Collisionless collision integrals, the bad, the worse, and the ugly

Robbie Ewart with Michael Nastac, Andrew Brown, Toby Adkins & Alex Schekochihin

13th Vienna conference,
Friday 29th July

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- Collisionless collision integrals à la Lynden-Bell

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- ‘Real’ collision integrals

Primer: Waterbags and Lynden Bell statistics

- We begin with the Vlasov equation

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = 0,$$

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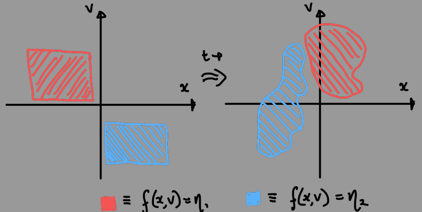
- This conserves phase volume:

$$\Gamma_\alpha(\eta) = \iint \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v}\, H\left(f_\alpha(\boldsymbol{r},\boldsymbol{v}) - \eta\right)$$

where, $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$$



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- Maximise the Entropy

$$S = -\frac{V}{\Delta\Gamma} \iint \mathrm{d}\boldsymbol{v}\, \mathrm{d}\eta\, P_0(\boldsymbol{v}, \eta) \ln P_0(\boldsymbol{v}, \eta), \quad f(\boldsymbol{v}) = \int \mathrm{d}\eta\, \eta P_0(\boldsymbol{v}, \eta)$$

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- This gives the equilibrium

$$P_0(\boldsymbol{v}, \eta) = \frac{e^{-\beta\Delta\Gamma\eta\left(\frac{1}{2}mv^2 - \mu(\eta)\right)}}{\int \mathrm{d}\eta'\, e^{-\beta\Delta\Gamma\eta\left(\frac{1}{2}mv^2 - \mu(\eta)\right)}}.$$

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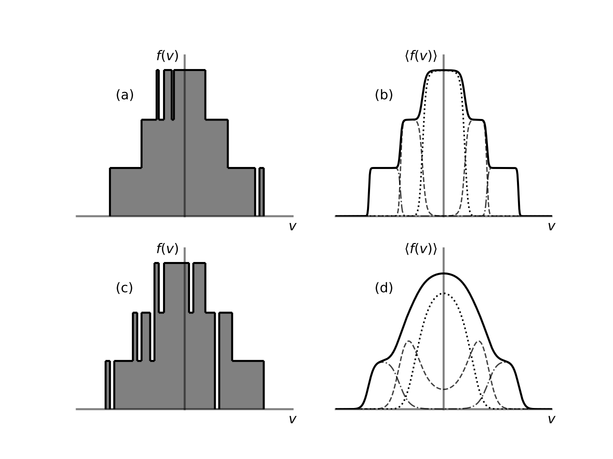

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$$\left\langle \delta f_\alpha(\boldsymbol{r},\boldsymbol{v})\delta f_{\alpha'}(\boldsymbol{r}',\boldsymbol{v}')\right\rangle \approx \Delta\Gamma_\alpha \delta_{\alpha\alpha'}\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(\boldsymbol{v}-\boldsymbol{v}')\left\langle \delta f_\alpha^2\right\rangle(\boldsymbol{v})$$

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- This gives a 'Proto-collision integral'

$$\frac{\partial f_{0\alpha}}{\partial t} = \sum_{\alpha''}\frac{16\pi^3 q_\alpha^2 q_{\alpha''}^2}{m_\alpha V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\cdot\int \mathrm{d}\boldsymbol{v}''\,\frac{\delta(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}''))}{\left|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}\right|^2}$$
$$\left[\frac{\Delta\Gamma_{\alpha''}}{m_\alpha}\left\langle \delta f_{\alpha''}^2\right\rangle(\boldsymbol{v}'')\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} - \frac{\Delta\Gamma_\alpha}{m_{\alpha''}}\left\langle \delta f_\alpha^2\right\rangle(\boldsymbol{v})\frac{\partial f_{0\alpha''}}{\partial \boldsymbol{v}''}\right].$$

The Closure Problem

- The microgranulation ansatz partially resolves the closure problem, but it is still present:

$$\begin{aligned}\left\langle \delta f_\alpha^2 \right\rangle(\boldsymbol{v}) &= \left\langle f_\alpha^2 \right\rangle(\boldsymbol{v}) - \left\langle f_\alpha \right\rangle^2(\boldsymbol{v}) \\ &= \int \mathrm{d}\eta\, \eta^2 P_{0\alpha}(\boldsymbol{v}, \eta) - \left(\int \mathrm{d}\eta\, \eta P_{0\alpha}(\boldsymbol{v}, \eta) \right)^2\end{aligned}$$

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- In general, the second moment isn't determined by the first, but we could ask what the maximum entropy assignment of $P_\alpha(\boldsymbol{v}, \eta)$ should be, subject to knowing the only the first moment.

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- I maximise that entropy subject to those constraints and I get the expression for the probability $P_0(\boldsymbol{v}, \eta)$

$$P_{0\alpha}(\boldsymbol{v}, \eta) = \frac{e^{-\psi_\alpha(\boldsymbol{v})\eta - \gamma_\alpha(\eta)}}{\int \mathrm{d}\eta' \, e^{-\psi_\alpha(\boldsymbol{v})\eta' - \gamma_\alpha(\eta')}}, \quad \langle f_\alpha^n \rangle(\boldsymbol{v}) = \int \mathrm{d}\eta \, \eta^n P_{0\alpha}(\boldsymbol{v}, \eta).$$

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What closure problem: Hyperkinetics

- It would be better if we could just write an evolution equation for $P(\boldsymbol{r}, \boldsymbol{v}, \eta)$

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- One plays all the same games

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$$\left\{\frac{\Delta\Gamma_{\alpha'}}{m_\alpha}\Big[\eta' - f_{0\alpha'}(\boldsymbol{v}')\Big]P_{0\alpha'}(\boldsymbol{v}',\eta')\left.\frac{\partial P_{0\alpha}}{\partial \boldsymbol{v}}\right|_\eta \right.$$
$$\left. - \frac{\Delta\Gamma_\alpha}{m_{\alpha'}}\Big[\eta - f_{0\alpha}(\boldsymbol{v})\Big]P_{0\alpha}(\boldsymbol{v},\eta)\left.\frac{\partial P_{0\alpha'}}{\partial \boldsymbol{v}'}\right|_{\eta'}\right\}.$$

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- What must it be quicker than:

$$\underbrace{\frac{\omega_{\text{pe}}}{n\lambda_{\text{De}}^3}}_{\text{True colls. on } f} \ll \underbrace{\frac{\omega_{\text{pe}}}{(n\lambda_{\text{De}}^3)^{1/3}}}_{\text{True colls. on} \delta f} \ll T_{\text{Nastac}}^{-1} \ll \Delta\Gamma \eta_{\text{eff}} \frac{\omega_{\text{pe}}}{n\lambda_{\text{De}}^3} \ll \omega_{\text{pe}}.$$

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$$C_{\mathrm{ei}} = \frac{\gamma_{\mathrm{ei}}}{m_{\mathrm{e}}} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \frac{\mathrm{d}\boldsymbol{v}'}{w} \left(\mathbf{I} - \frac{\boldsymbol{w}\boldsymbol{w}}{w^2} \right) \cdot \int \mathrm{d}\eta' \, \eta'$$

$$\left\{ \frac{\Delta\Gamma_{\mathrm{i}}}{m_{\mathrm{e}}} \left[\eta' - f_{\mathrm{i}}(\boldsymbol{v}')\right] P_{\mathrm{i}}(\boldsymbol{v}', \eta') \left. \frac{\partial P_{\mathrm{e}}}{\partial \boldsymbol{v}} \right|_{\eta} - \frac{\Delta\Gamma_{\mathrm{e}}}{m_{\mathrm{i}}} \left[\eta - f_{\mathrm{e}}(\boldsymbol{v})\right] P_{\mathrm{e}}(\boldsymbol{v}, \eta) \left. \frac{\partial P_{\mathrm{i}}}{\partial \boldsymbol{v}'} \right|_{\eta'} \right\},$$

$$\tilde{C}_{\mathrm{ei}} = \frac{\tilde{\gamma}_{\mathrm{ei}}}{m_{\mathrm{e}}} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \frac{\mathrm{d}\boldsymbol{v}'}{w} \left(\mathbf{I} - \frac{\boldsymbol{w}\boldsymbol{w}}{w^2} \right) \cdot \left[\frac{1}{m_{\mathrm{e}}} f_{\mathrm{i}}(\boldsymbol{v}') \frac{\partial f_{\mathrm{e}}}{\partial \boldsymbol{v}} - \frac{1}{m_{\mathrm{i}}} f_{\mathrm{e}}(\boldsymbol{v}) \frac{\partial P_{\mathrm{i}}}{\partial \boldsymbol{v}'} \right].$$

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- For the purpose of electron-ion collisions (to lowest order) we can port over a great deal of intuition from collisional theory with the simple change

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$$\begin{aligned} n_{\mathrm{i}} &= \int \mathrm{d}\boldsymbol{v}\, f_{\mathrm{i}}(\boldsymbol{v}) & & & n_{\mathrm{i}}^{\mathrm{a}} &= \iint \mathrm{d}\boldsymbol{v}\,\mathrm{d}\eta\, \left[\eta - f_{\mathrm{i}}(\boldsymbol{v})\right]^2 P_{\mathrm{i}}(\boldsymbol{v}, \eta) \\ \boldsymbol{u}_{\mathrm{i}} &= \frac{1}{n_{\mathrm{i}}} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_{\mathrm{i}}(\boldsymbol{v}) & & \to & \boldsymbol{u}_{\mathrm{i}}^{\mathrm{a}} &= \frac{1}{n_{\mathrm{i}}^{\mathrm{a}}} \iint \mathrm{d}\boldsymbol{v}\,\mathrm{d}\eta\, \boldsymbol{v} \left[\eta - f_{\mathrm{i}}(\boldsymbol{v})\right]^2 P_{\mathrm{i}}(\boldsymbol{v}, \eta) \end{aligned}$$

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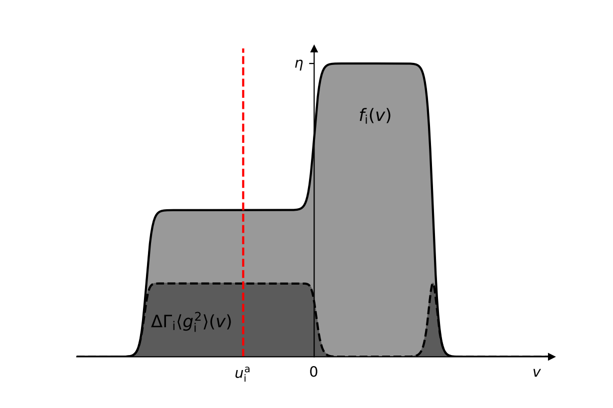
\boldsymbol{u}_{\mathrm{i}} &= \frac{1}{n_{\mathrm{i}}} \int \mathrm{d}\boldsymbol{v} \, \boldsymbol{v} f_{\mathrm{i}}(\boldsymbol{v}) & & \rightarrow & \boldsymbol{u}_{\mathrm{i}}^{\mathrm{a}} &= \frac{1}{n_{\mathrm{i}}^{\mathrm{a}}} \iint \mathrm{d}\boldsymbol{v} \, \mathrm{d}\eta \, \boldsymbol{v} \left[\eta - f_{\mathrm{i}}(\boldsymbol{v})\right]^2 P_{\mathrm{i}}(\boldsymbol{v}, \eta) \\
\boldsymbol{F}_{\mathrm{ei}} &\propto -\left(\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{i}}\right) & & & \boldsymbol{F}_{\mathrm{ei}} &\propto -\left(\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{i}}^{\mathrm{a}}\right)
\end{aligned}$$

Strange relaxation # 3

$$n_\mathrm{i}^\mathrm{a} = \iint \mathrm{d}\boldsymbol{v}\,\mathrm{d}\eta\;[\eta - f_\mathrm{i}(\boldsymbol{v})]^2\,P_\mathrm{i}(\boldsymbol{v},\eta)$$

$$\boldsymbol{u}_\mathrm{i}^\mathrm{a} = \frac{1}{n_\mathrm{i}^\mathrm{a}} \iint \mathrm{d}\boldsymbol{v}\,\mathrm{d}\eta\,\boldsymbol{v}\,[\eta - f_\mathrm{i}(\boldsymbol{v})]^2\,P_\mathrm{i}(\boldsymbol{v},\eta)$$



‘Real’ Collision integrals

- Back to the (forced) Vlasov-Poisson equation

$$\frac{\partial f_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v} f_{\boldsymbol{k}} = i\frac{q}{m}\varphi_{\boldsymbol{k}}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + i\frac{q}{m}\sum_{\boldsymbol{k}'\neq\boldsymbol{k}}\varphi_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial f_{\boldsymbol{k}-\boldsymbol{k}'}}{\partial \boldsymbol{v}}$$

$$\varphi_{\boldsymbol{k}} = \chi_{\boldsymbol{k}} + \phi_{\boldsymbol{k}} = \chi_{\boldsymbol{k}} + \frac{4\pi q_\alpha}{k^2}\int \mathrm{d}\boldsymbol{v}'\, f_{\boldsymbol{k}}(\boldsymbol{v}')$$

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$$\left\langle \chi_{\boldsymbol{k}}(t)\right\rangle = 0, \quad \left\langle \chi_{\boldsymbol{k}}(t)\chi^*_{\boldsymbol{k}'}(t')\right\rangle = \frac{2m^2}{q^2}\delta_{\boldsymbol{k}\boldsymbol{k}'}D_k\delta(t-t')$$

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- The evolution of the mean distribution function is

$$\frac{\partial\langle f_0\rangle}{\partial t} = \frac{q}{m}\frac{\partial}{\partial \boldsymbol{v}}\sum_{\boldsymbol{k}}\boldsymbol{k}\mathrm{Im}\langle\chi^*_k f_{\boldsymbol{k}}(\boldsymbol{v})\rangle + \frac{4\pi q^2}{m}\frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k^2}\int \mathrm{d}\boldsymbol{v}'\,\langle f_{\boldsymbol{k}}(\boldsymbol{v})f^*_{\boldsymbol{k}}(\boldsymbol{v}')\rangle$$

‘Real’ collision integrals # 2

- I need to know how to handle $\langle \chi_{\boldsymbol{k}} f_{\boldsymbol{k}}^* \rangle$ and $C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') = \langle f_{\boldsymbol{k}}(\boldsymbol{v}) f_{\boldsymbol{k}}(\boldsymbol{v}') \rangle$
- Gaussian white-noise forcing is easy to handle via the Furutsu-Novikov theorem

$$\langle \chi_{\boldsymbol{k}}^* f_{\boldsymbol{k}} \rangle = i \frac{m}{q} D_k \boldsymbol{k} \frac{\partial \langle f_0 \rangle}{\partial \boldsymbol{v}'}$$

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‘Real’ collision integrals # 2

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- We will try to compute $C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}')$ by writing down its evolution equation and looking for steady states

‘Real’ collision integrals # 3

$$\frac{\partial C_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}} = S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + N_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + i\frac{4\pi q^2}{mk^2}\int \mathrm{d}\boldsymbol{v}''\, C_{\boldsymbol{k}}(\boldsymbol{v}'',\boldsymbol{v}')\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} - C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v}'')\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}'}$$

$$C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',t) = C^{(0)}_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',\tau,T) + C^{(1)}_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',\tau,T) + ...$$

'Real' collision integrals # 3

$$\frac{\partial C_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot(\boldsymbol{v} - \boldsymbol{v}')C_{\boldsymbol{k}} = S_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') + N_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') + i\frac{4\pi q^2}{mk^2}\int \mathrm{d}\boldsymbol{v}''\, C_{\boldsymbol{k}}(\boldsymbol{v}'', \boldsymbol{v}')\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} - C_{\boldsymbol{k}}(\boldsymbol{v}', \boldsymbol{v}'')\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}'}$$

$$S_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') = 2D_k \boldsymbol{k}\boldsymbol{k} : \frac{\partial f_0}{\partial \boldsymbol{v}}\frac{\partial f_0}{\partial \boldsymbol{v}'}$$

$$N_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') = \tilde{D}\left(\frac{\partial}{\partial \boldsymbol{v}} + \frac{\partial}{\partial \boldsymbol{v}'}\right)^2 C_{\boldsymbol{k}} + 2\sum_{\boldsymbol{k}'} \tilde{D}_{k'}\boldsymbol{k}'\boldsymbol{k}' : \frac{\partial^2}{\partial \boldsymbol{v}\partial \boldsymbol{v}'}\left(C_{\boldsymbol{k}-\boldsymbol{k}'} - C_{\boldsymbol{k}}\right)$$

$$C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}', t) = C_{\boldsymbol{k}}^{(0)}(\boldsymbol{v}, \boldsymbol{v}', \tau, T) + C_{\boldsymbol{k}}^{(1)}(\boldsymbol{v}, \boldsymbol{v}', \tau, T) + ...$$

'Real' collision integrals # 3

$$\frac{\partial C_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot(\boldsymbol{v} - \boldsymbol{v}')C_{\boldsymbol{k}} = S_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') + N_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}')$$

$$+ i\frac{4\pi q^2}{mk^2} \int \mathrm{d}\boldsymbol{v}''\, C_{\boldsymbol{k}}(\boldsymbol{v}'', \boldsymbol{v}')\boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} - C_{\boldsymbol{k}}(\boldsymbol{v}', \boldsymbol{v}'')\boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}'}$$

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- For now, I'm going to aim to solve this within the Bogoliubov hypothesis

$$C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}', t) = C_{\boldsymbol{k}}^{(0)}(\boldsymbol{v}, \boldsymbol{v}', \tau, T) + C_{\boldsymbol{k}}^{(1)}(\boldsymbol{v}, \boldsymbol{v}', \tau, T) + ...$$

‘Real’ collision integrals: The schematic plan

- Solve the lowest order correlation function in steady state and plug it into the evolution equation for $f_0(\boldsymbol{v})$

$$\frac{\partial C_{\boldsymbol{k}}^{(0)}}{\partial \tau} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}}^{(0)} = S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + (...)^{(0)}$$

$$\frac{\partial\langle f_0\rangle}{\partial T} = D\frac{\partial^2\langle f_0\rangle}{\partial \boldsymbol{v}^2} + \frac{4\pi q^2}{m}\frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k^2}\int \mathrm{d}\boldsymbol{v}'\, C_{\boldsymbol{k}}^{(0)}(\boldsymbol{v},\boldsymbol{v}')$$

‘Real’ collision integrals: The schematic plan

- Solve the lowest order correlation function in steady state and plug it into the evolution equation for $f_0(\boldsymbol{v})$

$$\frac{\partial C^{(0)}_{\boldsymbol{k}}}{\partial \tau} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C^{(0)}_{\boldsymbol{k}} = S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + (...)^{(0)}$$

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- Use this expression to fake the long time evolution of $C^{(0)}_{\boldsymbol{k}}$, plug that back in to the next order to find the short time evolution of $C^{(1)}_{\boldsymbol{k}}$

$$\frac{\partial C^{(1)}_{\boldsymbol{k}}}{\partial \tau} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C^{(1)}_{\boldsymbol{k}} = -\frac{\partial C^{(0)}_{\boldsymbol{k}}}{\partial T} + (...)^{(1)}$$

‘Real’ collision integrals: The schematic plan

- Solve the lowest order correlation function in steady state and plug it into the evolution equation for $f_0(\boldsymbol{v})$

$$\frac{\partial C_{\boldsymbol{k}}^{(0)}}{\partial \tau} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}}^{(0)} = S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + (...)^{(0)}$$

$$\frac{\partial\langle f_0\rangle}{\partial T} = D\frac{\partial^2\langle f_0\rangle}{\partial \boldsymbol{v}^2} + \frac{4\pi q^2}{m}\frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k^2}\int \mathrm{d}\boldsymbol{v}'\, C_{\boldsymbol{k}}^{(0)}(\boldsymbol{v},\boldsymbol{v}')$$

- Use this expression to fake the long time evolution of $C_{\boldsymbol{k}}^{(0)}$, plug that back in to the next order to find the short time evolution of $C_{\boldsymbol{k}}^{(1)}$

$$\frac{\partial C_{\boldsymbol{k}}^{(1)}}{\partial \tau} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}}^{(1)} = -\frac{\partial C_{\boldsymbol{k}}^{(0)}}{\partial T} + (...)^{(1)}$$

- Iterate until satisfied

$$\frac{\partial\langle f_0\rangle}{\partial T} = D\frac{\partial^2\langle f_0\rangle}{\partial \boldsymbol{v}^2} + \frac{4\pi q^2}{m}\frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k^2}\int \mathrm{d}\boldsymbol{v}'\, C_{\boldsymbol{k}}^{(0)}(\boldsymbol{v},\boldsymbol{v}') + C_{\boldsymbol{k}}^{(1)}(\boldsymbol{v},\boldsymbol{v}')$$

Bonus slides: H-theorem

$$S = -\sum_\alpha \frac{1}{\Delta\Gamma_\alpha} \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\eta\, P_{0\alpha} \ln P_{0\alpha}.$$

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\sum_\alpha \frac{V}{\Delta\Gamma_\alpha} \int \mathrm{d}\boldsymbol{v}\, \mathrm{d}\eta \left(1 + \ln P_{0\alpha}\right) \frac{\partial P_{0\alpha}}{\partial t} \\
&= \sum_\alpha \sum_{\alpha''} \frac{16\pi^3 q_\alpha^2 q_{\alpha''}^2}{\Delta\Gamma_\alpha \Delta\Gamma_{\alpha''}} \int \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{v}'' \sum_{\boldsymbol{k}} \frac{\delta(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}''))}{k^4 |\epsilon_{\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{v}}|^2} \int \mathrm{d}\eta\, \mathrm{d}\eta''\, \eta'' \\
&\left[\frac{\Delta\Gamma_{\alpha''}^2}{m_\alpha^2} (\eta'' - f_{0\alpha''}(\boldsymbol{v}'')) \frac{P_{0\alpha''}(\boldsymbol{v}'', \eta'')}{P_{0\alpha}(\boldsymbol{v}, \eta)} \left(\boldsymbol{k} \cdot \left.\frac{\partial P_{0\alpha}}{\partial \boldsymbol{v}}\right|_\eta \right)^2 - \right. \\
&\left. \frac{\Delta\Gamma_\alpha \Delta\Gamma_{\alpha''}}{m_\alpha m_{\alpha''}} (\eta - f_{0\alpha}(\boldsymbol{v})) \boldsymbol{k} \cdot \left.\frac{\partial P_{0\alpha}}{\partial \boldsymbol{v}}\right|_\eta \boldsymbol{k} \cdot \left.\frac{\partial P_{0\alpha}}{\partial \boldsymbol{v}''}\right|_{\eta''} \right].
\end{aligned} \tag{1}$$

Bonus slides: H-theorem # 2

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{\alpha}\sum_{\alpha''}\frac{16\pi^3 q_\alpha^2 q_{\alpha''}^2}{\Delta\Gamma_\alpha \Delta\Gamma_{\alpha''}}\int \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}''\sum_{\boldsymbol{k}}\frac{\delta(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}''))}{k^4|\epsilon_{\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{v}}|^2}\int \mathrm{d}\eta\,\mathrm{d}\eta''$$
$$\left(\frac{\Delta\Gamma_{\alpha''}}{m_\alpha}(\eta''-f_{0\alpha''}(\boldsymbol{v}''))\sqrt{\frac{P_{0\alpha''}(\boldsymbol{v}'',\eta'')}{P_{0\alpha}(\boldsymbol{v},\eta)}}\boldsymbol{k}\cdot\frac{\partial P_{0\alpha}}{\partial \boldsymbol{v}}\bigg|_{\eta}\right.$$
$$\left.-\frac{\Delta\Gamma_{\alpha}}{m_{\alpha''}}(\eta-f_{0\alpha}(\boldsymbol{v}))\sqrt{\frac{P_{0\alpha}(\boldsymbol{v},\eta)}{P_{0\alpha''}(\boldsymbol{v}'',\eta'')}}\boldsymbol{k}\cdot\frac{\partial P_{0\alpha''}}{\partial \boldsymbol{v}''}\bigg|_{\eta''}\right)^2 \geq 0.$$