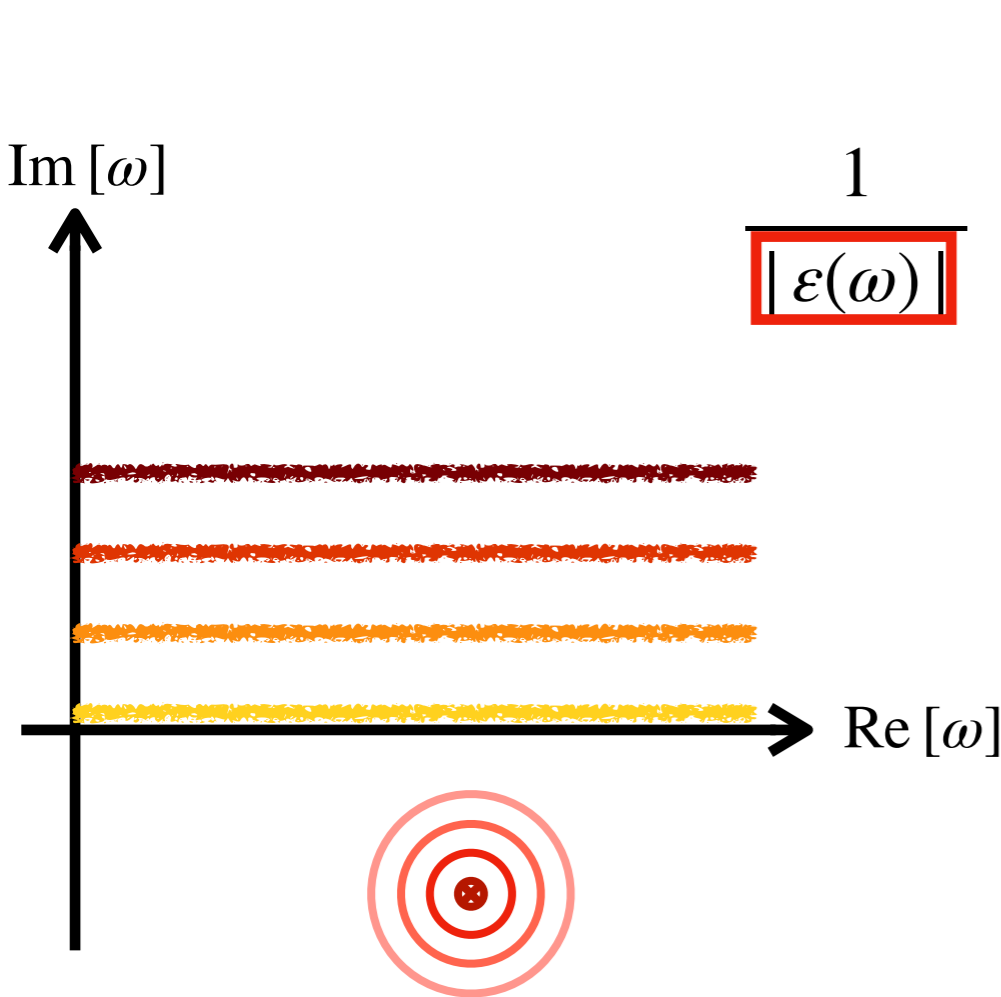
Thermalisation

$$[\delta H(t)]_{\text{trans.}} \simeq e^{\gamma_M t}$$

How strong is the amplification?

Amplification eigenvalue

$$\lambda(\omega) = \text{EigMax} \left[1/\epsilon(\omega) \right]$$



“Natural” and symmetric ansatz

$$\lambda(\omega) \propto \frac{1}{|(\omega - \omega_M)(\omega + \omega_M^*)|}$$

Why is such a simple ansatz so effective?

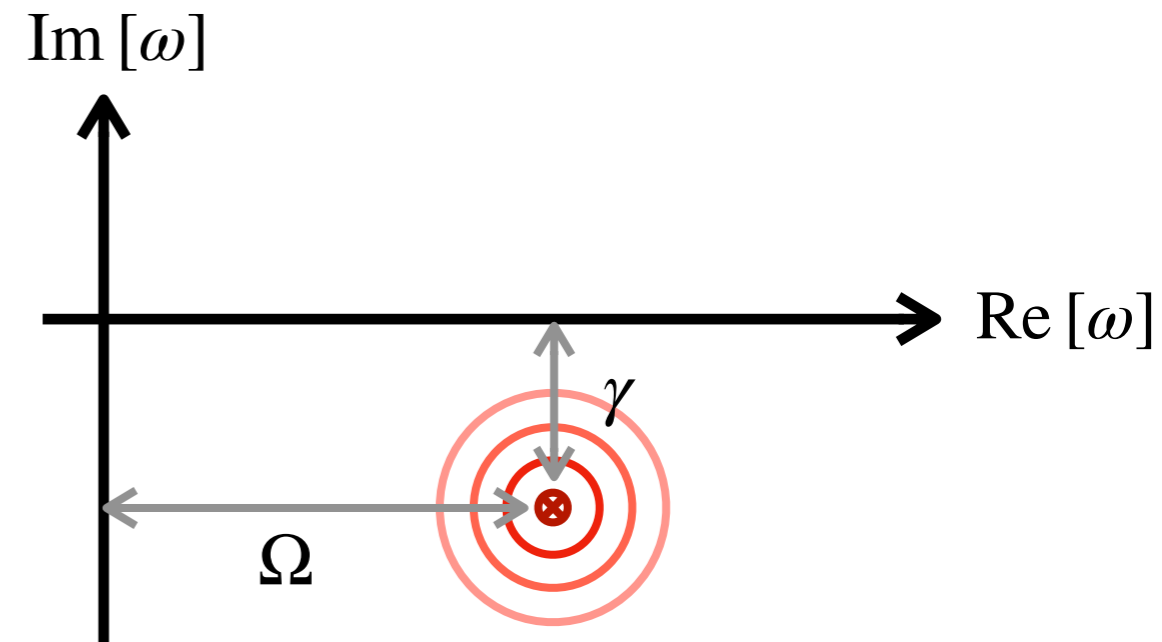
Weakly damped modes and Landau's trick

Root of the **dispersion function**

$$\varepsilon(\Omega + i\gamma) = 0$$

The mode is **weakly damped** $\gamma \ll \Omega$

$$\varepsilon(\Omega) + i\gamma \frac{\partial}{\partial \Omega} \varepsilon(\Omega) = 0$$



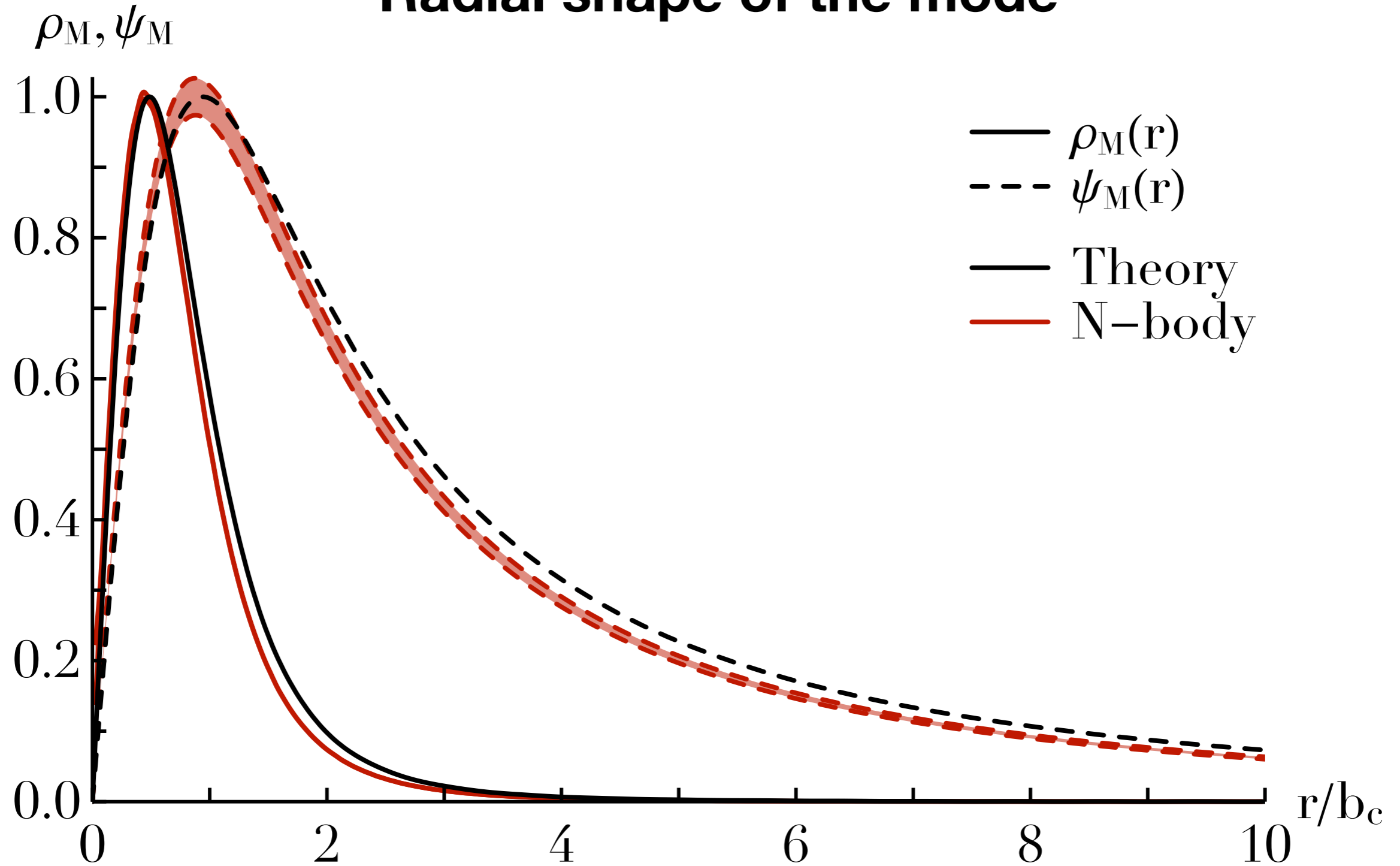
Self-consistent constraints for the mode's frequency

$$\text{Re}[\varepsilon(\Omega)] = 0$$

$$\gamma = - \frac{\text{Im}[\varepsilon(\Omega)]}{\partial \varepsilon(\Omega) / \partial \Omega}$$

Can one infer the modes
without ever going in the lower half of the complex plane?

Radial shape of the mode



To estimate the mode's shape from **N-body simulations**

Radial shell projection for $\rho_M(r)$

Heggie+(2020)

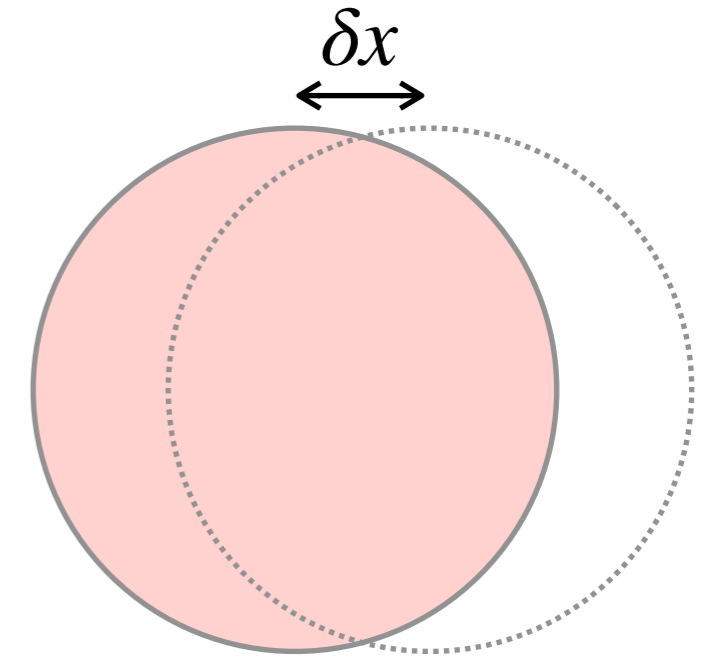
Multipole projection for $\psi_M(r)$

Lau+(2020)

Mode vs overall shift

Whole cluster shift by δx

$$\begin{aligned} \delta\rho(x, y, z) &= \rho_0(x - \delta x, y, z) - \rho_0(x, y, z) \\ &= \frac{d\rho_0}{dr} \frac{x}{r} \delta x \end{aligned}$$

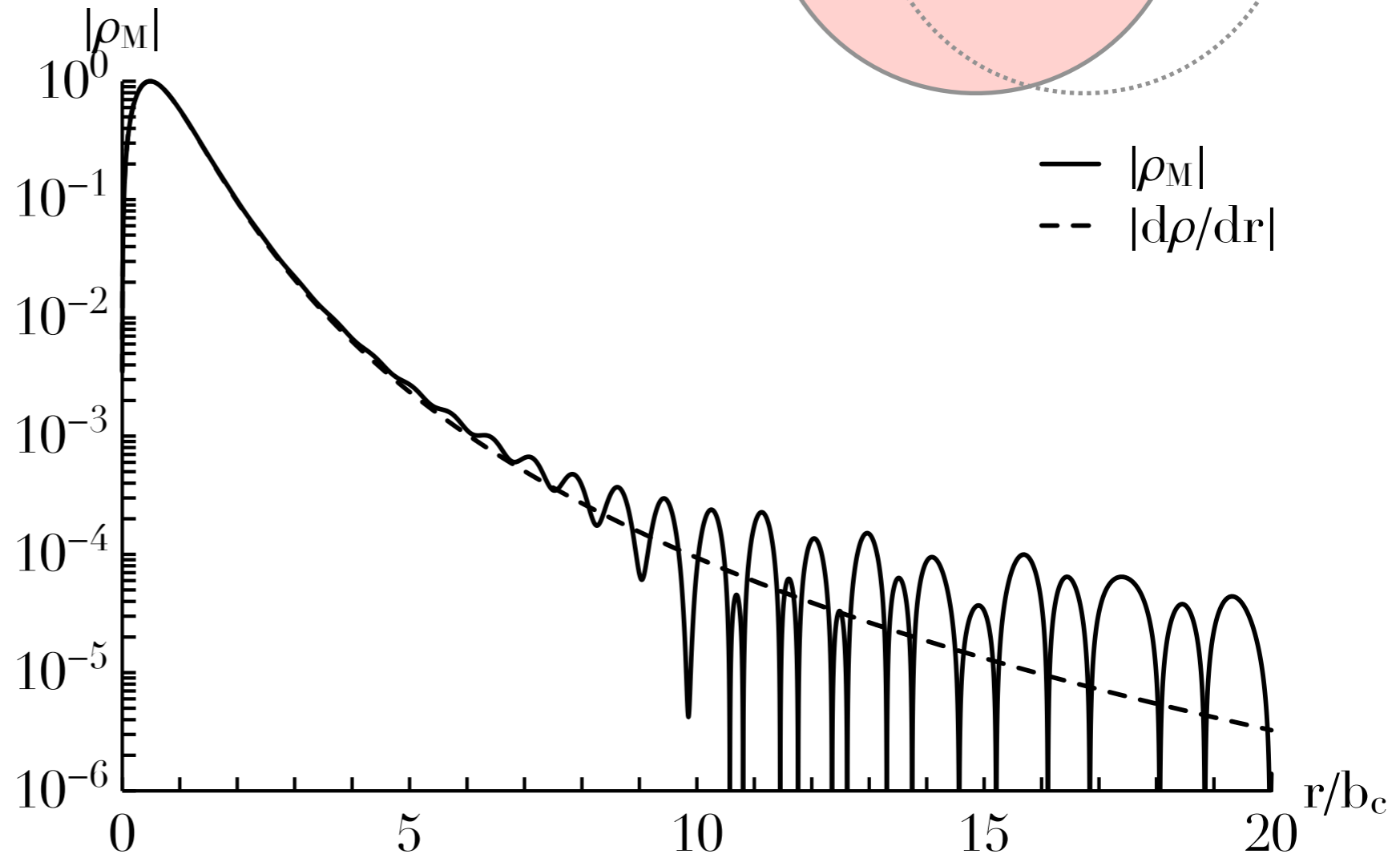


Shift's **perturbation**

$$\delta\rho \propto \frac{d\rho_0}{dr} Y_{\ell m}(\hat{\mathbf{r}})$$

Mode's **perturbation**

$$\delta\rho \propto \rho_M(r) Y_{\ell m}(\hat{\mathbf{r}})$$



Why is the mode so similar to the density gradient?

Constraints on the radial shape

Conserving the **linear momentum**

$$\delta \mathbf{P} = \int d\mathbf{r} \mathbf{r} \delta \rho(\mathbf{r}, t) = 0$$

 \implies

$$\int dr r^3 \rho_M(r) = 0$$

Mode's **node**

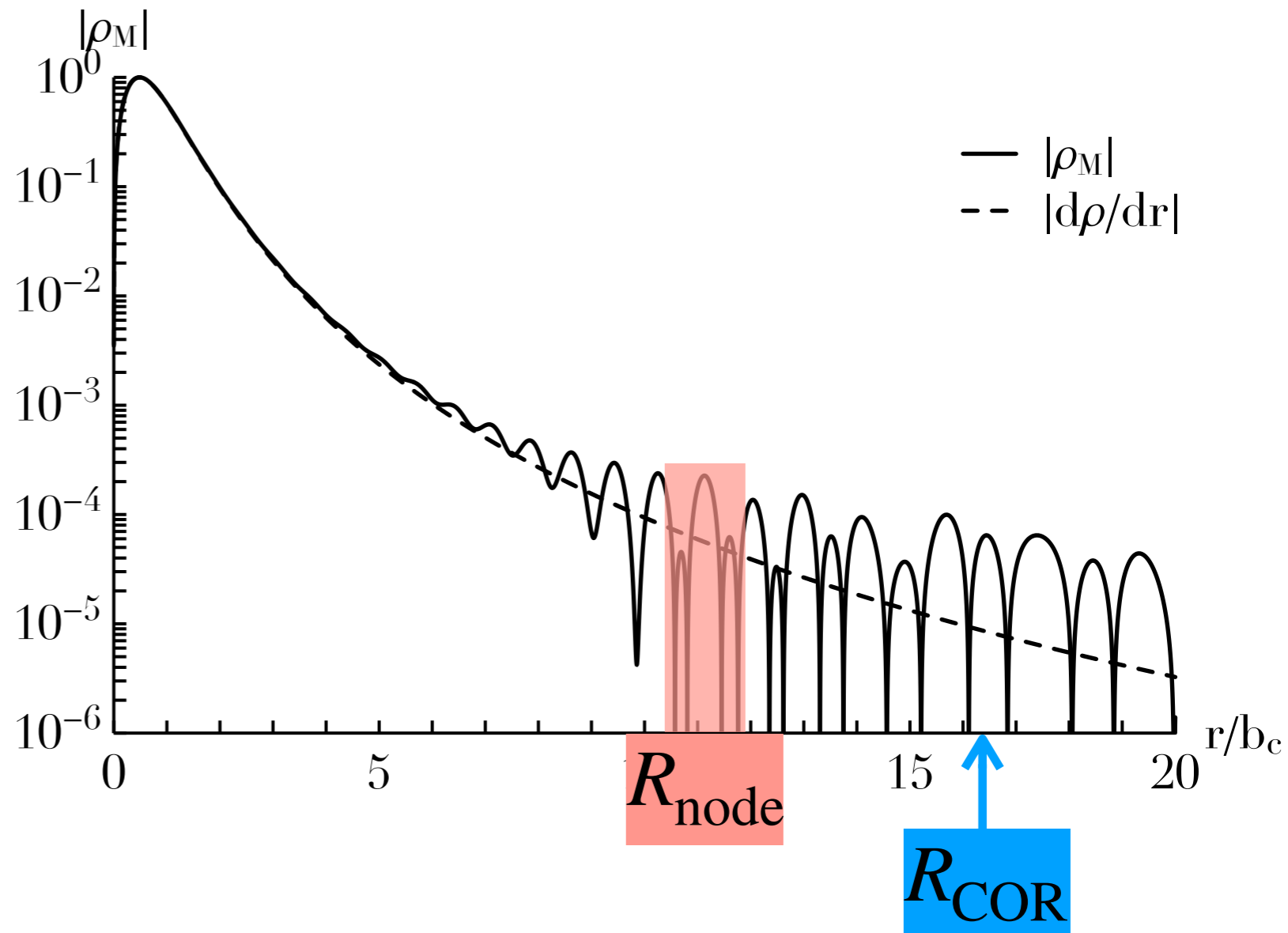
$$\rho_M(R_{\text{node}}) = 0$$

Mode's **corotation radius**

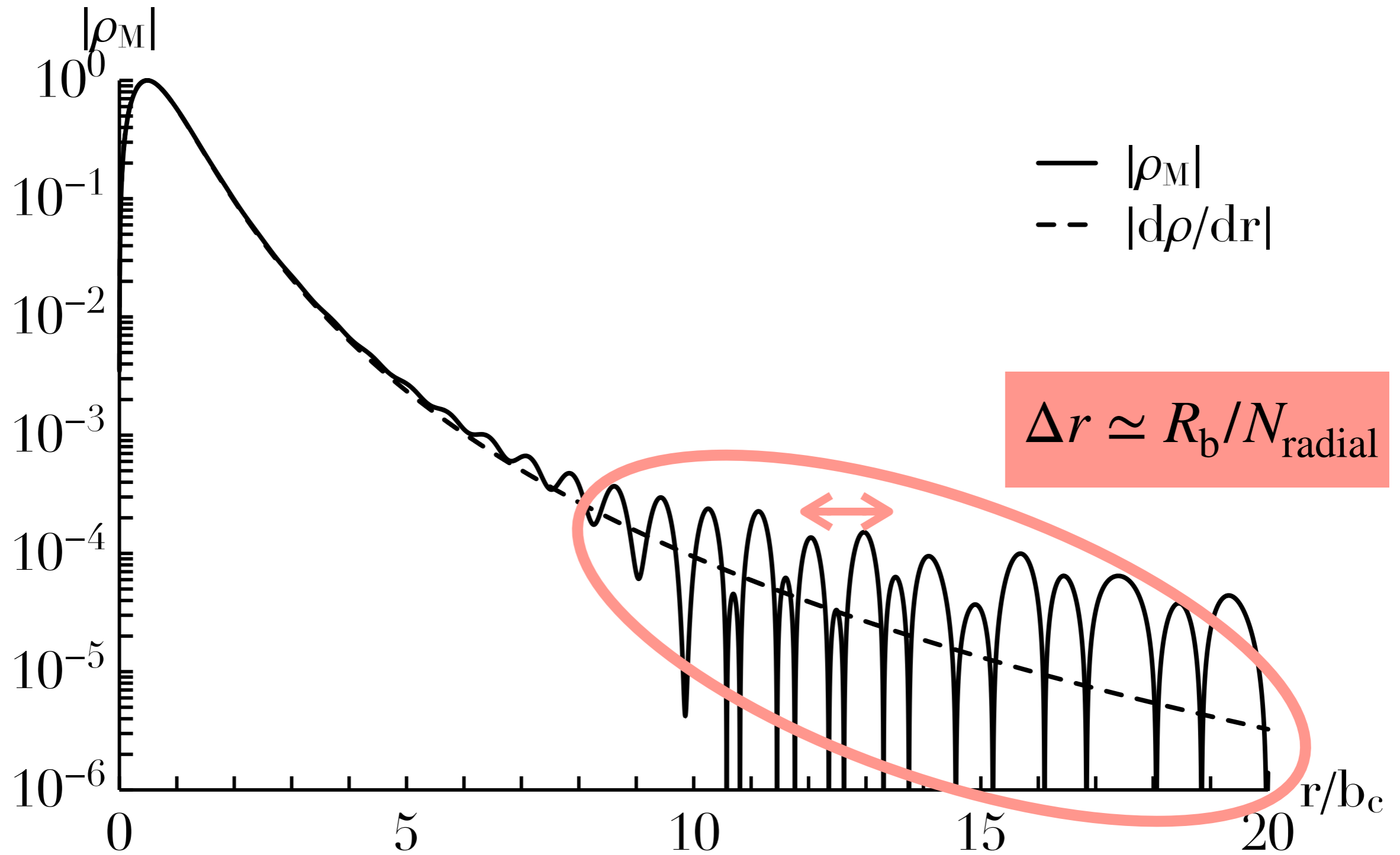
$$\Omega_2^{\text{circ}}(R_{\text{COR}}) = \text{Re}[\omega_M]$$

Constraint on **rotation**

$$R_{\text{node}} \leq R_{\text{COR}}$$



Mode's node



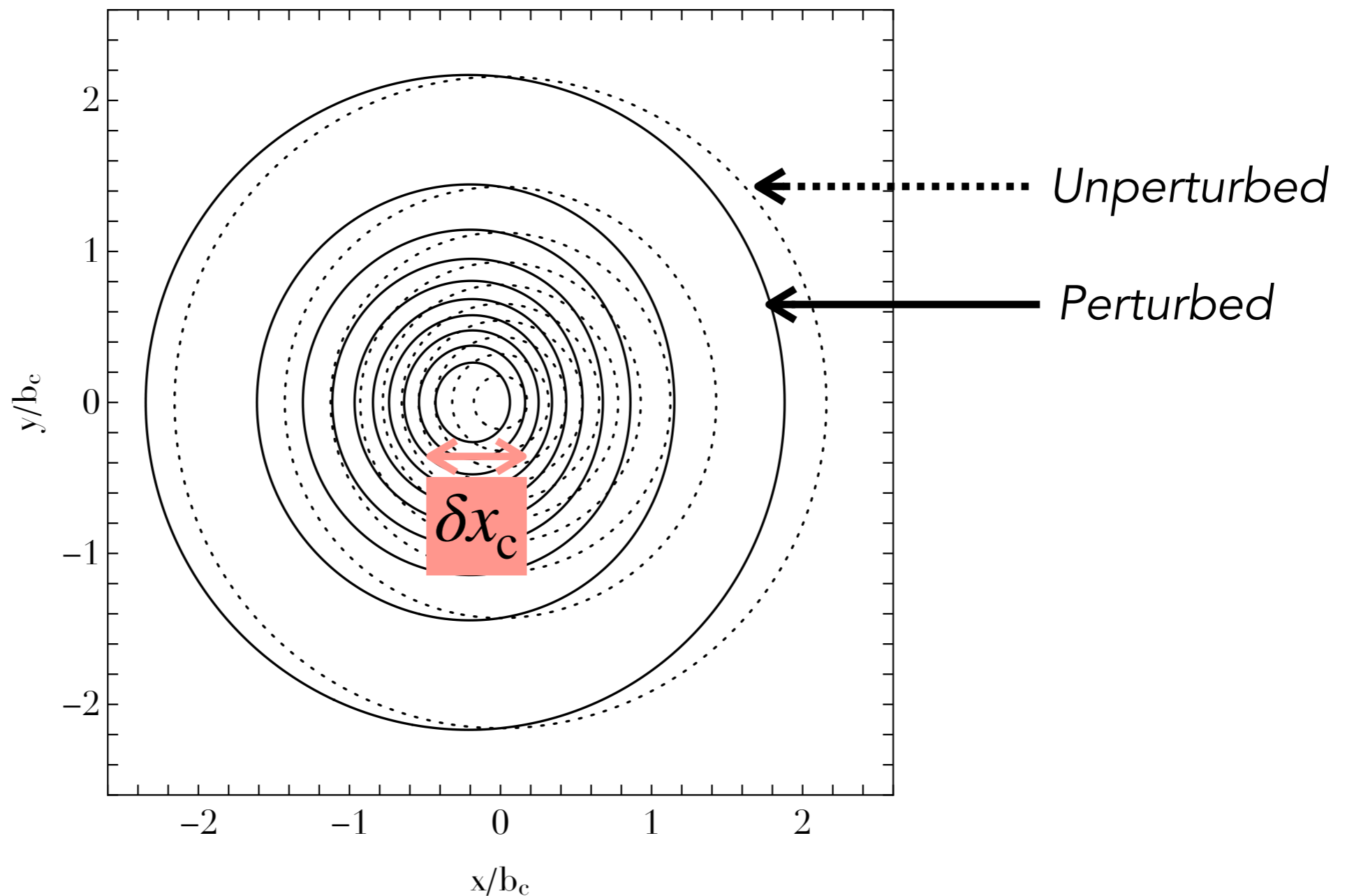
How to reduce spurious oscillations from the basis elements?

Dynamics of the perturbation

Typical perturbation

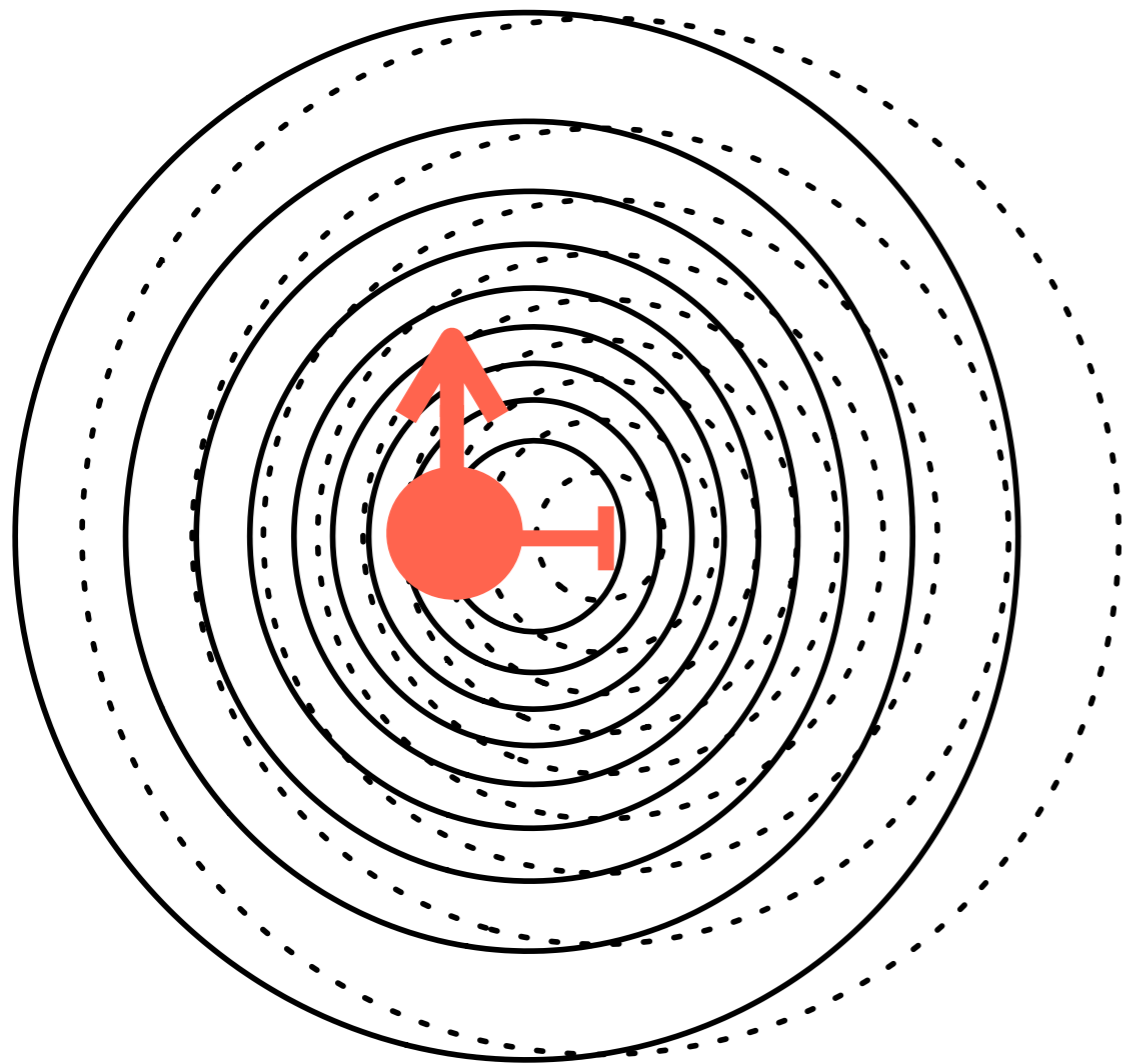
$$\delta\rho(\mathbf{r}, t) = A_M(t) \rho_M(r) Y_{\ell m}(\hat{\mathbf{r}})$$

Wobble of the **density centre**



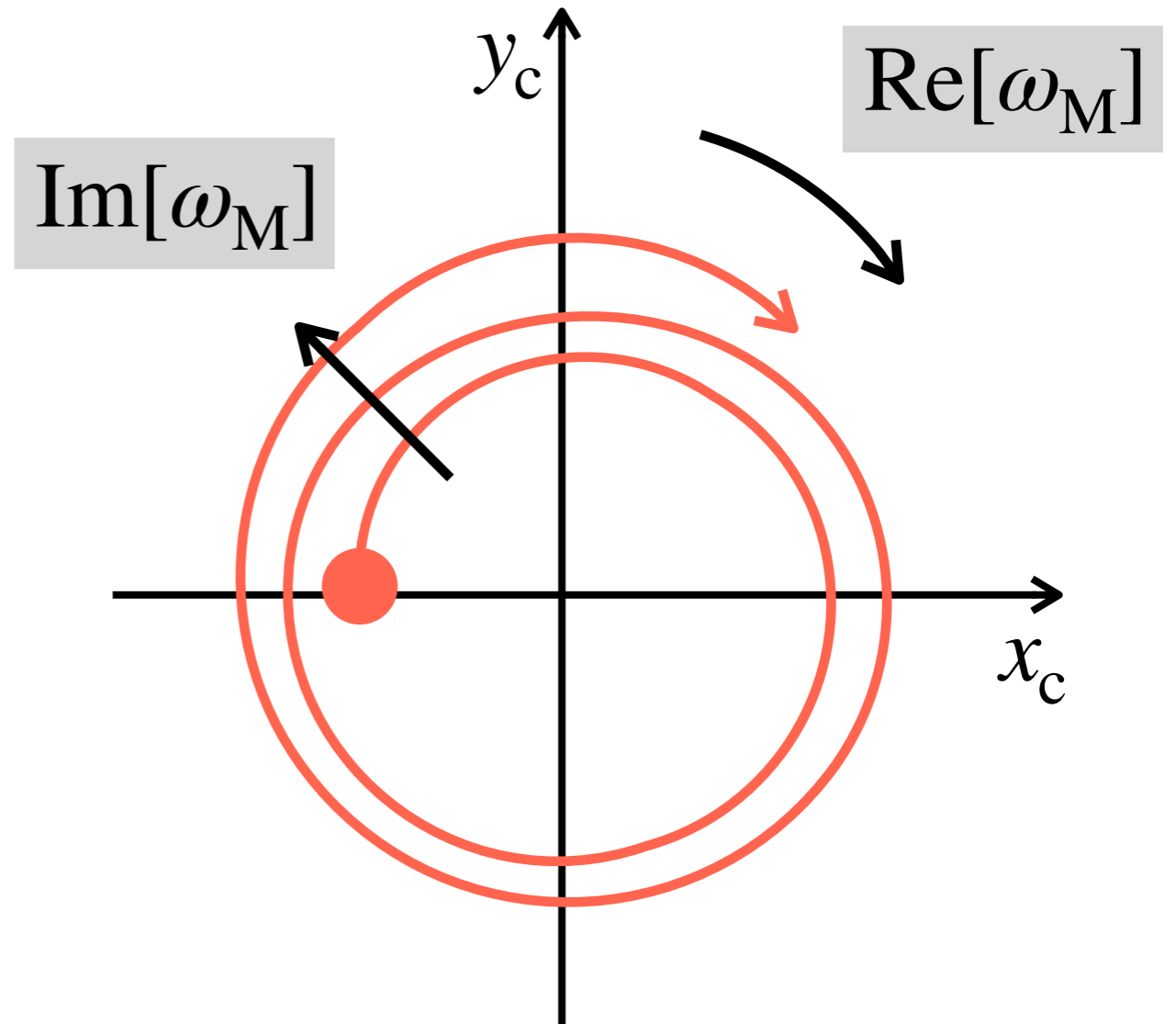
Dynamics of the density centre

Time evolution of the density centre



Rotation timescale

$$T = 2\pi/\text{Re}[\omega_M] \simeq 50 \text{ HU}$$

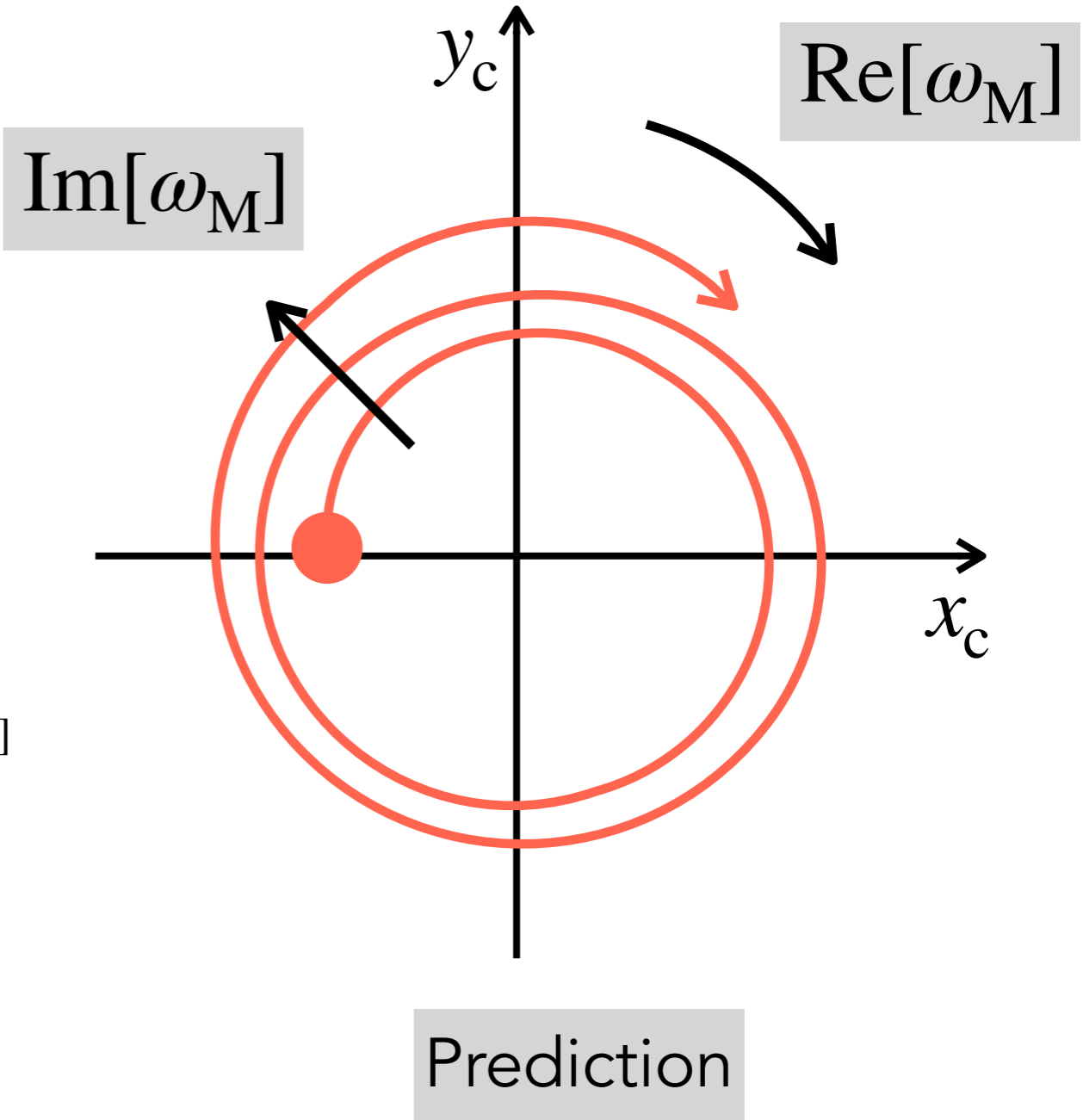
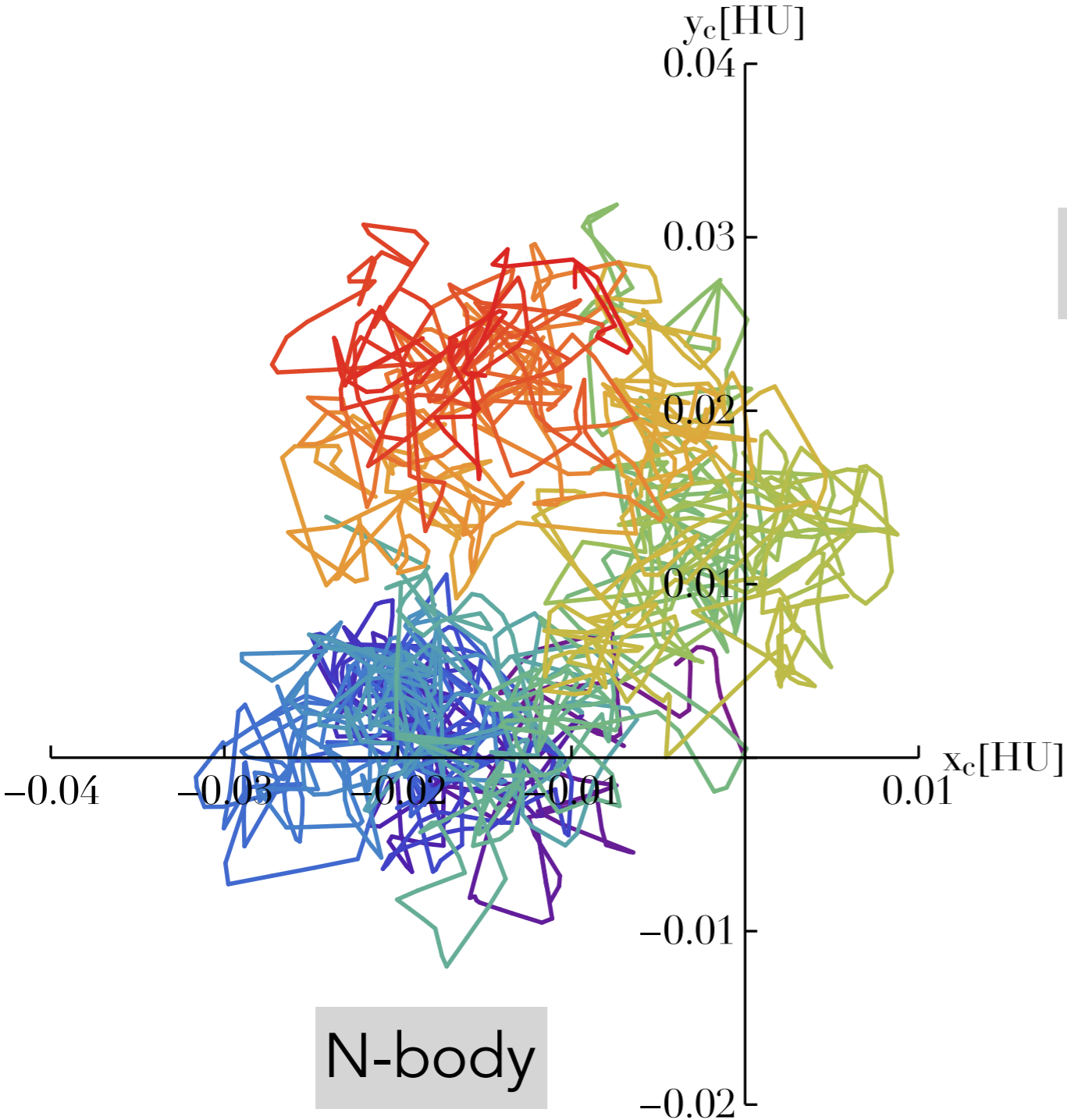


Growth timescale

$$T = 3/\text{Im}[\omega_M] \simeq 250 \text{ HU}$$

Dynamics of the density centre

Time evolution of the density centre



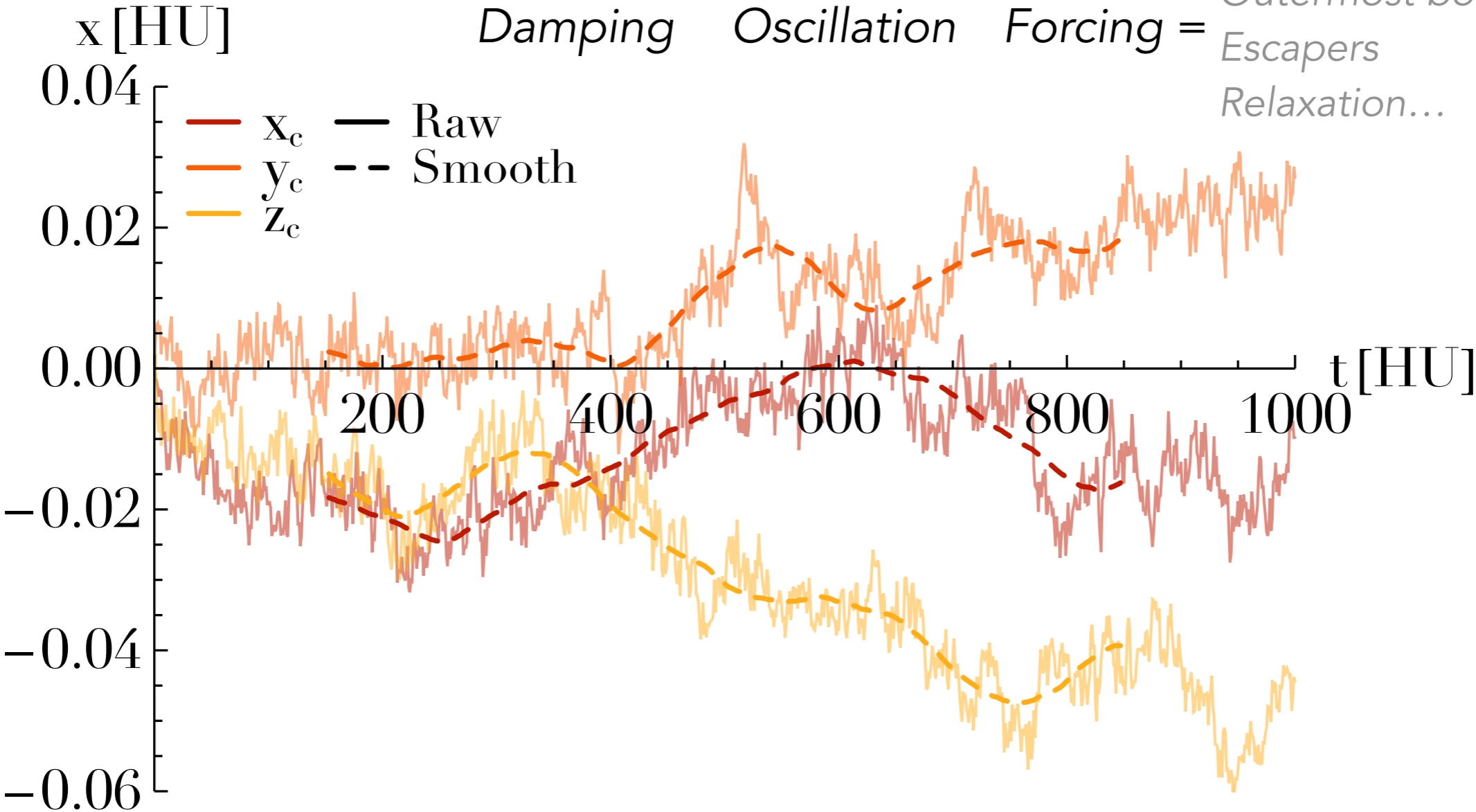
Why is the centre's dynamics so messy?

Dynamics of the density centre

Stochastic dynamics

$$\frac{d^2 x_c}{dt^2} - \gamma_M \frac{dx_c}{dt} + \Omega_M^2 x_c = \eta(t)$$

Perpetual Poisson noise
Outermost bound particles
Escapers
Relaxation...



Damping *Oscillation* *Forcing =*

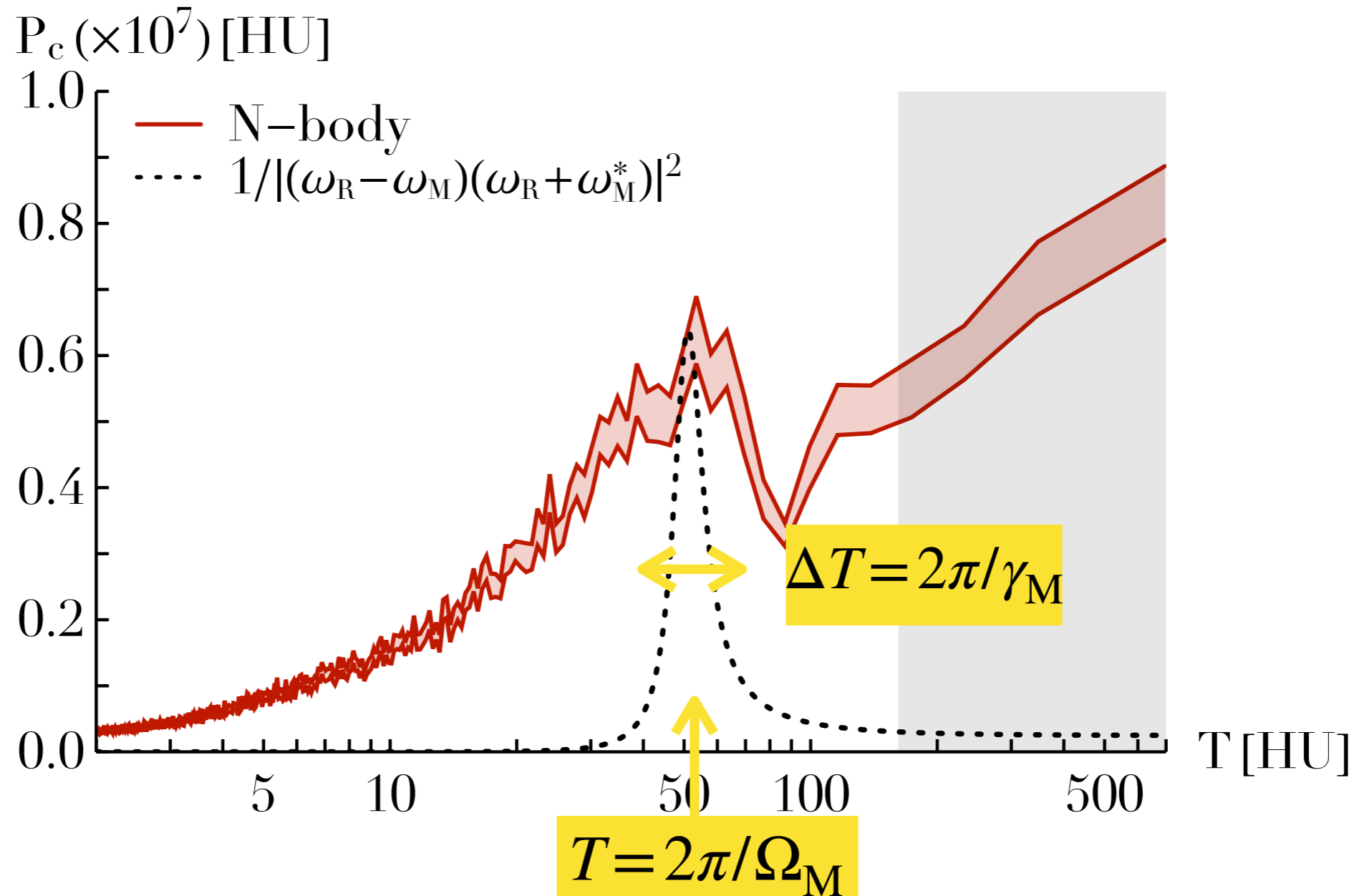
How to understand these large-scale excursions?

Dynamics of the density centre

Power spectrum

$$\langle |\hat{x}_c(\omega)|^2 \rangle \propto \frac{1}{|(\omega - \omega_M)(\omega + \omega_M^*)|^2}$$

In **N-body simulations** (*with time-filtering*)



What about QL diffusion?

Long-term dynamics

Decomposing the **fluctuations**

$$\delta\Phi_{\text{tot}}(t) = \delta\Phi_{\text{BL}}(t) + \delta\Phi_{\text{M}}(t)$$

Total *Drives* *Drives*
fluctuations *BL* *QL*

Two sources of **evolution**

$$\frac{\partial F}{\partial t} = \left(\frac{\partial F}{\partial t} \right)_{\text{BL}} + \left(\frac{\partial F}{\partial t} \right)_{\text{QL}}$$

Requires a **splitting** of the perturbations

$$\{ \mathbf{x}_i(t), \mathbf{v}_i(t) \} \mapsto \{ \delta\Phi_{\text{BL}}(t), \delta\Phi_{\text{M}}(t) \}$$

How to measure the waves' amplitude in N-body runs?

Mode's energy

Typical perturbation

$$\delta\rho(\mathbf{r}, t) = A_M(t) \rho_M(r) Y_{\ell m}(\hat{\mathbf{r}})$$

Energy in the mode

$$E_M(t) = |A_M(t)|^2$$

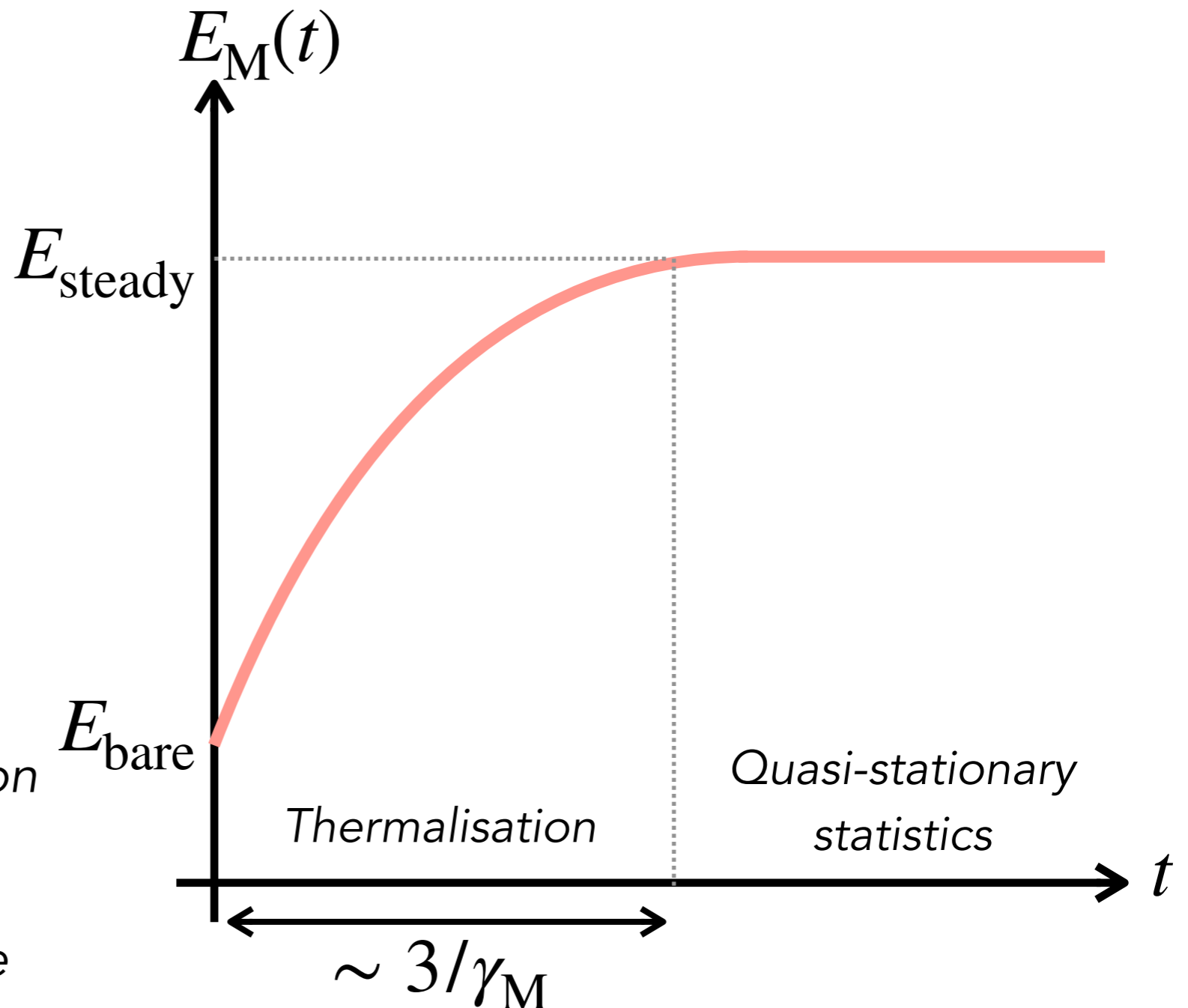
Wave equation Hamilton+(2020)

$$\frac{dE_M}{dt} = 2\gamma_M E_M + S_M$$

Energy budget

$2\gamma_M E_M$ Landau damping
Resonant interaction

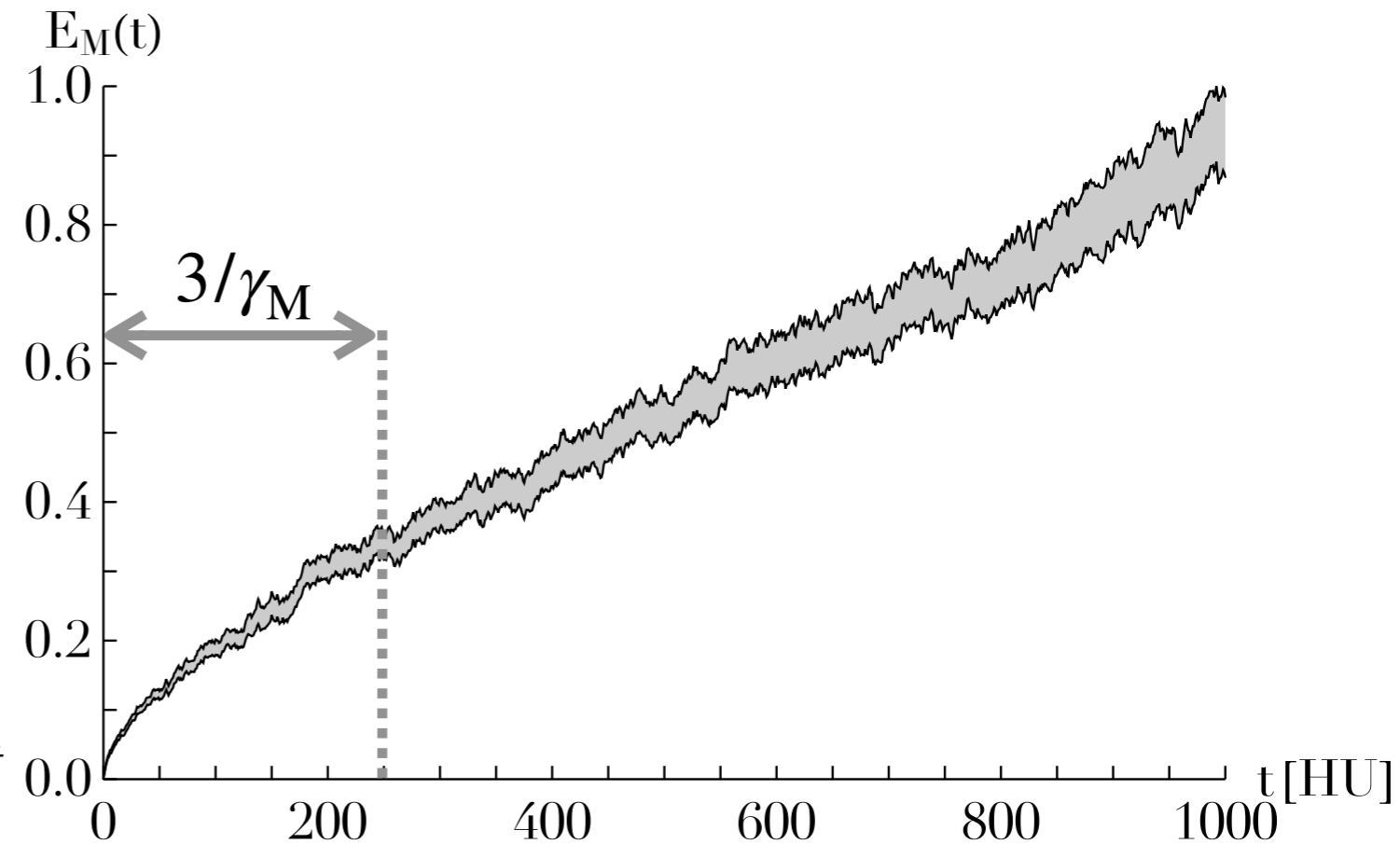
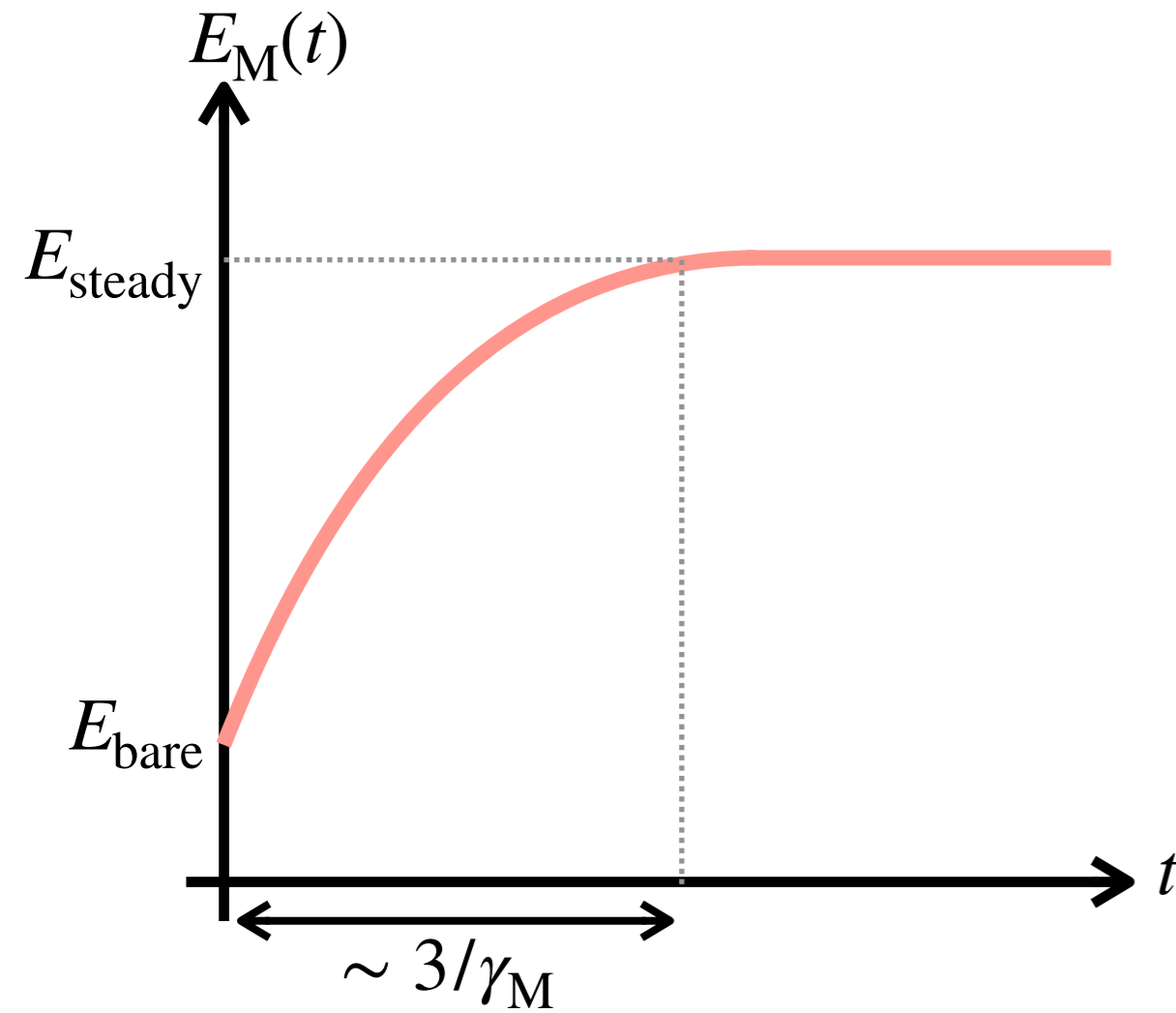
S_M Spontaneous emission
Perpetual Poisson noise



Mode's energy

Prediction

N-body simulations



No saturation
Diffusive-like dynamics

How to check the wave equation in N-body simulations?

Dominating mode

Wave equation

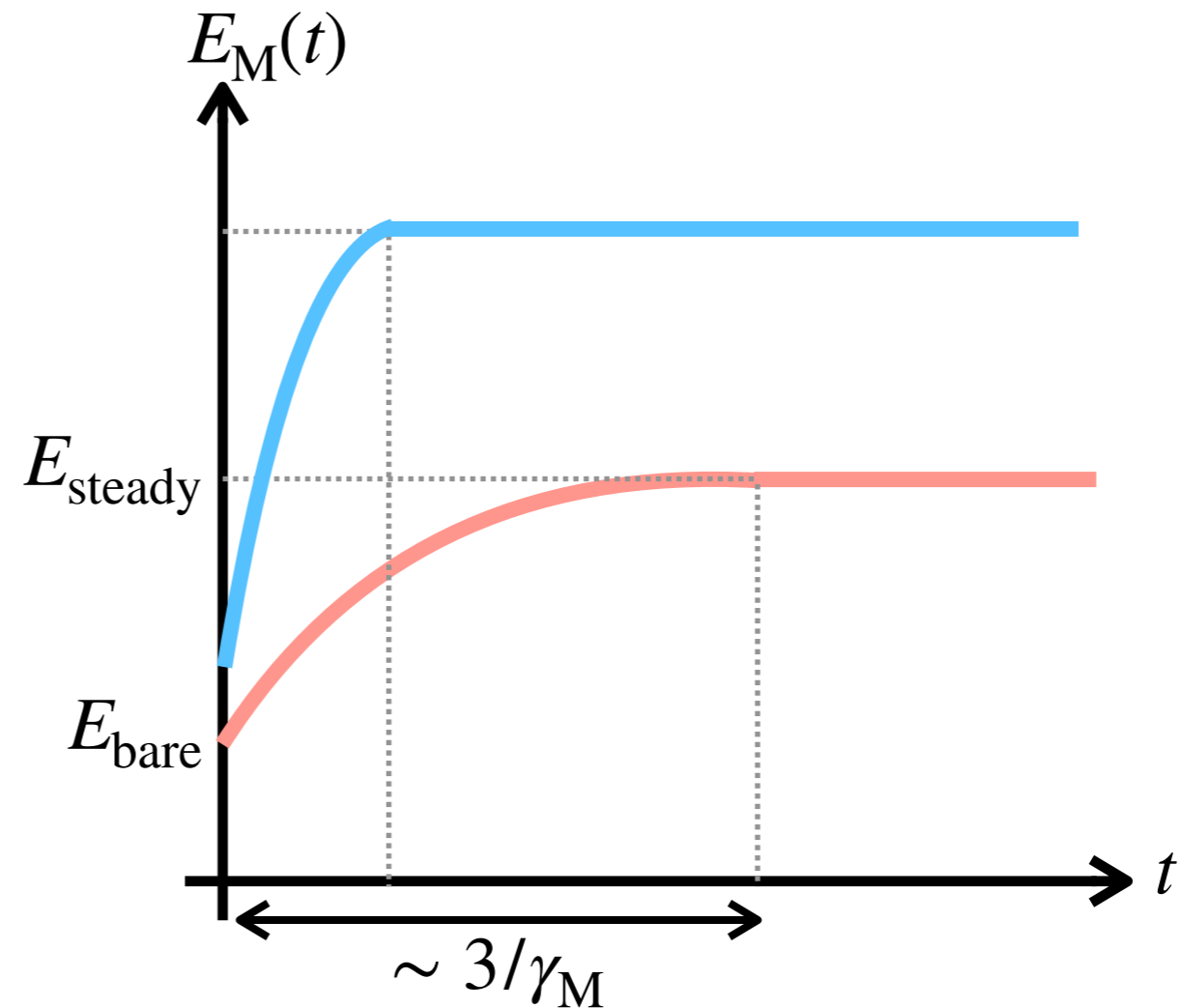
$$\frac{dE_M}{dt} = 2\gamma_M E_M + S_M$$

Steady energy

$$E_{\text{steady}} = -\frac{S_M}{2\gamma_M}$$

Spontaneous emission

$$S_M = \frac{1}{N} \sum_{\mathbf{k}} \int d\mathbf{J} \delta_D[\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \Omega_M] F(\mathbf{J})$$



Which mode is dominating?

What happens after thermalisation?

(Weakly damped) **Quasilinear Theory** *Hamilton+(2020)*

$$\frac{\partial F(\mathbf{J})}{\partial t} = - \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}} \mathbf{k} \delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \Omega_M) \left\{ \frac{1}{N} F(\mathbf{J}) - E_M(t) \mathbf{k} \cdot \frac{\partial F}{\partial \mathbf{J}} \right\} \right]$$

$$\delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \Omega_M)$$

Resonant **particle-wave** interaction

$$F(\mathbf{J})$$

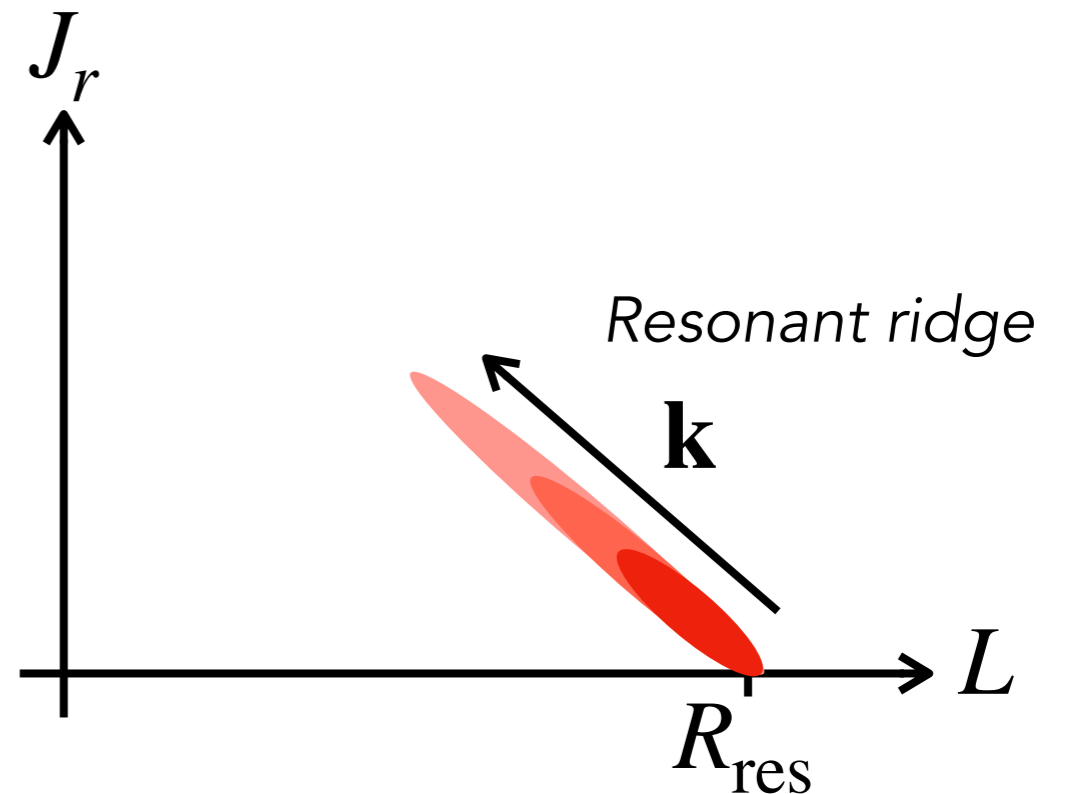
Friction

Cost of spontaneous emission

$$E_M(t) \mathbf{k} \cdot \frac{\partial F}{\partial \mathbf{J}}$$

Diffusion

Resonant absorption



How to integrate over the QL resonance condition?

BL vs QL**Balescu-Lenard equation**

$$\frac{\partial F(\mathbf{J})}{\partial t} = \frac{1}{N} \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}, \mathbf{k}'} \int d\mathbf{J}' \frac{\delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}' \cdot \boldsymbol{\Omega}(\mathbf{J}'))}{|\varepsilon(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}))|^2} \times \dots \right]$$

QL diffusion equation

$$\frac{\partial F(\mathbf{J})}{\partial t} = \frac{1}{N} \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k}} \mathbf{k} \delta_D(\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \Omega_M) \times \dots \right]$$

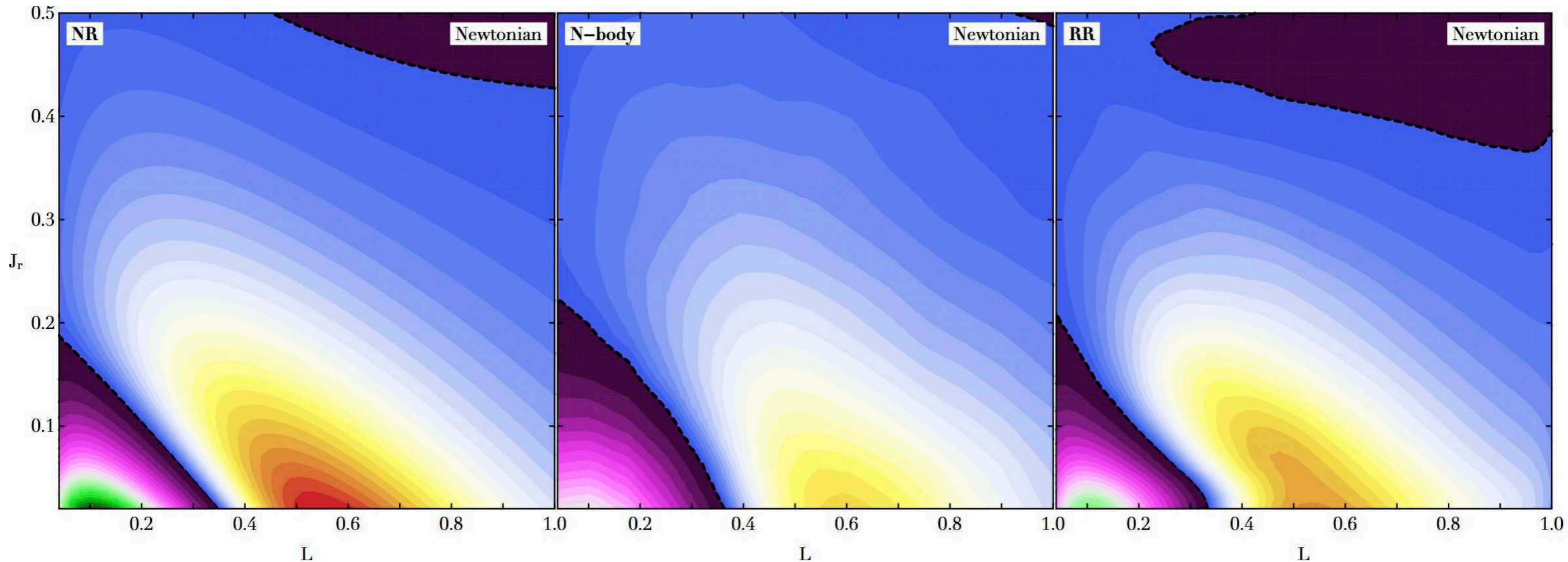
Close to a (weakly) **damped mode**

$$\text{Im}[\omega_M] \ll \text{Re}[\omega_M] \quad \Longrightarrow \quad \frac{1}{|\varepsilon(\Omega_M)|} \gg 1$$

Can QL ever dominate over BL?

Long-term relaxation

Diffusion in orbital space $\partial F(\mathbf{J})/\partial t$



*Orbit-averaged
homogeneous Landau*

Direct N-body

*Inhomogeneous
Balescu-Lenard*

Why don't we find any trace of QL in numerical simulations?

Conclusion

Alternative approach to **analytic continuation**

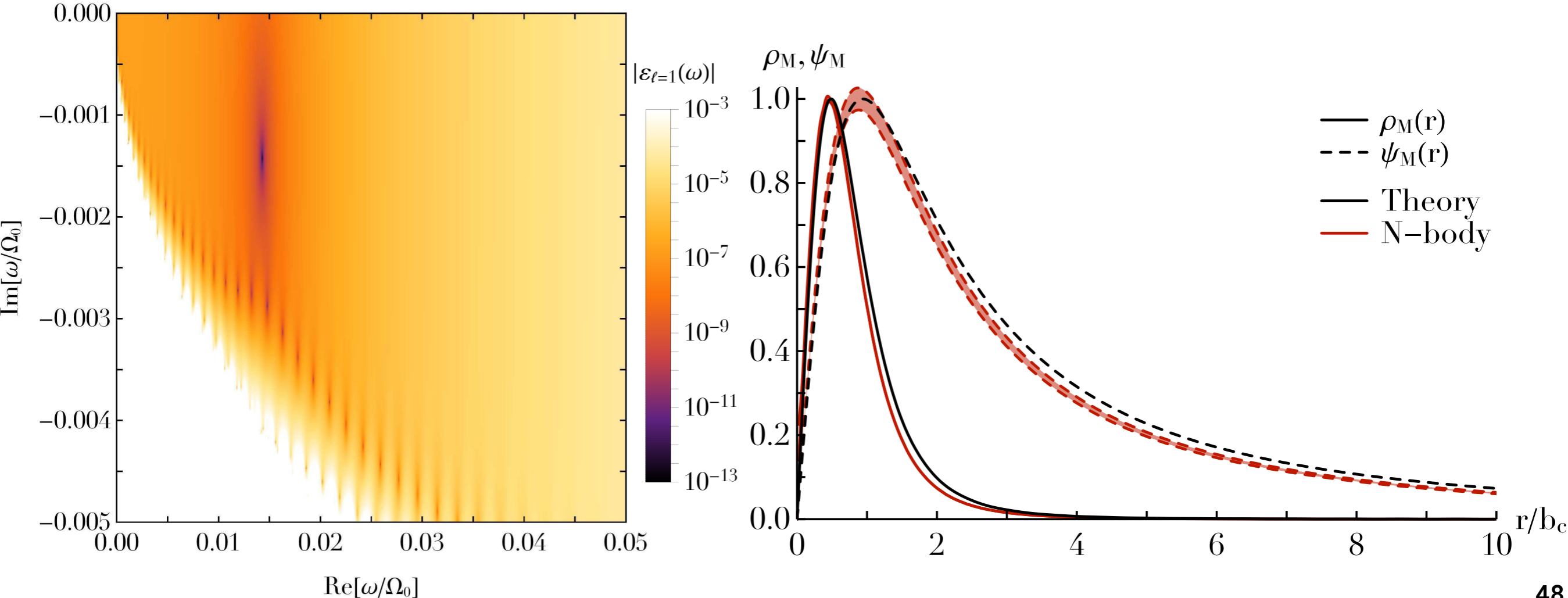
$$M(\omega) = \sum_k a_k D_k(\omega)$$

Legendre series

$$M(\omega) = \frac{P(\omega)}{Q(\omega)}$$

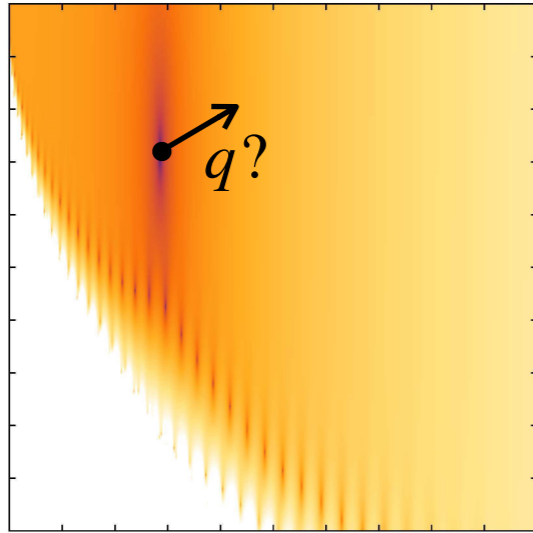
Rational functions Weinberg(1994)

(Weakly) damped modes are unavoidable in globular clusters



Future works

Impact of **anisotropy**



Impact of **potential**

Less puffy (e.g., *Plummer*)

Truncated (e.g., *King*)

Cuspy (e.g., *Hernquist*)

Degenerate (e.g., *quasi-Keplerian*)

Thermalisation timescale

$$1/\gamma_M \text{ vs } F_{\text{vK}}(\mathbf{J}, \omega)_{\text{Lau+(2020)}}$$

Landau *van Kampen*

Others

Other harmonics

What is so special with $\ell = 1$?

Disc dynamics

Swing amplification

QL theory and escapers