Kinetic blocking and $1/N^2$ kinetic theory

Jean-Baptiste Fouvry, IAP
fouvry@iap.fr

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Collisional dynamics

Total (specific) Hamiltonian

$$H = \sum_{i=1}^{N} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m\, U(\mathbf{w}_i, \mathbf{w}_j)$$

Scaling

$$m = \frac{M_{\mathrm{tot}}}{N}; \qquad \frac{1}{N} \ll 1$$

Plasma

$$\begin{cases} U_{\mathrm{ext}} = \frac{1}{2}\mathbf{v}^2 \\ U = 1/|\mathbf{x} - \mathbf{x}'| \\ \Phi_0 = 0 \\ \mathbf{w} = (\mathbf{x}, \mathbf{v}) \\ F = F(\mathbf{v}) \end{cases}$$

Galaxy

$$\begin{cases} U_{\mathrm{ext}} = \frac{1}{2}\mathbf{v}^2 \\ U = -1/|\mathbf{x} - \mathbf{x}'| \\ \Phi_0 = \Phi_0(\mathbf{x}) \\ \mathbf{w} = (\boldsymbol{\theta}, \mathbf{J}) \\ F = F(\mathbf{J}) \end{cases}$$

Balescu-Lenard equation

Homogeneous Balescu-Lenard equation

$$\frac{\partial F(\mathbf{v})}{\partial t} = m \frac{\partial}{\partial \mathbf{v}} \cdot \left[\int \mathrm{d}\mathbf{k}\, \mathbf{k} \int \mathrm{d}\mathbf{v}'\, \delta_{\mathrm{D}}[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\, \frac{|\psi_{\mathbf{k}\mathbf{k}}|^2}{|\varepsilon_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{v})|^2} \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}'} \right) F(\mathbf{v})\, F(\mathbf{v}') \right]$$

Inhomogeneous Balescu-Lenard equation

Orbital frequencies

$$\frac{\partial F(\mathbf{J})}{\partial t} = m \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{k},\mathbf{k}'} \mathbf{k} \int \mathrm{d}\mathbf{J}'\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}')]\, |\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')|^2 \times \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} \right) F(\mathbf{J})\, F(\mathbf{J}') \right]$$

Balescu-Lenard in 1D

Homogeneous Balescu-Lenard equation

$$\frac{\partial F(v)}{\partial t} = m \frac{\partial}{\partial v}\left[\int \mathrm{d}k\,|k| \int \mathrm{d}v'\,\delta_{\mathrm{D}}[v-v']\,\frac{|\psi_{kk}|^2}{|\varepsilon_k(kv)|^2} \times \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'}\right) F(v)\,F(v')\right]$$

Local resonance condition

$$v' = v$$

Exact cancellation of the flux

$$\frac{\partial F(v)}{\partial t} = \frac{1}{N} \times 0$$

Kinetic blocking

$$T_{\mathrm{relax}} \simeq N^2\, T_{\mathrm{dyn}}$$

Balescu-Lenard in 1D

Inhomogeneous Balescu-Lenard equation

$$\frac{\partial F(J)}{\partial t} = m \frac{\partial}{\partial J} \left[\sum_{k,k'} k \int \mathrm{d}J' \, \delta_{\mathrm{D}}[k\Omega(J) - k'\Omega(J')] \, |\, \psi^{\mathrm{d}}_{kk'}(J,J') \,|^2 \times \left(k \frac{\partial}{\partial J} - k' \frac{\partial}{\partial J'} \right) F(J) \, F(J') \right]$$

Additional **symmetry** constraints

$$\psi^{\mathrm{d}}_{kk'} \propto \delta_{kk'}$$

1:1 resonance

$$J \mapsto \Omega(J) \quad \text{monotonic}$$

Local resonances

$$J' = J$$

Vanishing of the flux

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N} \times 0$$

Kinetic blocking

$$T_{\mathrm{relax}} \simeq N^2 \, T_{\mathrm{dyn}}$$

What is the $1/N^2$ kinetic theory?

Balescu-Lenard from BBGKY

N **identical particles** of mass $m = \frac{M_{\mathrm{tot}}}{N}$ in phase space $\mathbf{w}_i = (\mathbf{x}_i, \mathbf{v}_i)$

Total specific **Hamiltonian**

$$H_N = \sum_{i=1}^{N} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m\, U(\mathbf{w}_i, \mathbf{w}_j)$$

3D self-gravitating systems

$$U_{\mathrm{ext}} = \frac{|\mathbf{v}|^2}{2}$$

$$U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}$$

System characterised by the **N-body PDF** $P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$

Continuity equation in phase space

$$\frac{\partial P_N}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{w}_i} \cdot \left(P_N\, \dot{\mathbf{w}}_i \right) = 0$$

Exact Liouville equation

$$\frac{\partial P_N}{\partial t} + \left[P_N, H_N \right]_N = 0$$

Poisson bracket

BBGKY hierarchy

Reduced DFs

$$F_n(\mathbf{w}_1, \ldots, \mathbf{w}_n, t) = m^n \frac{N!}{(N-n)!} \int \mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \, P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$$

BBGKY hierarchy

$$\frac{\partial F_n}{\partial t} + \left[F_n, H_n\right]_n + \int \mathrm{d}\mathbf{w}_{n+1} \left[F_{n+1}, \delta H_{n+1}\right]_n = 0$$

With

$$H_n = \sum_{i=1}^{n} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{n} m \, U(\mathbf{w}_i, \mathbf{w}_j)$$

n-body system

$$\delta H_{n+1} = \sum_{i=1}^{n} U(\mathbf{w}_i, \mathbf{w}_{n+1})$$

Interactions with (n+1)

BBGKY at 1/N

Cluster representation

$$\begin{cases} F_2(\mathbf{w},\mathbf{w}') = F_1(\mathbf{w})\,F_1(\mathbf{w}') + G_2(\mathbf{w},\mathbf{w}') \\ F_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') = \ldots + G_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') \end{cases} \Longrightarrow \begin{cases} G_2 \sim 1/N \\ G_3 \sim 1/N^2 \end{cases}$$

Truncation at **order 1/N**: 2 dynamical quantities

$F(\mathbf{w},t)$ **1-body DF**

$G(\mathbf{w},\mathbf{w}',t)$ **2-body correlation**

BBGKY - 1

$$\frac{\partial F}{\partial t} + \left[F, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w},\mathbf{w}')\right]_{\mathbf{w}} = 0$$

BBGKY - 2

$$\frac{\partial G}{\partial t} + \left[G, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}''\, G(\mathbf{w}',\mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w},\mathbf{w}'')\right]_{\mathbf{w}}$$
$$+ m \left[F(\mathbf{w})\,F(\mathbf{w}'), U(\mathbf{w},\mathbf{w}')\right]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

BBGKY - 1

$$\frac{\partial F}{\partial t} + \left[F, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} = 0$$

$\left[F, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ **Collision** term

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BBGKY - 2

$$\frac{\partial G}{\partial t} + \left[G, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}''\, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}} + m \left[F(\mathbf{w})\, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

$\left[G, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}''\, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}}$ **Collective effects**

$m \left[F(\mathbf{w})\, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ 1-body DF sourcing

How to solve BBGKY

Adiabatic approximation

i.e. evolution along **quasi-stationary states**

$$F = F(\mathbf{J}, t) \;\; ; \;\; H_0 = H_0(\mathbf{J}, t) \;\; \Longrightarrow \;\; \left[F_0(\mathbf{J})\,, H_0(\mathbf{J})\right]_{\mathbf{w}} = 0$$

Mean-field equilibrium

Timescale separation

$$\frac{\partial G}{\partial t} + \left[G, H_0\right]_{\mathbf{w}} + (\ldots) = 0$$

$$\frac{\partial F}{\partial t} = - \int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$$

Collision operator

$$\longrightarrow \begin{cases} T_G \simeq T_{\mathrm{dyn}} \\ T_F \simeq N \times T_G \end{cases}$$

Bogoliubov's Ansatz

$$\frac{\partial G}{\partial t} = \mathrm{BBGKY}_2\left[F = \mathrm{cst}\,, G\right]$$

$$\frac{\partial F}{\partial t} = \mathrm{BBGKY}_1\left[F\,, G(t \rightarrow +\infty)\right]$$

BBGKY at order 1/N²

Describing the system

$$F(\mathbf{w}_1) \propto 1$$

1-body DF

$$G_2(\mathbf{w}_1, \mathbf{w}_2) \propto 1/N$$

2-body correlation

$$G_3(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \propto 1/N^2$$

3-body correlation

Correlation splitting

$$G_2 = \frac{1}{N} G_2^{(1)} + \frac{1}{N^2} G_2^{(2)}$$

Neglecting **collective effects**

$$\int \mathrm{d}w_3 \, G_2^{(1)}(\mathbf{w}_2, \mathbf{w}_3) \, U(\mathbf{w}_1, \mathbf{w}_3) \to 0$$

Hot limit

Neglecting the **cold crossed term**

$$G_2^{(1)} \times G_2^{(1)} \to 0 \;\; \text{in} \;\; \partial G_3 / \partial t$$

Hot limit

A well-posed hierarchy

Solving in **sequence**

$$\frac{\partial G_2^{(1)}}{\partial t} + \left[G_2^{(1)}, H_0\right] = S[F]$$ (2 terms in the rhs)

$$\frac{\partial G_3}{\partial t} + \left[G_3, H_0\right] = S[F, G_2^{(1)}]$$ (24 terms in the rhs)

$$\frac{\partial G_2^{(2)}}{\partial t} + \left[G_2^{(2)}, H_0\right] = S[F, G_3]$$ (2 terms in the rhs)

$$\frac{\partial F}{\partial t} = C[G_2^{(2)}]$$ (4 terms in the rhs)

After **re-injecting** and **expanding** all the derivatives

~1,000 terms

Simplifying the collision operator

Interaction potential

$$U(\mathbf{w},\mathbf{w}') = \sum_k U_k[J,J']\,\mathrm{e}^{\mathrm{i}k(\theta-\theta')}$$

Monotonic frequency profile

$$J \mapsto \Omega(J)$$

Large time limit

$$\lim_{t\to+\infty} \int_0^t \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}(t-t')\omega_\mathrm{R}} = \pi\delta_\mathrm{D}(\omega_\mathrm{R}) + \mathrm{i}\,\mathscr{P}\left(\frac{1}{\omega_\mathrm{R}}\right)$$

Number of terms keeps growing: ~10,000 terms

Do NOT perform these calculations by hand!

Using a custom grammar in **Mathematica**

Simplifying the collision operator

Typical shape

$$\frac{\partial F(J)}{\partial t} = \frac{\partial}{\partial J}\left[\sum_{k_1,k_2}\int \mathrm{d}J_1 \mathrm{d}J_2 \ldots\right]$$

Relabellings

$$\{J_1, J_2, k_1, k_2\} \quad \rightarrow \quad \omega_{\mathrm{R}} = (k_1 + k_2)\Omega[J] - k_1\Omega[J_1] - k_2\Omega[J_2]$$

Integration by parts

$$\delta'_{\mathrm{D}} \;\rightarrow\; \delta_{\mathrm{D}}; \quad \partial^2_{J_1} F \;\rightarrow\; \partial_{J_1} F$$

Scaling relations

$$\delta_{\mathrm{D}}[\alpha\omega_{\mathrm{R}}] = \delta_{\mathrm{D}}[\omega_{\mathrm{R}}]/|\alpha|$$

Monotonic frequency profile

$$\int \mathrm{d}J_2\, f(J_1, J_2)\, \delta_{\mathrm{D}}(\Omega[J_1] - \Omega[J_2]) = f(J_1, J_1)/|\Omega'[J_1]|$$

Final equation

The 1D 1/N² inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J}\Bigg[\sum_{k_1,k_2} \frac{1}{k_1^2(k_1+k_2)} \mathscr{P}\!\!\int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \times \int \mathrm{d}J_2\, \mathscr{U}_{k_1k_2}(\mathbf{J})\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}] \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F_3(\mathbf{J})\Bigg]$$

The DF appears three times

$$\mathbf{J} = \big(J, J_1, J_2\big); \quad F_3(\mathbf{J}) = F(J)\, F(J_1)\, F(J_2)$$

3-body resonances

$$\mathbf{k} = \big(k_1+k_2, -k_1, -k_2\big); \quad \mathbf{\Omega} = \big(\Omega[J], \Omega[J_1], \Omega[J_2]\big)$$

Coupling coefficient

Kinetic equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J}\Bigg[\sum_{k_1,k_2} \frac{1}{k_1^2(k_1+k_2)} \mathscr{P}\!\!\int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \times \int \mathrm{d}J_2\, \mathscr{U}_{k_1k_2}(\mathbf{J})\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}] \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F_3(\mathbf{J})\Bigg]$$

Interaction potential

$$U(\mathbf{w},\mathbf{w}') = \sum_k U_k[J,J']\,\mathrm{e}^{\mathrm{i}k(\theta-\theta')}$$

Coupling coefficient

$$\mathscr{U}_{k_1k_2}(\mathbf{J}) = \Big[\big(\Omega[J]-\Omega[J_1]\big)\,\mathscr{U}^{(1)}_{k_1k_2}(\mathbf{J}) + k_2\,\mathscr{U}^{(2)}_{k_1k_2}(\mathbf{J})\Big]^2$$

Coupling coefficient

Coupling coefficient

$$\mathcal{U}_{k_1 k_2}(\mathbf{J}) = \Big[\Big(\Omega[J]-\Omega[J_1]\Big)\,\mathcal{U}^{(1)}_{k_1 k_2}(\mathbf{J}) + k_2\,\mathcal{U}^{(2)}_{k_1 k_2}(\mathbf{J})\Big]^2$$

Two parts

$$\begin{aligned}
\mathcal{U}^{(1)}_{k_1 k_2}(\mathbf{J}) = {} & k_2(k_1+k_2)\left\{U_{k_1+k_2}(J,J_2)\,\partial_{J_2}U_{k_1}(J_1,J_2) - U_{k_2}(J,J_2)\,\partial_J U_{k_1}(J,J_1)\right\} \\
& +k_1(k_1+k_2)\left\{U_{k_1}(J,J_1)\,\partial_J U_{k_2}(J,J_2) - U_{k_1+k_2}(J,J_1)\,\partial_{J_1}U_{k_2}(J_1,J_2)\right\} \\
& -k_1 k_2\left\{U_{k_2}(J_1,J_2)\,\partial_{J_1}U_{k_1+k_2}(J,J_1) - U_{k_1}(J_1,J_2)\,\partial_{J_2}U_{k_1+k_2}(J,J_2)\right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}^{(2)}_{k_1 k_2}(\mathbf{J}) = {} & (k_1+k_2)\,\frac{\mathrm{d}\Omega}{\mathrm{d}J}\,U_{k_1}(J,J_1)\,U_{k_2}(J,J_2) \\
& -k_1\,\frac{\mathrm{d}\Omega}{\mathrm{d}J_1}\,U_{k_1+k_2}(J,J_1)\,U_{k_2}(J_1,J_2) \\
& -k_2\,\frac{\mathrm{d}\Omega}{\mathrm{d}J_2}\,U_{k_1}(J_1,J_2)\,U_{k_1+k_2}(J,J_2)
\end{aligned}$$

Homogeneous limit easily follows

Properties

Conservation laws

$$\begin{cases} M(t) = \int \mathrm{d}J\, F(J,t) & \text{(total mass)} \\ P(t) = \int \mathrm{d}J\, J\, F(J,t) & \text{(total momentum)} \\ E(t) = \int \mathrm{d}J\, H_0(J)\, F(J,t) & \text{(total energy)} \end{cases}$$

Dynamical **temperature**

$$\varepsilon(\omega) = 1 - \frac{1}{Q} Z'(\omega)$$

$$Q \propto k^2 \lambda_{\mathrm{D}}^2 \propto \frac{J_0 \Omega_0}{G M_{\mathrm{tot}}}$$

Q >> 1 is hot

Relaxation time

$$T_{\mathrm{relax}} \propto N^2\, Q^4\, T_{\mathrm{dyn}}$$

Well-posedness

High-order resonant denominator

$$\frac{\partial F(J)}{\partial t} \propto \mathscr{P}\int \mathrm{d}J_1 \, \frac{K(J, J_1)}{(\Omega[J] - \Omega[J_1])^4}$$
$$\propto \mathscr{P}\int \mathrm{d}\Omega_1 \, \frac{K(\Omega, \Omega_1)}{(\Omega - \Omega_1)^4}$$

Symmetrisation using **fundamental resonances**

$$K(\Omega, \Omega+\delta\Omega) = \mathcal{O}\left[(\delta\Omega)^3\right]$$

Most deft calculation

Principal value is **well-posed**

$$\mathscr{P}\int \mathrm{d}\Omega_1 \, \frac{K(\Omega, \Omega_1)}{(\Omega - \Omega_1)^4} \simeq \mathscr{P}\int \frac{\mathrm{d}\delta\Omega}{\delta\Omega}$$

H-Theorem

Boltzmann entropy

$$S(t) = -\int \mathrm{d}J\, s\big[F(J,t)\big] \quad \text{with} \quad s(F) = F\ln(F)$$

Rate of entropy growth

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{2\pi^3 m^2}{3} \sum_{k_1,k_2} \int \mathrm{d}\mathbf{J} \frac{1}{k_1^2(k_1+k_2)^2}\, \mathcal{P}\!\left(\frac{1}{(\Omega[J]-\Omega[J_1])^4}\right) \mathcal{U}_{k_1k_2}(\mathbf{J})$$
$$\times \frac{\delta_{\mathrm{D}}\big[\mathbf{k}\cdot\mathbf{\Omega}\big]}{F_3(\mathbf{J})} \left\{ (k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)} \right\}^2$$

H-Theorem

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq 0$$

Steady states

Boltzmann DF

$$F_{\mathrm{B}}(J) \propto \mathrm{e}^{-\beta H_0(J)+\gamma J}$$

$$\frac{\partial F_{\mathrm{B}}}{\partial t} \propto -\beta\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\, (\mathbf{k}\cdot\mathbf{\Omega}) = 0$$

Constraint from **H-Theorem** (with a non-vanishing coupling)

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}] \left\{ (k_1+k_2)\frac{F'(J)}{F(J)} - k_1 \frac{F'(J_1)}{F(J_1)} - k_2 \frac{F'(J_2)}{F(J_2)} \right\}^2$$

Line constraint

$$G(\Omega) = \frac{F'(J[\Omega])}{F(J[\Omega])} \;\Rightarrow\; \forall \Omega_1, \Omega_2 : G\left(\frac{k_1\Omega_1+k_2\Omega_2}{k_1+k_2} \right) = \frac{k_1 G(\Omega_1)+k_2 G(\Omega_2)}{k_1+k_2}$$

Recovering the Boltzmann DF

$$\frac{F'(J)}{F(J)} = -\beta\Omega(J) + \gamma \;\Rightarrow\; F(J) \propto \mathrm{e}^{-\beta H_0(J)+\gamma J}$$

Does it work?

Spin dynamics

$$H = \sum_{i=1}^{N} U_{\text{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m\, U(\mathbf{w}_i, \mathbf{w}_j)$$

Interaction potentials

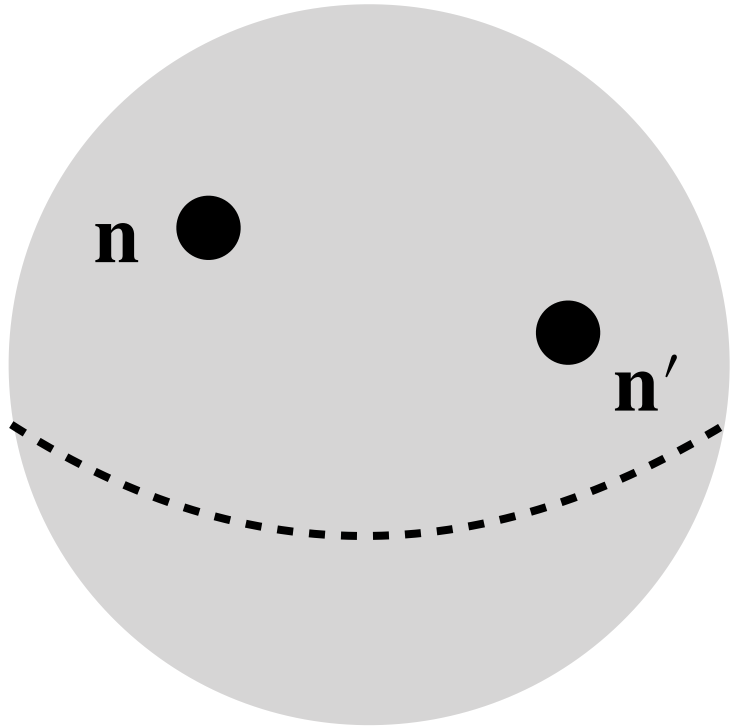
$$U_{\text{ext}}(\mathbf{w}) \propto \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}\right)^2; \quad U(\mathbf{w}, \mathbf{w}') \propto \mathbf{w} \cdot \mathbf{w}'$$

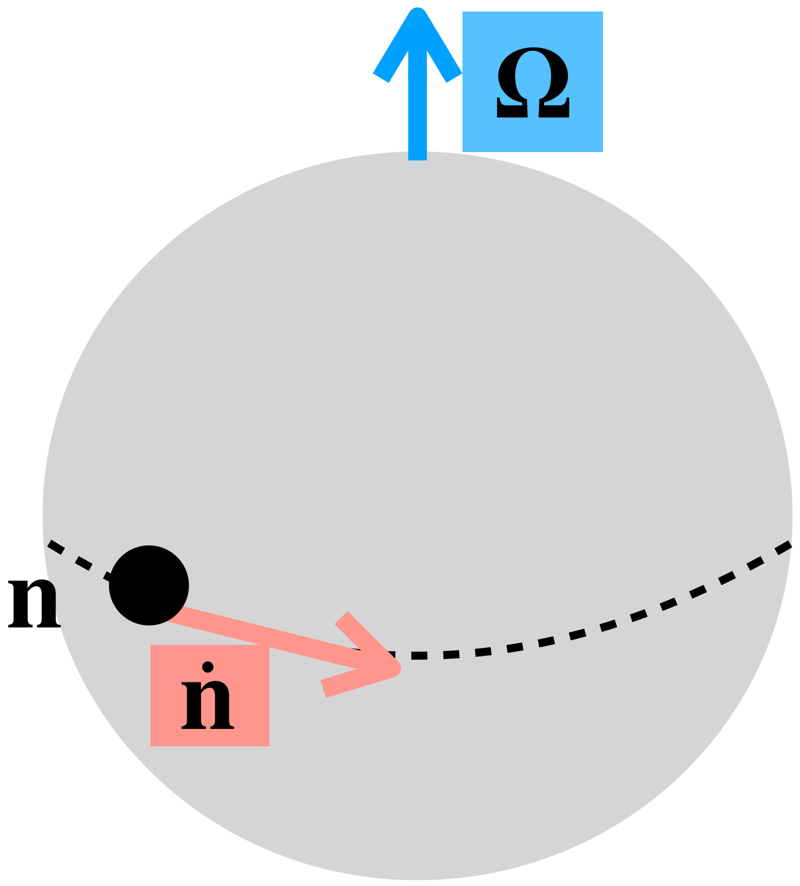
Dynamics

$$\dot{\mathbf{n}} = \frac{\partial H}{\partial \mathbf{n}} \times \mathbf{n}$$

Rotations preserve the geometric structure

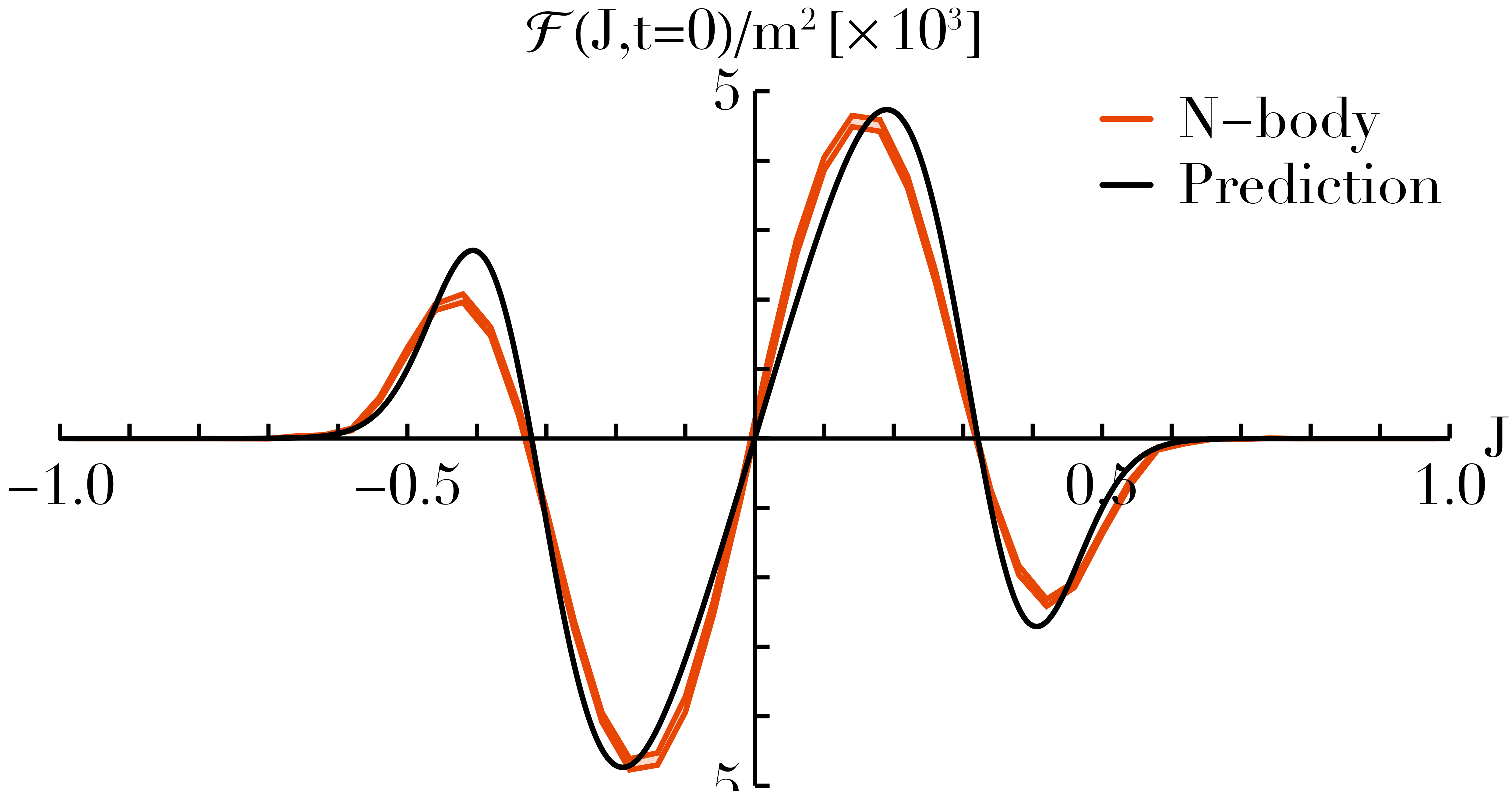
$$|\mathbf{n}| = 1$$





Does it match?

Diffusion flux $$\frac{\partial F}{\partial t} = \frac{\partial}{\partial J}\mathscr{F}(J)$$



Second-order kinetic blocking

Constraint from **H-Theorem**

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,\mathcal{U}_{k_1k_2}(\mathbf{J})\left\{(k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)}\right\}^2$$

Killing the interaction

$$\forall k_1, k_2, J, J_1:\ \mathcal{U}_{k_1k_2}(J, J_1, J_2^{\mathrm{res}}) = 0$$

Simple **frequency profile**

$$\Omega[J] \propto J$$

Extremely slow relaxation

$$T_{\mathrm{relax}} \propto N^3\, Q^6\, T_{\mathrm{dyn}}$$

Second-order kinetic blocking

$$U(\mathbf{w},\mathbf{w}') \propto |J-J'|^{\alpha} \sum_{k=1}^{+\infty} \frac{1}{|k|^{\alpha}} \cos\left[k(\theta-\theta')\right]$$

Second-order kinetic blocking

Constraint from **H-Theorem**

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\, \mathcal{U}_{k_1 k_2}(\mathbf{J}) \left\{ (k_1+k_2)\frac{F'(J)}{F(J)} - k_1 \frac{F'(J_1)}{F(J_1)} - k_2 \frac{F'(J_2)}{F(J_2)} \right\}^2$$

Killing the interaction

$$\forall k_1, k_2, J, J_1 : \;\; \mathcal{U}_{k_1 k_2}(J, J_1, J_2^{\mathrm{res}}) = 0$$

Simple **frequency profile**

$$\Omega[J] \propto J$$

Extremely slow relaxation

$$T_{\mathrm{relax}} \propto N^3\, Q^6\, T_{\mathrm{dyn}}$$

Second-order kinetic blocking (Challenging simulations)

$$U(\mathbf{w}, \mathbf{w}') \propto |J - J'|^{2n}\, B_{2n}\Big[\frac{1}{2\pi}\, w_{2\pi}(\theta - \theta')\Big]$$

Bernoulli polynomial

Angle wrapping

Next steps

Second-order kinetic blocking

$$\frac{\partial F(J)}{\partial t} \propto \frac{1}{N^3} \times \cdots$$

Crossed term

$$\frac{\partial G_3}{\partial t} \propto G_2^{(1)} \times G_2^{(1)}$$

Collective effects

$$\psi_{kk} \to \psi_{kk}/\varepsilon_k(k\,\Omega)$$

Klimontovich approach

$$\frac{\partial F}{\partial t} \propto \left\langle \left[\delta F, \delta\Phi\right] \right\rangle$$

1/N² inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J} \left[\sum_{k_1,k_2} \frac{1}{k_1^2(k_1+k_2)} \mathscr{P} \int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \right.$$
$$\left. \times \int \mathrm{d}J_2 \, \mathscr{U}_{k_1 k_2}(\mathbf{J}) \, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}] \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} \right) F_3(\mathbf{J}) \right]$$

with

$$\mathbf{J} = \big(J, J_1, J_2\big); \quad F_3(\mathbf{J}) = F(J)\, F(J_1)\, F(J_2)$$

$$\mathbf{k} = \big(k_1+k_2, -k_1, -k_2\big); \quad \mathbf{\Omega} = \big(\Omega[J], \Omega[J_1], \Omega[J_2]\big)$$