Quasilinear theory: adding inhomogeneities and collisions from first principles

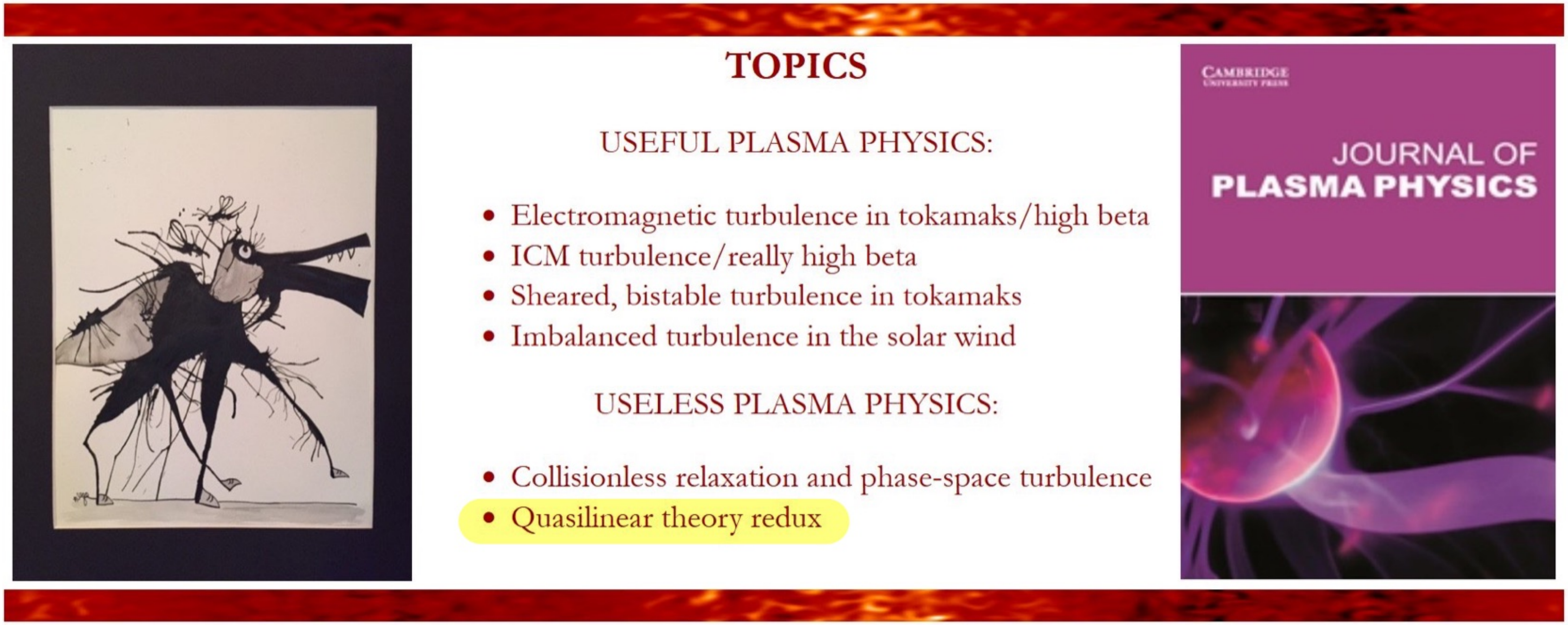
Ilya Y. Dodin

Princeton Plasma Physics Laboratory
Princeton University

13th Plasma Kinetics Working Meeting
Wolfgang Pauli Institute, Vienna

arxiv:2201.08562v4, to appear in J. Plasma Phys.

Protest & Outline: QL theory redux is not useless!



- What's wrong with the textbook theory?

  Fails to conserve the wave action, misses inhomogeneities and collisions.

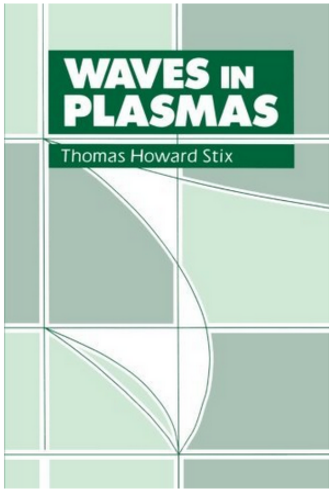
- How do we fix this?

  Hmm... *Weyl calculus?!* Duh!

- Examples: grand unification

  Electrostatic turbulence, electromagnetic turbulence, relativistic gravity

Quasilinear theory: the textbook version



- For simplicity, consider 1D Langmuir turbulence:

$$\partial_t f + v\partial_x f + ({}^e\!/_m)\,\widetilde{E}\partial_v f = 0, \quad f = \overline{f} + \widetilde{f}, \quad \widetilde{f} \ll \overline{f}$$

- **QL approximation:** keep the nonlinearity in $\partial_t \overline{f}$ but linearize the equation for $\widetilde{f}$

$$\frac{\partial \overline{f}}{\partial t} + \left\langle \frac{e}{m}\widetilde{E}\frac{\partial \widetilde{f}}{\partial v}\right\rangle = 0, \qquad \frac{\partial \widetilde{f}}{\partial t} + v\frac{\partial \widetilde{f}}{\partial x} + \frac{e}{m}\widetilde{E}\frac{\partial \overline{f}}{\partial v} \cancel{+ \frac{e}{m}\widetilde{E}\frac{\partial \widetilde{f}}{\partial v} - \left\langle \frac{e}{m}\widetilde{E}\frac{\partial \widetilde{f}}{\partial v}\right\rangle} = 0$$

- Use global-mode decomposition $\widetilde{E} = \sum_n \widetilde{E}_k e^{ikx}$, where $k = 2\pi n/L$. Solve for $\widetilde{f} = \sum_n \widetilde{f}_k e^{ikx}$ to the zeroth order in ∂_t, neglect the initial conditions:

$$\widetilde{f}_k = \cancel{g_k} - \frac{i({}^e\!/_m)\widetilde{E}_k}{\omega_k - kv}\frac{\partial \overline{f}}{\partial v}, \qquad \omega_k \doteq i\,\frac{\mathrm{d}\ln \widetilde{E}_k}{\mathrm{d}t}$$

Vedenov *et al.* (1961); Drummond and Pines (1962)

Quasilinear theory: the textbook version (continued)

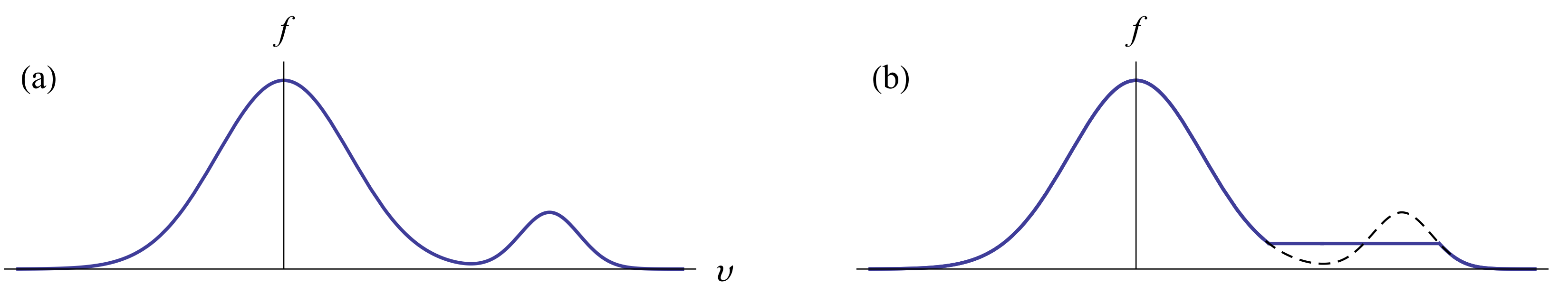
- The average distribution satisfies a diffusion equation:

$$\frac{\partial \overline{f}}{\partial t} = \frac{\partial}{\partial v}\left(\mathsf{D}(t,v)\,\frac{\partial \overline{f}(t,v)}{\partial v}\right), \qquad \mathsf{D} = \frac{e^2}{m^2}\int_{\mathsf{L}} \frac{\mathrm{d}k}{2\pi L}\frac{|\widetilde{E}_k(t)|^2}{i\,[kv-\omega_k(t)]}$$

- To close the system, one assumes that the local "frequency" $\omega_k \doteq i\,\mathrm{d}_t \ln \widetilde{E}_k$ satisfies the linear dispersion relation (with $\gamma_k \doteq \mathrm{im}\,\omega_k$):

$$1 - \frac{4\pi e^2}{mk^2}\int_{\mathsf{L}} \mathrm{d}v\,\frac{\partial_v \overline{f}(t,v)}{v-\omega_k(t)/k} = 0, \qquad \frac{\mathrm{d}|\widetilde{E}_k|^2}{\mathrm{d}t} = 2\gamma_k|\widetilde{E}_k|^2$$

- This closed model conserves particles, momentum and energy, and accurately describes effects like broadband bump-on-tail instability... *right?*



Quasilinear theory: the textbook version (continued)

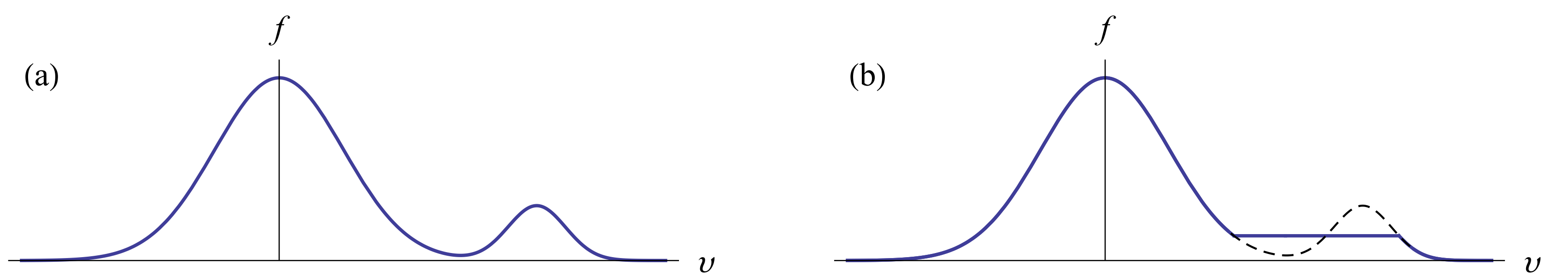
- The average distribution satisfies a diffusion equation:

$$\frac{\partial \overline{f}}{\partial t} = \frac{\partial}{\partial v}\left(\mathsf{D}(t,v)\,\frac{\partial \overline{f}(t,v)}{\partial v}\right), \qquad \mathsf{D} = \frac{e^2}{m^2}\int_{\mathsf{L}} \frac{\mathrm{d}k}{2\pi L}\frac{|\widetilde{E}_k(t)|^2}{i\,[kv-\omega_k(t)]}$$

- To close the system, one assumes that the local "frequency" $\omega_k \doteq i\,\mathrm{d}_t \ln \widetilde{E}_k$ satisfies the linear dispersion relation (with $\gamma_k \doteq \mathrm{im}\,\omega_k$):

$$1 - \frac{4\pi e^2}{mk^2}\int_{\mathsf{L}} \mathrm{d}v\,\frac{\partial_v \overline{f}(t,v)}{v-\omega_k(t)/k} = 0, \qquad \frac{\mathrm{d}|\widetilde{E}_k|^2}{\mathrm{d}t} = 2\gamma_k|\widetilde{E}_k|^2$$

- This closed model conserves particles, momentum and energy, and accurately describes effects like broadband bump-on-tail instability... *right?* **Not so much.**



Vedenov *et al.* (1961); Drummond and Pines (1962)

The traditional QL theory is conservative only accidentally.

- **First error:** the correct equation for the linear-wave amplitude is the *action-conservation law* (introduced after the original QL theory,* ignored in the canon):

$$\xcancel{\frac{\mathrm{d}|\widetilde{E}_k|^2}{\mathrm{d}t} = 2\gamma_k|\widetilde{E}_k|^2}$$

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = 2\gamma_k\mathcal{I}, \qquad \mathcal{I} = |\widetilde{E}_k|^2 \left.\frac{\partial_\omega(\omega^2\epsilon_{\mathrm{H}})}{16\pi\omega^2}\right|_{(\omega_{\boldsymbol{k}},\boldsymbol{k})}$$

- The equations for $\mathcal{I}$ and for $|E_k|^2$ agree only in the cold limit.

- But then, why is the QL theory *exactly* conservative? Answer: it contains a **second error** that partly compensates the first one!

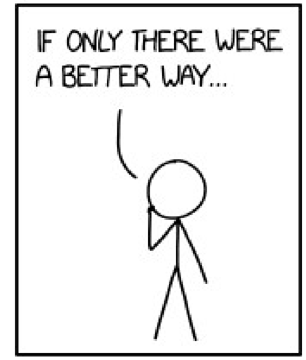
$$\widetilde{f}_k = \cancel{g_k} - \frac{i(e/m)\widetilde{E}_k}{\omega_k - kv}\frac{\partial\overline{f}}{\partial v} + \underbrace{\mathcal{O}(\partial_t\overline{f})}_{\text{non-negligible}} + (\text{may as well include } \partial_x)$$

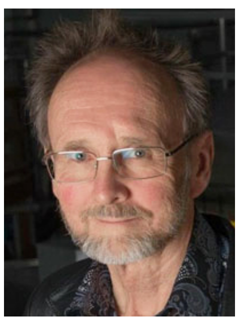
- The 'ponderomotive' effects due to $\mathcal{O}(\overline{\partial}_t, \overline{\partial}_x)$ determine the conservation laws. Those critically affect, for example, RF current drive and turbulence saturation.

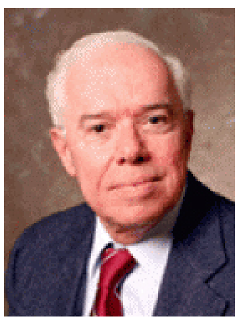
* Whitham (1965); Bretherton and Garrett (1968); Dewar (1972); Dodin *et al.* (2019) . . .

‘Kind-of-known’ approach: oscillation-center (OC) QL theory

- With $\mathcal{O}(\overline{\partial}_t, \overline{\partial}_x)$ retained, the only known approach is heuristic:
  - ignore $\mathcal{O}(\overline{\partial}_t, \overline{\partial}_x)$ near resonances separated out arbitrarily
  - OC coordinate transformation for non-resonant particles
  - *assume* the proper action equation to connect the dots







$$\frac{\partial F_0}{\partial t} + \frac{\partial \langle K \rangle}{\partial \mathbf{p}} \cdot \frac{\partial F_0}{\partial \mathbf{x}} - \frac{\partial \langle K \rangle}{\partial \mathbf{x}} \cdot \frac{\partial F_0}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathsf{D} \cdot \frac{\partial F_0}{\partial \mathbf{p}} \right)$$

$$\frac{\partial n_k{}^l}{\partial t} + \frac{\partial \omega_k{}^l}{\partial \mathbf{k}} \cdot \frac{\partial n_k{}^l}{\partial \mathbf{x}} - \frac{\partial \omega_k{}^l}{\partial \mathbf{x}} \cdot \frac{\partial n_k{}^l}{\partial \mathbf{k}} = 2\gamma_k{}^l n_k{}^l, \qquad n_k{}^l \equiv \frac{\partial \epsilon_r}{\partial \omega_k{}^l} \frac{k^2 \mid \phi_k{}^l \mid^2}{8\pi L^6}$$

$$\mathsf{D} = 4\pi e^2 \sum_{k,l} (\mid \phi_k{}^l \mid^2 / 8\pi L^6) \mathbf{k}\mathbf{k} 2\pi\delta(\omega_k{}^l - \mathbf{k}\cdot\mathbf{p}/m), \qquad \langle K \rangle = H_0 - \sum_{k,l} \frac{4\pi e^2}{m} \frac{k^2 \mid \phi_k{}^l \mid^2}{8\pi L^6} \frac{\partial}{\partial \omega_k{}^l} \frac{\Pi(\omega_k{}^l - \mathbf{k}\cdot\mathbf{p}/m)}{\omega_k{}^l - \mathbf{k}\cdot\mathbf{p}/m}$$

- Oscillation-center QL theory is not actually proven, not expressed in terms of measurable quantities, not extendable to off-shell waves and collisional plasmas.

Dewar (1973); McDonald *et al.* (1985)

Image: adapted from xkcd.com

A first-principle approach

- Use $\partial_t f = \{\overline{H} + \widetilde{H}, f\}$, split $f = \overline{f} + \widetilde{f}$, and linearize the equation for fluctuations $\widetilde{f}$:

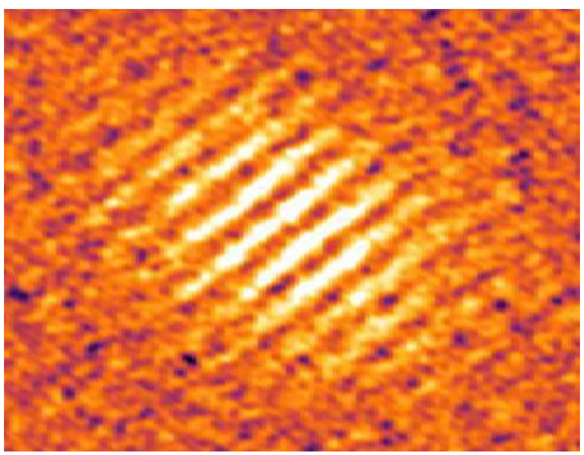
$$\partial_t \widetilde{f} - \{\overline{H}, \widetilde{f}\} = \{\widetilde{H}, \overline{f}\}, \qquad \partial_t \overline{f} - \{\overline{H}, \overline{f}\} = \langle\{\widetilde{H}, \widetilde{f}\}\rangle$$

- Define phase-space velocities in general *canonical* coordinates $\boldsymbol{z}$, with $\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$$v^\alpha(t, \boldsymbol{z}) \doteq J^{\alpha\beta} \underbrace{\partial_\beta \overline{H}(t, \boldsymbol{z})}_{\mathcal{O}(1)} \qquad u^\alpha(t, \boldsymbol{z}) \doteq J^{\alpha\beta} \underbrace{\partial_\beta \widetilde{H}(t, \boldsymbol{z})}_{\mathcal{O}(\varepsilon)}$$

- In terms of the Green's operator $\widehat{G}$ and unperturbed microscopic fluctuations g:

$$\widetilde{f} = g - \widehat{G}\widehat{u}^\alpha \partial_\alpha \overline{f}, \qquad \widehat{u}^\alpha = u^\alpha(t, \widehat{\boldsymbol{z}}), \qquad \widehat{G} = \lim_{\nu \to 0+} \int_0^\infty \mathrm{d}\tau \, \mathrm{e}^{-\nu\tau - \tau(\partial_{\boldsymbol{t}} + \boldsymbol{v}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}})}$$



- Define $\widehat{D}^{\alpha\beta} = \overline{\widehat{u}^\alpha \widehat{G} \widehat{u}^\beta}$ and $\mathfrak{F}^\alpha = \overline{u^\alpha g}/\overline{f}$. Then,

$$\boxed{\partial_t \overline{f} - \{\overline{H}, \overline{f}\} = \partial_\alpha (\widehat{D}^{\alpha\beta} \partial_\beta \overline{f} - \mathfrak{F}^\alpha \overline{f})}$$

  - macroscopic part of u^α → **generalized QLT**
  - microscopic part of u^α → **collision operator***

Dodin (2022)

*No need to consider two-particle distributions!

To approximate an operator, approximate its Weyl symbol.

- Any operator $\widehat{\mathsf{A}}\psi(\mathsf{x}) = \int \mathrm{d}\mathsf{x}'\,\mathcal{A}(\mathsf{x},\mathsf{x}')\,\psi(\mathsf{x}')$ on "any" space x can be expressed through its *symbol* using $\widehat{\mathsf{x}} = \mathsf{x}$ and $\widehat{\mathsf{k}} = -i\partial_{\mathsf{x}}$:

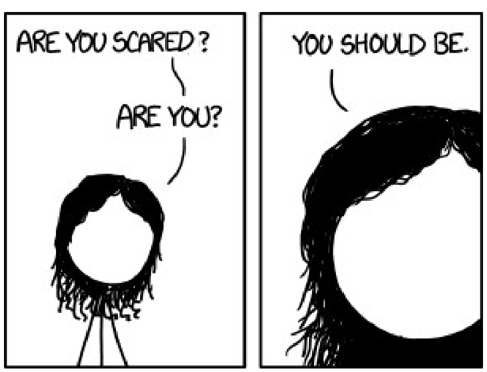
$$\mathsf{A}(\mathsf{x},\mathsf{k}) = \int \mathrm{d}\mathsf{s}\,\mathcal{A}(\mathsf{x}+\mathsf{s}/2,\mathsf{x}-\mathsf{s}/2)\,\mathrm{e}^{-i\mathsf{k}\cdot\mathsf{s}}$$

$$\widehat{\mathsf{A}} = \frac{1}{(2\pi)^{2\mathsf{n}}}\int \mathrm{d}\mathsf{x}'\,\mathrm{d}\mathsf{k}'\,\mathrm{d}\mathsf{x}''\,\mathrm{d}\mathsf{k}''\,\mathsf{A}(\mathsf{x}',\mathsf{k}')\,\mathrm{e}^{i\mathsf{k}''\cdot(\mathsf{x}'-\widehat{\mathsf{x}})-i\mathsf{x}''\cdot(\mathsf{k}'-\widehat{\mathsf{k}})}$$

$$\widehat{1} \Leftrightarrow 1$$
$$\widehat{\mathsf{x}} \Leftrightarrow \mathsf{x}$$
$$\widehat{\mathsf{k}} \Leftrightarrow \mathsf{k}$$
$$\widehat{\mathsf{A}}^\dagger \Leftrightarrow \mathsf{A}^\dagger$$
$$\widehat{\mathsf{A}}\widehat{\mathsf{B}} \Leftrightarrow \mathsf{A} \star \mathsf{B}$$

- $\widehat{\boldsymbol{D}}$ acts on functions of 7D $\boldsymbol{X} \doteq (t,\boldsymbol{x},\boldsymbol{p})$, so our $\widehat{\mathsf{x}}$ operator is $\widehat{\boldsymbol{X}}$. Correspondingly, our $\widehat{\mathsf{k}}$ operator is $\widehat{\boldsymbol{K}} \doteq (i\partial_t, -i\partial_{\boldsymbol{x}}, -i\partial_{\boldsymbol{p}})$, it induces a 7D space $\boldsymbol{K} \doteq (-\omega,\boldsymbol{k},\boldsymbol{r})$.

$$\boldsymbol{D}(\boldsymbol{X},\boldsymbol{K}) = \int \mathrm{d}\boldsymbol{K}'\,\overline{\boldsymbol{W}}_{\boldsymbol{u}}(\boldsymbol{X},\boldsymbol{K}')\,G(\boldsymbol{X},\boldsymbol{K}-\boldsymbol{K}'), \qquad \boldsymbol{W}_{\boldsymbol{u}} \sim \text{symb}\,|\boldsymbol{u}\rangle\langle\boldsymbol{u}|$$



- The following approximation will be enough (tough to prove!):

$$\boldsymbol{D}(\boldsymbol{X},\boldsymbol{K}) \approx \underbrace{\boldsymbol{D}(\boldsymbol{X},\boldsymbol{0})}_{\text{usual QLT}} + \underbrace{(\boldsymbol{K}\cdot\partial_{\boldsymbol{K}})\boldsymbol{D}(\boldsymbol{X},\boldsymbol{0})}_{\text{non-negligible correction}}$$

$$G(\boldsymbol{X},\boldsymbol{K}) \approx \pi\,\delta(\Omega) + i\,\lim_{\nu\to 0}\frac{\Omega}{\Omega^2+\nu^2}, \qquad \Omega \approx \omega - \boldsymbol{k}\cdot\boldsymbol{v}$$

Image: adapted from xkcd.com

$\boldsymbol{v} \doteq \partial_{\boldsymbol{p}}\overline{H}$

PRINCETON PLASMA PHYSICS LABORATORY

To approximate an operator, approximate its Weyl symbol.

- Any operator $\widehat{\mathsf{A}}\psi(\mathsf{x}) = \int \mathrm{d}\mathsf{x}'\, \mathcal{A}(\mathsf{x},\mathsf{x}')\, \psi(\mathsf{x}')$ on "any" space x can be expressed through its *symbol* using $\widehat{\mathsf{x}} = \mathsf{x}$ and $\widehat{\mathsf{k}} = -i\partial_{\mathsf{x}}$:

$$\mathsf{A}(\mathsf{x},\mathsf{k}) = \int \mathrm{d}\mathsf{s}\, \mathcal{A}(\mathsf{x}+\mathsf{s}/2, \mathsf{x}-\mathsf{s}/2)\, \mathrm{e}^{-i\mathsf{k}\cdot\mathsf{s}}$$

$$\widehat{\mathsf{A}} = \frac{1}{(2\pi)^{2\mathsf{n}}} \int \mathrm{d}\mathsf{x}'\, \mathrm{d}\mathsf{k}'\, \mathrm{d}\mathsf{x}''\, \mathrm{d}\mathsf{k}''\, \mathsf{A}(\mathsf{x}',\mathsf{k}')\, \mathrm{e}^{i\mathsf{k}''\cdot(\mathsf{x}'-\widehat{\mathsf{x}})-i\mathsf{x}''\cdot(\mathsf{k}'-\widehat{\mathsf{k}})}$$

$$\widehat{1} \Leftrightarrow 1$$
$$\widehat{\mathsf{x}} \Leftrightarrow \mathsf{x}$$
$$\widehat{\mathsf{k}} \Leftrightarrow \mathsf{k}$$
$$\widehat{\mathsf{A}}^\dagger \Leftrightarrow \mathsf{A}^\dagger$$
$$\widehat{\mathsf{A}}\widehat{\mathsf{B}} \Leftrightarrow \mathsf{A} \star \mathsf{B}$$

- $\widehat{\boldsymbol{D}}$ acts on functions of 7D $\boldsymbol{X} \doteq (t, \boldsymbol{x}, \boldsymbol{p})$, so our $\widehat{\mathsf{x}}$ operator is $\widehat{\boldsymbol{X}}$. Correspondingly, our $\widehat{\mathsf{k}}$ operator is $\widehat{\boldsymbol{K}} \doteq (i\partial_t, -i\partial_{\boldsymbol{x}}, -i\partial_{\boldsymbol{p}})$, it induces a 7D space $\boldsymbol{K} \doteq (-\omega, \boldsymbol{k}, \boldsymbol{r})$.

$$\boldsymbol{D}(\boldsymbol{X},\boldsymbol{K}) = \int \mathrm{d}\boldsymbol{K}'\, \overline{\boldsymbol{W}}_{\boldsymbol{u}}(\boldsymbol{X},\boldsymbol{K}')\, G(\boldsymbol{X},\boldsymbol{K}-\boldsymbol{K}'), \qquad \boldsymbol{W}_{\boldsymbol{u}} \sim \text{symb } |\boldsymbol{u}\rangle\langle\boldsymbol{u}|$$

- The following approximation will be enough (tough to prove!):

$$\boldsymbol{D}(\boldsymbol{X},\boldsymbol{K}) \approx \underbrace{\boldsymbol{D}(\boldsymbol{X},\boldsymbol{0})}_{\text{usual QLT}} + \underbrace{(\boldsymbol{K}\cdot\partial_{\boldsymbol{K}})\boldsymbol{D}(\boldsymbol{X},\boldsymbol{0})}_{\text{non-negligible correction}}$$

$$G(\boldsymbol{X},\boldsymbol{K}) \approx \pi\,\delta(\Omega) + i\, \lim_{\nu\to 0} \frac{\Omega}{\Omega^2+\nu^2}, \qquad \Omega \approx \omega - \boldsymbol{k}\cdot\boldsymbol{v}$$

Inspiration for doing the Weyl calculus on phase space: Michael's talk at Vienna-2019

$\boldsymbol{v} \doteq \partial_{\boldsymbol{p}}\overline{H}$

Collisionless equation for the OC distribution, $\varepsilon^2 \ll \lambda/L \ll \varepsilon \ll 1$

- Use* $|\boldsymbol{u}\rangle = i\boldsymbol{J}\widehat{\boldsymbol{q}}\,|\widetilde{H}\rangle$ to express $\boldsymbol{W}_{\boldsymbol{u}} \sim$ symb $|\boldsymbol{u}\rangle\langle\boldsymbol{u}|$ through $W_{\widetilde{H}} \sim$ symb $|\widetilde{H}\rangle\langle\widetilde{H}|$:

$$W_{\widetilde{H}} = (2\pi)^{-(n+1)} \int \mathrm{d}\tau\,\mathrm{d}\boldsymbol{s}\,\mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}\,\overline{\widetilde{H}(t+\tau/2, \boldsymbol{x}+\boldsymbol{s}/2, \boldsymbol{p})\,\widetilde{H}(t-\tau/2, \boldsymbol{x}-\boldsymbol{s}/2, \boldsymbol{p})}$$

- For the OC distribution $F \doteq \overline{f} + \partial_{\boldsymbol{p}} \cdot (\boldsymbol{\Theta}\,\partial_{\boldsymbol{p}}\overline{f})$, we get an equation that is applicable **both to on-shell and off-shell waves** and captures **ponderomotive effects**:

$$\frac{\partial F}{\partial t} = \{\mathcal{H}, F\} + \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\mathsf{D}\,\frac{\partial F}{\partial \boldsymbol{p}}\right)$$

$$\mathcal{H} = \overline{H} + \frac{\partial}{\partial \boldsymbol{p}} \cdot \int \mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}\overline{W}_{\widetilde{H}}}{2(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}$$

$$\boldsymbol{\Theta} = \frac{\partial}{\partial \vartheta} \int \mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}\,\frac{\boldsymbol{k}\boldsymbol{k}^{\dagger}\overline{W}_{\widetilde{H}}}{2(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \vartheta)}\bigg|_{\vartheta=0}$$

$$\mathsf{D} = \pi \int \mathrm{d}\boldsymbol{k}\;\boldsymbol{k}\boldsymbol{k}^{\dagger}\overline{W}_{\widetilde{H}}(t, \boldsymbol{x}, \boldsymbol{k}\cdot\boldsymbol{v}, \boldsymbol{k}; \boldsymbol{p})$$

- No coordinate transformations → **no singularities**. Also, when averaging over phase-space volume $\Delta x\,\Delta k \gtrsim 1$, D is *proven* positive-semidefinite → ***H*-theorem**.

*In the $\boldsymbol{z}$-representation, $\widehat{\boldsymbol{q}} = -i\partial_{\boldsymbol{z}}$.

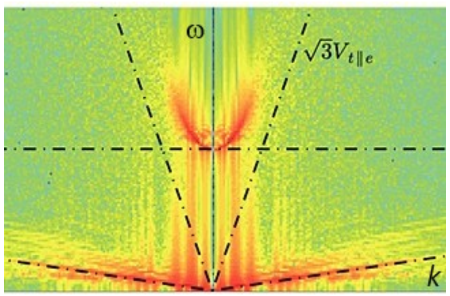
Image: adapted from xkcd.com

Let's make the fields self-consistent (but not necessarily on-shell).

- Use a generic linear-wave action for vacuum and a generic particle Hamiltonian. Then $\widetilde{\boldsymbol{\Psi}}$ satisfies a linear equation with initial conditions g_s as sources.

$$S_0 = \frac{1}{2}\int \widetilde{\boldsymbol{\Psi}}^\dagger \widehat{\boldsymbol{\Xi}}_0 \widetilde{\boldsymbol{\Psi}}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s \approx H_{0s} + \widehat{\boldsymbol{\alpha}}_s^\dagger \widetilde{\boldsymbol{\Psi}} + \frac{1}{2}(\widehat{\boldsymbol{L}}_s\widetilde{\boldsymbol{\Psi}})^\dagger(\widehat{\boldsymbol{R}}_s\widetilde{\boldsymbol{\Psi}}) \quad \rightarrow \quad \widehat{\boldsymbol{\Xi}}\widetilde{\boldsymbol{\Psi}} = \sum_s \int \mathrm{d}\boldsymbol{p}\,\widehat{\boldsymbol{\alpha}}_s g_s$$

$$\boldsymbol{\Xi}(\omega,\boldsymbol{k}) \approx \boldsymbol{\Xi}_0(\omega,\boldsymbol{k}) - \sum_s \int \mathrm{d}\boldsymbol{p}\,(\boldsymbol{L}_s^\dagger \boldsymbol{R}_s)_{\mathsf{H}}(\omega,\boldsymbol{k};\boldsymbol{p})F_s(\boldsymbol{p}) + \sum_s \fint \mathrm{d}\boldsymbol{p}\,\frac{\boldsymbol{\alpha}_s(\omega,\boldsymbol{k};\boldsymbol{p})\boldsymbol{\alpha}_s^\dagger(\omega,\boldsymbol{k};\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s + i0}\,\boldsymbol{k}\cdot\frac{\partial F_s(\boldsymbol{p})}{\partial \boldsymbol{p}}$$



- The general solution is $\widetilde{\boldsymbol{\Psi}} = \widetilde{\boldsymbol{\Psi}}^{(\text{macro})} + \widetilde{\boldsymbol{\Psi}}^{(\text{micro})}$. The correspondingng Wigner tensors are U and $\mathsf{S}/(2\pi)^{n+1}$.

$$\widehat{\boldsymbol{\Xi}}\widetilde{\boldsymbol{\Psi}}^{(\text{macro})} = 0, \qquad \widetilde{\boldsymbol{\Psi}}^{(\text{micro})} = \sum_s \int \mathrm{d}\boldsymbol{p}\,\widehat{\boldsymbol{\Xi}}^{-1}\widehat{\boldsymbol{\alpha}}_s g_s$$

$$\mathsf{U}(\omega,\boldsymbol{k}) = \int \frac{\mathrm{d}\tau}{2\pi}\frac{\mathrm{d}\boldsymbol{s}}{(2\pi)^n}\,\langle \widetilde{\boldsymbol{\Psi}}^{(\text{macro})}(t+\tau/2,\boldsymbol{x}+\boldsymbol{s}/2)\,\widetilde{\boldsymbol{\Psi}}^{(\text{macro})\dagger}(t-\tau/2,\boldsymbol{x}-\boldsymbol{s}/2)\rangle\,\mathrm{e}^{i\omega\tau - i\boldsymbol{k}\cdot\boldsymbol{s}}$$

$$\mathsf{S}(\omega,\boldsymbol{k}) = 2\pi\sum_{s'}\int \mathrm{d}\boldsymbol{p}'\,\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'})F_{s'}(\boldsymbol{p}')\boldsymbol{\Xi}^{-1}(\omega,\boldsymbol{k})(\boldsymbol{\alpha}_{s'}\boldsymbol{\alpha}_{s'}^\dagger)(\omega,\boldsymbol{k};\boldsymbol{p}')\boldsymbol{\Xi}^{-\dagger}(\omega,\boldsymbol{k})$$

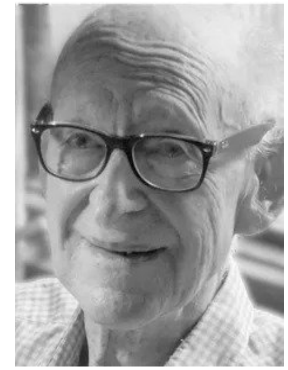
Fluctuation-dissipation theorem: $\mathsf{S}_{\text{eq}} = -{}^{2T}\!/\!_{\omega}\,(\boldsymbol{\Xi}^{-1})_{\mathsf{A}}$

Image: adapted from Kakad *et al.* (2014)

Equation for the OC distribution with self-consistent fields

- In a self-consistent field, a collision operator emerges:

$$\frac{\partial F_s}{\partial t} = \{\mathcal{H}_s, F_s\} + \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\mathsf{D}_s \frac{\partial F_s}{\partial \boldsymbol{p}} \right) + \mathcal{C}_s$$



- $\mathcal{C}_s$ has a Balescu–Lenard form, satisfies ***H*-theorem**, conserves particles and energy–momentum.

$$\mathcal{C}_s = \frac{\partial}{\partial \boldsymbol{p}} \cdot \sum_{s'} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^n} \mathrm{d}\boldsymbol{p}' \, \pi \, \delta(\boldsymbol{k}\cdot\boldsymbol{v}_s - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'}) \, |\boldsymbol{\alpha}_s^\dagger(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k}; \boldsymbol{p}) \boldsymbol{\Xi}^{-1}(\omega, \boldsymbol{k}) \boldsymbol{\alpha}_{s'}(\boldsymbol{k}\cdot\boldsymbol{v}'_{s'}, \boldsymbol{k}; \boldsymbol{p}')|^2 \times \boldsymbol{k}\boldsymbol{k} \cdot \left(\frac{\partial F_s(\boldsymbol{p})}{\partial \boldsymbol{p}} F_{s'}(\boldsymbol{p}') - F_s(\boldsymbol{p}) \frac{\partial F_{s'}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} \right)$$

- ***K*–χ theorem**[*] for the ponderomotive energy $\Delta_s \doteq \mathcal{H}_s - H_{0s}$:

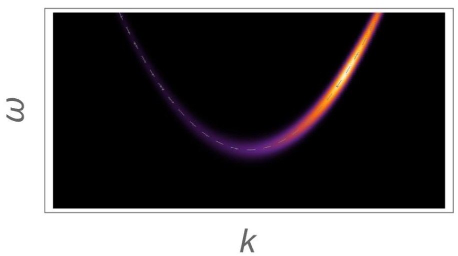
$$\Delta_s = -\frac{1}{2} \frac{\delta}{\delta F_s} \int \boldsymbol{\Xi}_{\mathsf{H}} : \mathsf{U} \, \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k}$$

$$\Delta_s = \frac{1}{2} \frac{\partial}{\partial \boldsymbol{p}} \cdot \int \boldsymbol{k} \frac{(\boldsymbol{\alpha}_s^\dagger \mathsf{U} \boldsymbol{\alpha}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s} \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k} + \frac{1}{2} \int \mathsf{U} : (\boldsymbol{L}_s^\dagger \boldsymbol{R}_s)_{\mathsf{H}} \, \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k}$$

[*] e.g. in Kaufman (1987)

Photos: Radu Balescu and Andrew Lenard. Image: xkcd.com

Special case: on-shell waves and the associated conservation laws



- Wave kinetic equation (WKE) for a complexified wave field:*

$$\widehat{\boldsymbol{\Xi}}\,|\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}\rangle = |\mathbf{0}\rangle \quad\rightarrow\quad \widehat{\boldsymbol{\Xi}}\,|\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}\rangle\langle\widetilde{\boldsymbol{\Psi}}_{\mathsf{c}}| = \widehat{\mathbf{0}} \quad\rightarrow\quad \mathbf{tr}\int \mathrm{d}\boldsymbol{\omega}\,\boldsymbol{\Xi}\star\mathbf{U}_{\mathsf{c}} = 0$$

- On-shell waves have $\mathbf{U}_{\mathsf{c}} \approx \delta(\Lambda)\,J(t,\boldsymbol{x},\boldsymbol{k})\,\boldsymbol{\eta}\boldsymbol{\eta}^{\dagger}$. In terms of the action density J, the WKE is

$$\boxed{\partial_t J + \boldsymbol{v}_{\mathrm{g}}\cdot\partial_{\boldsymbol{x}}J - \partial_{\boldsymbol{x}}\omega\cdot\partial_{\boldsymbol{k}}J = 2\gamma J}$$

conserves the sign of J

$\int \mathrm{d}\boldsymbol{p}\,\overline{H}_s F_s$	OC energy density
$\int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}F_s$	OC momentum density
$\int \mathrm{d}\boldsymbol{k}\,\omega J$	wave energy density
$\int \mathrm{d}\boldsymbol{k}\,\boldsymbol{k}J$	wave momentum density

- Combined together, the equations for F_s and J conserve the energy–momentum:

$$\frac{\partial}{\partial t}\left(\sum_s\int \mathrm{d}\boldsymbol{p}\,\overline{H}_s F_s + \int \mathrm{d}\boldsymbol{k}\,\omega J\right) + \frac{\partial}{\partial x^i}\left(\sum_s\int \mathrm{d}\boldsymbol{p}\,(\overline{H}_s + \Delta_s)v_s^i F_s + \int \mathrm{d}\boldsymbol{k}\,\omega v_{\mathrm{g}}^i J\right) = +\sum_s\int \mathrm{d}\boldsymbol{p}\,\frac{\partial \overline{H}_s}{\partial t}\,F_s$$

$$\frac{\partial}{\partial t}\left(\sum_s\int \mathrm{d}\boldsymbol{p}\,p_l F_s + \int \mathrm{d}\boldsymbol{k}\,k_l J\right) + \frac{\partial}{\partial x^i}\left(\sum_s\int \mathrm{d}\boldsymbol{p}\,(p_l v_s^i + \Delta_s\delta_l^i)F_s + \int \mathrm{d}\boldsymbol{k}\,k_l v_{\mathrm{g}}^i J\right) = -\sum_s\int \mathrm{d}\boldsymbol{p}\,\frac{\partial \overline{H}_s}{\partial x^l}\,F_s$$

- F_s and J are fundamental objects, the oscillating fields *per se* are not needed.

*cf. McDonald and Kaufman (1985)

So how does one apply all this stuff?

- Need to represent the vacuum-field action and the particle Hamiltonians in the form

$$S_0 = \frac{1}{2}\int \widetilde{\boldsymbol{\Psi}}^\dagger \widehat{\boldsymbol{\Xi}}_0 \widetilde{\boldsymbol{\Psi}}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s \approx H_{0s} + \widehat{\boldsymbol{\alpha}}_s^\dagger \widetilde{\boldsymbol{\Psi}} + \frac{1}{2}(\widehat{\boldsymbol{L}}_s \widetilde{\boldsymbol{\Psi}})^\dagger (\widehat{\boldsymbol{R}}_s \widetilde{\boldsymbol{\Psi}})$$

- **Non-relativistic electrostatic interactions:** Dewar's theory, Balescu–Lenard theory, and the formulas for electrostatic fluctuations are subsumed *(see paper)*.

$$S_0 = \int \frac{(\nabla\widetilde{\varphi})^2}{8\pi}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int \widetilde{\varphi}\,\frac{(-\nabla^2)}{4\pi}\,\widetilde{\varphi}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x}, \qquad H_s = \frac{p^2}{2m_s} + e_s\overline{\varphi} + e_s\widetilde{\varphi}$$

$$\Xi_0(\omega,\boldsymbol{k}) = k^2/4\pi, \qquad \alpha_s(\omega,\boldsymbol{k}) = e_s, \qquad L_s(\omega,\boldsymbol{k}) = R_s(\omega,\boldsymbol{k}) = 0$$

  - The 'dressing' $F-\overline{f} = \partial_{\boldsymbol{p}}\cdot(\boldsymbol{\Theta}\,\partial_{\boldsymbol{p}}\overline{f})$ carries energy–momentum:

$$\sum_s \int \mathrm{d}\boldsymbol{p}\, H_{0s}F_s + \int \mathrm{d}\boldsymbol{k}\, wJ = \sum_s \int \mathrm{d}\boldsymbol{p}\, H_{0s}\overline{f}_s + \frac{1}{8\pi}\overline{\widetilde{E}^2}$$

$$\sum_s \int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p}F_s + \underbrace{\int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}J}_{\text{non-negligible}} = \sum_s \int \mathrm{d}\boldsymbol{p}\, \boldsymbol{p}\overline{f}_s$$

Image: adapted from xkcd.com

Relativistic electromagnetic interactions

- Let's adopt $\widetilde{\boldsymbol{E}} = i\widehat{\omega}\widetilde{\boldsymbol{A}}/c$ as the interaction field (Weyl gauge) and $\widetilde{\boldsymbol{B}} = (c\widehat{\boldsymbol{k}}/\widehat{\omega}) \times \widetilde{\boldsymbol{E}}$:

$$S_0 = \int \frac{\widetilde{E}^2 - \widetilde{B}^2}{8\pi} \, \mathrm{d}t \, \mathrm{d}\boldsymbol{x} = \frac{1}{2} \int \widetilde{\boldsymbol{E}}^\dagger \underbrace{\frac{1}{4\pi} \left[\mathbf{1} + \frac{c^2}{\widehat{\omega}^2} \left(\widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}}^\dagger - \mathbf{1}\widehat{k}^2 \right) \right]}_{\widehat{\boldsymbol{\Xi}}_0} \widetilde{\boldsymbol{E}} \, \mathrm{d}t \, \mathrm{d}\boldsymbol{x},$$

- Relativistic-particle Hamiltonian can be Taylor-expanded and expressed through $\widetilde{\boldsymbol{E}}$:

$$H_s = \sqrt{m_s^2 c^4 + (\boldsymbol{p}c - e_s\overline{\boldsymbol{A}} - e_s\widetilde{\boldsymbol{A}})^2} + e_s\overline{\varphi} + e_s\overline{\varphi}$$

$$= H_{0s} + \underbrace{\frac{ie_s}{\widehat{\omega}} \overline{\boldsymbol{v}}_s^\dagger}_{\widehat{\boldsymbol{\alpha}}_s} \widetilde{\boldsymbol{E}} + \frac{1}{2} \Bigg(\underbrace{\frac{e_s^2}{\widehat{\omega}} \widetilde{\boldsymbol{E}}}_{\widehat{L}_s \widetilde{\boldsymbol{E}}} \Bigg)^\dagger \Bigg(\underbrace{\frac{\mathbf{1} - \boldsymbol{v}_s\boldsymbol{v}_s^\dagger/c^2}{m_s\gamma_s} \frac{1}{\widehat{\omega}} \widetilde{\boldsymbol{E}}}_{\widehat{R}_s \widetilde{\boldsymbol{E}}} \Bigg)$$

- Energy–momentum conservation, with $f_s^{(\mathrm{kin})}(\boldsymbol{p}) \doteq f_s(\boldsymbol{p} + e_s\widetilde{\boldsymbol{A}}/c)$:

$$\sum_s \int \mathrm{d}\boldsymbol{p} \, H_{0s} F_s + \int \mathrm{d}\boldsymbol{k} \, wJ = \sum_s \int \mathrm{d}\boldsymbol{p} \, H_{0s} \overline{f_s^{(\mathrm{kin})}} + \frac{1}{8\pi} \overline{(\widetilde{\boldsymbol{E}}^\dagger\widetilde{\boldsymbol{E}} + \widetilde{\boldsymbol{B}}^\dagger\widetilde{\boldsymbol{B}})}$$

$$\sum_s \int \mathrm{d}\boldsymbol{p} \, \boldsymbol{p} F_s + \int \mathrm{d}\boldsymbol{k} \, \boldsymbol{k} J = \sum_s \int \mathrm{d}\boldsymbol{p} \, \boldsymbol{p} \, \overline{f_s^{(\mathrm{kin})}} + \frac{\overline{\widetilde{\boldsymbol{E}} \times \widetilde{\boldsymbol{B}}}}{4\pi c}$$

Relativistic electromagnetic interactions: the formulas you were dying to see

- Relativistic nonlinear potentials (U is the average Wigner tensor of $\widetilde{\boldsymbol{E}}$:)

$$\mathsf{D}_s = \pi e_s^2 \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}\boldsymbol{k}\, \frac{\boldsymbol{v}_s^\dagger \mathsf{U}(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k})\boldsymbol{v}_s}{(\boldsymbol{k}\cdot\boldsymbol{v}_s)^2}$$

$$\boldsymbol{\Theta}_s = e_s^2 \frac{\partial}{\partial\vartheta} \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}\boldsymbol{k}}{\omega^2} \frac{(\boldsymbol{v}_s^\dagger \mathsf{U} \boldsymbol{v}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s + \vartheta}\bigg|_{\vartheta=0}$$

$$\Delta_s = \frac{e_s^2}{2}\frac{\partial}{\partial\boldsymbol{p}} \cdot \fint \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\boldsymbol{k}}{\omega^2}\frac{(\boldsymbol{v}_s^\dagger \mathsf{U}\boldsymbol{v}_s)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s} + \frac{e_s^2}{2}\int \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}\, \frac{\mathrm{tr}(\mathsf{U}\boldsymbol{\mu}_s^{-1})}{\omega^2}$$

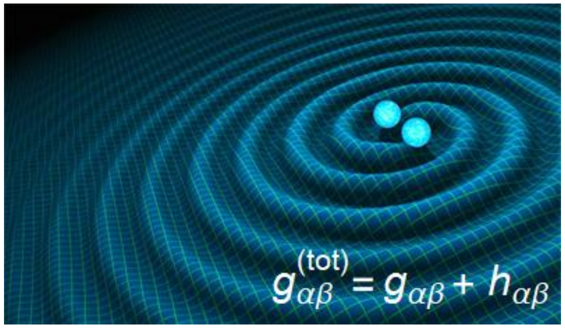
- Fluctuation spectrum and collision operator* ($\boldsymbol{\epsilon}$ is the dielectric tensor):

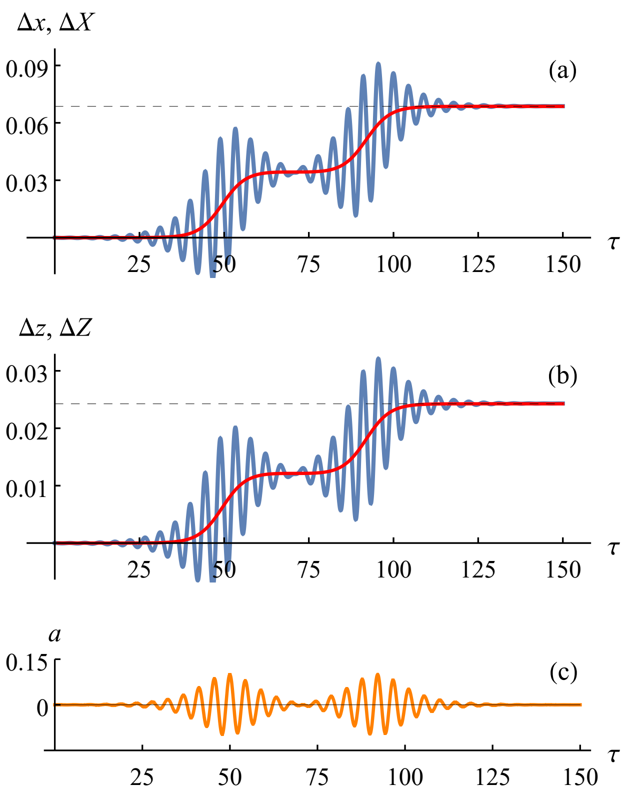
$$\mathsf{S}(\omega, \boldsymbol{k}) = 2\pi \sum_s \left(\frac{4\pi e_s}{\omega}\right)^2 \int \mathrm{d}\boldsymbol{p}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_s) F_s(\boldsymbol{p}) \boldsymbol{\epsilon}^{-1}(\omega, \boldsymbol{k})\, \boldsymbol{v}_s \boldsymbol{v}_s^\dagger\, \boldsymbol{\epsilon}^{-\dagger}(\omega, \boldsymbol{k})$$

$$\mathcal{C}_s = \frac{\partial}{\partial\boldsymbol{p}} \cdot \sum_{s'} 2e_s^2 e_{s'}^2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \mathrm{d}\boldsymbol{p}'\, \frac{|\boldsymbol{v}_s^\dagger \boldsymbol{\epsilon}^{-1}(\boldsymbol{k}\cdot\boldsymbol{v}_s, \boldsymbol{k})\boldsymbol{v}'_{s'}|^2}{(\boldsymbol{k}\cdot\boldsymbol{v}_s)^4}\, \delta(\boldsymbol{k}\cdot\boldsymbol{v}_s - \boldsymbol{k}\cdot\boldsymbol{v}'_{s'})$$
$$\times\, \boldsymbol{k}\boldsymbol{k} \cdot \left(\frac{\partial F_s(\boldsymbol{p})}{\partial\boldsymbol{p}} F_{s'}(\boldsymbol{p}') - F_s(\boldsymbol{p}) \frac{\partial F_{s'}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}\right)$$

*cf. Hizanidis *et al.* (1983); Silin (1961)

Quasilinear theory for gravitational waves





- The same formalism can be readily applied to particle interactions with gravitational waves.

$$\mathsf{H}(\mathsf{x}, \mathsf{p}) = m^2 + g^{\alpha\beta}(\mathsf{x}) p_\alpha p_\beta, \quad g_{\alpha\beta} = \overline{g}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}$$

- The QL coefficients are explicitly found in terms of $\mathfrak{E} \doteq p_\alpha p_\beta p_\gamma p_\delta \mathsf{U}^{\alpha\beta\gamma\delta}$. QL diffusion is gauge-invariant.

$$\mathsf{D} = \frac{\pi}{4P^0} \int \mathrm{dk}\, \boldsymbol{k}\boldsymbol{k} \mathfrak{E}\, \delta(k^\rho p_\rho)$$

$$\boldsymbol{\Theta} = \frac{1}{4P^0} \frac{\partial}{\partial \vartheta} \fint \mathrm{dk}\, \frac{\boldsymbol{k}\boldsymbol{k}\mathfrak{E}}{\vartheta P^0 - k^\rho p_\rho}\bigg|_{\vartheta = 0}$$

$$\Delta = \frac{p_\alpha p_\beta}{2P^0} \int \mathrm{dk}\, \mathsf{U}^{\alpha}{}_{\gamma}{}^{\gamma\beta} - \frac{1}{8P^0} \frac{\partial}{\partial p_\lambda} \fint \mathrm{dk}\, \frac{k_\lambda \mathfrak{E}}{k^\rho p_\rho}$$

- Vacuum GWs:* effective 'ponderomotive' metric

$$\mathcal{H}' = m^2 + g^{\alpha\beta}_{\mathrm{eff}} p_\alpha p_\beta, \qquad g^{\alpha\beta}_{\mathrm{eff}} \doteq \overline{g}^{\alpha\beta} + \int \mathrm{dk}\, \mathsf{U}^{\alpha}{}_{\gamma}{}^{\gamma\beta}.$$

Garg and Dodin (2020, 2021a, 2021b, 2022)

*Lorenz gauge assumed, $\nabla_\alpha \widetilde{g}^{\alpha\beta} = 0$

Local quasilinear theory for magnetized plasma

- The *canonical* $W_{\widetilde{H}}$ cannot be expanded in L^{-1} because $\overline{\boldsymbol{A}}$ (as opposed to $\overline{\boldsymbol{B}}$) depends on $\boldsymbol{x}$ rapidly. In *non-canonical* variables, the derivation is too cumbersome.
- **Fix #1 (boring)**: find *global* angle–action coordinates $(\boldsymbol{\phi}, \mathbf{J})$, Fourier-expand in $\boldsymbol{\phi}$, treat each $\widetilde{f}_n$ as a separate $\widetilde{f}$. Since $\widetilde{f}_n$ are $\boldsymbol{\phi}$-independent, there is no problem left.

$$\widetilde{f} = \sum_{\boldsymbol{n}} \widetilde{f}_n(\mathbf{J}) \mathrm{e}^{i\boldsymbol{n}\cdot\boldsymbol{\phi}}$$

- **Fix #2 (fun)**: find *local* canonical coordinates in which the theory works.
  - Homogeneous field: $(Q_3, P_3) = (\theta, \mu)$, $(Q_2, P_2) = (z, p_z)$, $(Q_1, P_1) = (x, y)$.
  - Inhomogeneous field: similar coordinates with $\dot{P}_1 \ll \dot{Q}_1 = v_{\text{drift}}$ and $\dot{P}_2 \ll \dot{Q}_2$.*
  - Fourier-expand in θ but retain weak dependence on the local Q_1 and Q_2:

$$\widetilde{f} = \sum_{n} \widetilde{f}_n(Q_1, Q_2, P_1, P_2; \mu) \mathrm{e}^{in\theta}$$

  - $\mathbf{D}$, $\boldsymbol{\Theta}$, and Δ are needed only to the zeroth order. Thus, they are the same as in a homogeneous field except $\boldsymbol{v}$ includes $\boldsymbol{v}_{\text{drift}}$.
  - This agrees with and generalizes the findings of Catto *et al.* (2017).

* Wong (2000); cf. Kennel and Engelmann (1966)

Summary

- **Result:** QL theory is corrected and derived from first principles as a *local theory*.
  Wigner tensors vs. global-mode decomposition
  - general Hamiltonian, any interaction field;
  - inhomogeneity, collisions and off-shell waves;
  - H-theorem for inhomogeneous plasma;
  - generalized conservative Balescu–Lenard collision operator;
  - conservation of the action, energy, and momentum for on-shell fields;
  - many known results are subsumed as special cases.
- **Take-home message:** $\mathcal{O}(\bar{\partial}_t, \bar{\partial}_x)$ is non-negligible on $t \gg \omega^{-1}$ and $\ell \gg k^{-1}$. Calculations ignoring this are unreliable. Weyl calculus is *the* way to get things right.

$$\widetilde{f}_k = -\frac{i(e/m)\widetilde{E}_k}{\omega_k - kv}\frac{\partial \overline{f}}{\partial v} + \boldsymbol{\mathcal{O}}(\partial_t \overline{\boldsymbol{f}}, \partial_x \overline{\boldsymbol{f}}), \qquad F - \overline{f} = \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\boldsymbol{\Theta}\,\frac{\partial \overline{f}}{\partial \boldsymbol{p}}\right) = \boldsymbol{\mathcal{O}}(\widetilde{\boldsymbol{E}}^{\mathbf{2}})$$

- **Potential applications:** RF current drive, models of turbulence saturation

References

[Bretherton and Garrett (1968)]
F. P. Bretherton and C. J. R. Garrett, *Wavetrains in inhomogeneous moving media*, Proc. R. Soc. A **302**, 529 (1968).

[Catto *et al.* (2017)]
P. J. Catto, J. Lee, and A. K. Ram, *A quasilinear operator retaining magnetic drift effects in tokamak geometry*, J. Plasma Phys. **83**, 905830611 (2017).

[Dewar (1972)]
R. L. Dewar, *A Lagrangian derivation of the action-conservation theorem for density waves*, Astrophys. J. **174**, 301 (1972).

[Dewar (1973)]
R. L. Dewar, *Oscillation center quasilinear theory*, Phys. Fluids **16**, 1102 (1973).

[Dodin (2022)]
I. Y. Dodin, *Quasilinear theory for inhomogeneous plasma*, 2201.08562 (2022), to appear in J. Plasma Phys.

[Dodin *et al.* (2019)]
I. Y. Dodin, D. E. Ruiz, K. Yanagihara, Y. Zhou, and S. Kubo, *Quasioptical modeling of wave beams with and without mode conversion. I. Basic theory*, Phys. Plasmas **26**, 072110 (2019).

[Drummond and Pines (1962)]
W. E. Drummond and D. Pines, *Non-linear stability of plasma oscillations*, Nucl. Fusion Suppl. **3**, 1049 (1962).

[Garg and Dodin (2020)]
D. Garg and I. Y. Dodin, *Average nonlinear dynamics of particles in gravitational pulses: effective Hamiltonian, secular acceleration, and gravitational susceptibility*, Phys. Rev. D **102**, 064012 (2020).

[Garg and Dodin (2021)a]
G. Garg and I. Y. Dodin, *Gauge-invariant gravitational waves in matter beyond linearized gravity*, 2106.05062 (2021).

[Garg and Dodin (2021)b]
G. Garg and I. Y. Dodin, *Gauge invariants of linearized gravity with a general background metric*, 2105.04680 (2021).

[Garg and Dodin (2022)]
G. Garg and I. Y. Dodin, *Gravitational wave modes in matter*, 2204.09095 (2022), to appear in J. Cosmol. Astropart. Phys.

[Hizanidis *et al.* (1983)]
K. Hizanidis, K. Molvig, and K. Swartz, *A retarded time superposition principle and the relativistic collision operator*, J. Plasma Phys. **30**, 223 (1983).

[Kakad *et al.* (2014)]
B. Kakad, A. Kakad, and Y. Omura, *Nonlinear evolution of ion acoustic solitary waves in space plasmas: Fluid and particle-in-cell simulations*, J. Geophys. Res. Space Phys. **119**, 5589 (2014).

[Kaufman (1987)]
A. N. Kaufman, *Phase-space-Lagrangian action principle and the generalized* K-χ *theorem*, Phys. Rev. A **36**, 982 (1987).

[Kennel and Engelmann (1966)]
C. F. Kennel and F. Engelmann, *Velocity space diffusion from weak plasma turbulence in a magnetic field*, Phys. Fluids **9**, 2377 (1966).

[McDonald *et al.* (1985)]
S. W. McDonald, C. Grebogi, and A. N. Kaufman, *Locally coupled evolution of wave and particle distribution in general magnetoplasma geometry*, Phys. Lett. A **111**, 19 (1985).

[McDonald and Kaufman (1985)]
S. W. McDonald and A. N. Kaufman, *Weyl representation for*

Back

References

electromagnetic waves: The wave kinetic equation, Phys. Rev. A **32**, 1708 (1985).

[Silin (1961)]
V. P. Silin, *Collision integral for charged particles*, Zh. Eksp. Teor. Fiz. **40**, 1768 (1961).

[Vedenov *et al.* (1961)]
A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, *Nonlinear oscillations of rarified plasma*, Nucl. Fusion **1**, 82 (1961).

[Whitham (1965)]
G. B. Whitham, *A general approach to linear and non-linear dispersive waves using a Lagrangian*, J. Fluid Mech. **22**, 273 (1965).

[Wong (2000)]
H. V. Wong, *Particle canonical variables and guiding center Hamiltonian up to second order in the Larmor radius*, Phys. Plasmas **7**, 73 (2000).