A collisional interpretation of the quasilinear delta function & other observations

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Peter Catto

Plasma Science and Fusion Center, MIT

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Libby Tolman

Institute for Advanced Study, Princeton

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Quasilinear (QL) used for rf heating & current drive

*Applied rf acting on resonant particles allows a QL theory

*QL operator has negative definite entropy production

*QL theory requires a velocity dependent resonance

*Resonant particles lead to a delta function in QL operator

*Need physics to estimate the size of a delta function

*Collisions simplest and relevant for magnetic fusion

*Will other interpretations have similar properties?

Preliminaries

*Transport coefficients & standard quasilinear operators often appear to be independent of collisions

*Can be misleading as these results may be due to collisional (ν) resonant plateau (RP) behavior

*RP behavior is due to narrow collisional layers about wave-particle resonances in velocity space: $\omega = k_{\|} v_{\|}$

*RP behavior occurs when ν cancels out: "plateau" regime

*Are there other physical interpretations?

Quick & dirty rf QL operator

*Vlasov eq. with $\vec{a}$ = acceleration due to rf

$$-i(\omega - k_{\parallel} v_{\parallel}) f_1 = -\vec{a} \cdot \nabla_v (f_0 + f_1)$$

*Neglecting f_1 on right yields QL operator

$$Q\{f_0\} = -\langle \nabla_v \cdot (\vec{a} f_1) \rangle_{\substack{\text{average} \\ \text{over} \\ \text{everything}}} = -\langle \nabla_v \cdot [\frac{\vec{a}\vec{a}}{i(\omega - k_{\parallel} v_{\parallel})} \cdot \nabla_v f_0] \rangle_{\text{ave}}$$

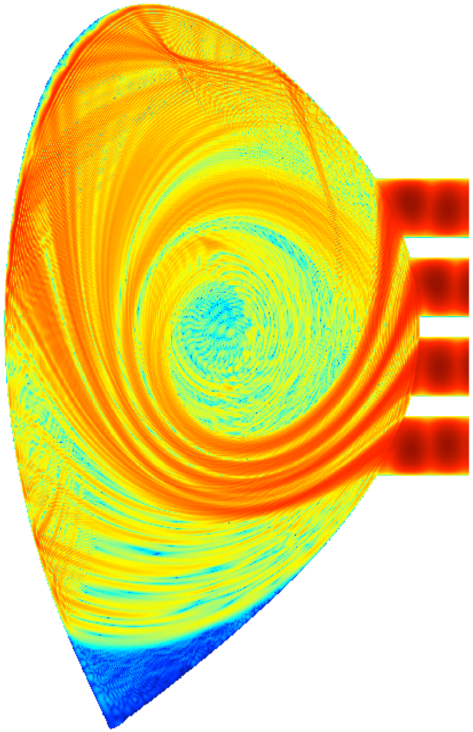
*Use $(\omega - k_{\parallel} v_{\parallel})^{-1} \rightarrow i\pi\delta(\omega - k_{\parallel} v_{\parallel})$ and do averages, to find

$$Q\{f_0\} = \nabla_v \cdot (\overleftrightarrow{D} \cdot \nabla_v f_0)$$

*Interaction time in $\overleftrightarrow{D} \propto |\vec{a}|^2 \delta(\omega - k_{\parallel} v_{\parallel}) \sim |\vec{a}|^2 \tau_{\text{int}}$???

RF heating works!

RF & CD simulations

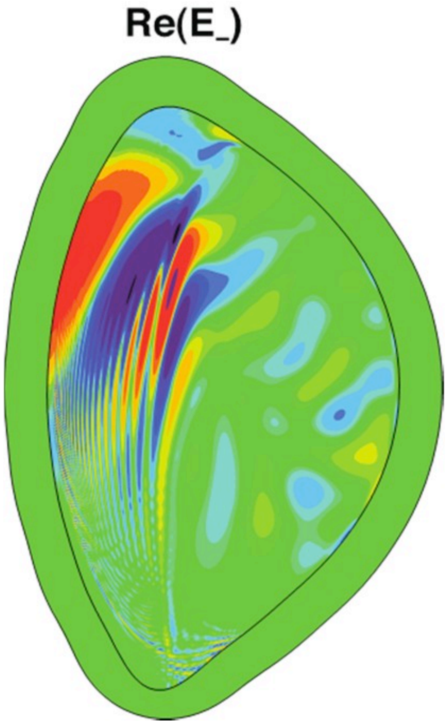


Outside launch

LHCD: $\omega > k_{||}v_e$

Simulations iterate between a linear full wave solver for f_1 and EM fields and a combined quasilinear + Fokker-Planck solver evolving f_0

Shiraiwa   Wright



Inside launch

LH: FW$\Rightarrow$IBW

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Enhanced collision frequency

*To find wave-resonant particle interaction time use

$$i(\omega - k_{\|}v_{\|})f_1 \rightarrow i(\omega - k_{\|}v_{\|})f_1 + \nu v_{\perp}^2 \partial^2 f_1/\partial v_{\|}^2$$

*Collisions matter in a narrow boundary layer of width w

$$k_{\|}v_{\|}wf_1 \sim (\omega - k_{\|}v_{\|})f_1 \sim \nu v_{\perp}^2 \partial^2 f_1/\partial v_{\|}^2 \sim \nu f_1/w^2$$

*Find

$$w \sim (\nu/k_{\|}v_{\|})^{1/3} < (\nu qR/v)^{1/3} << 1$$

and

$$\nu_{\text{eff}} \sim \nu / w^2 \sim \nu(k_{\|}v_{\|}/\nu)^{2/3} = 1/\tau_{\text{int}}$$

*Interaction time estimate is τ_{int}

$$\delta(\omega - k_{\|}v_{\|}) \sim \tau_{\text{int}} << \nu^{-1}$$

QL diffusivity

*QL diffusivity for $\vec{a} \sim Ze\vec{e}/M$ with $\vec{e}$ applied rf field is

$$D \sim (\text{accel.})^2(\text{int. time}) = |\vec{a}|^2 \frac{w^2}{\nu} = \frac{|\vec{a}|^2}{wk_\parallel v_\parallel} = \frac{|\vec{a}|^2}{\omega - k_\parallel v_\parallel}$$

*And integral over $v_\parallel$ boundary layer of width w gives

"effective" diffusivity independent of ν

$$D_w \sim wD = \frac{|\vec{a}|^2}{k_\parallel v_\parallel},$$

even though collisions are "essential"

Crude nonlinear estimates

*Expect QL to begin to fail when $\nabla_v f_1/\nabla_v f_0 \sim f_1/f_0 w \sim 1$

Using $f_1 \sim w f_0$ in $k_{\|} v_{\|} w f_1 \sim \vec{a} \cdot \nabla_v f_0$ with $v_e^2 = 2T_e/m$ gives

$$|\vec{a}| \sim k_{\|} v_e^2 w^2 \quad \& \quad \text{therefore} \quad D \sim |\vec{a}|^2 w^2/\nu \sim \nu v_e^2$$

As $Q\{f_0\} \sim D f_0/v_e^2$ & $C\{f_0\} \sim \nu f_0$, QL theory could fail once

$$Q\{f_0\}/C\{f_0\} \sim D/\nu v_e^2 \sim 1$$

*Suggests should order $Q\{f_0\} << C\{f_0\}$, implying f_0 Maxwellian to lowest order!

Trapping time reminder

*Without collisions let Δ = width of resonant region, then

$$k_{\|}v_{\|}\Delta f_1 \sim \vec{a}\cdot\nabla_v(f_0+f_1) \sim |\vec{a}|(f_0/v_{\|}+f_1/\Delta v_{\|})$$

*Define trapping time τ_{trap} by using nonlinear term

$$f_1/\tau_{trap} = |\vec{a}|\,f_1/\Delta v_{\|}$$

*Balancing resonant and nonlinear terms gives Δ & τ_{trap}

$$\Delta \sim \sqrt{|\vec{a}|/k_{\|}v_{\|}^2}$$

$$\tau_{trap} = \Delta v_{\|}/|\vec{a}| \sim (k_{\|}|\vec{a}|)^{-1/2}$$

*Linearization requires $f_1/f_0 << \Delta \sim \sqrt{|\vec{a}|/k_{\|}v_{\|}^2} << 1$ or long τ_{trap}

Even cruder nonlinear estimate

*Collisional treatment assumes $1 >> w > \Delta$ or $\nu_{eff}\tau_{trap} > 1$:

$$1 >> (\nu / k_{\parallel} v_{\parallel})^{1/3} > (|\vec{a}| / k_{\parallel} v_{\parallel}^2)^{1/2}$$

*Nonlinear term also perturbs trajectories

*Imagine there is diffusion due to nonlinearity

$$D_{\Delta} = (\text{fraction})(\text{accel.})^2(\text{trap.time}) = \Delta |\vec{a}|^2 \tau_{trap} \sim |\vec{a}|^2 / k_{\parallel} v_{\parallel} \sim D_w \sim wD$$

*Can $\vec{a} \cdot \nabla_v f_1$ give "diffusion" in $v_{\parallel}$ or $k_{\parallel}$???

Electron QL operator: $\Omega_i << \omega << \Omega_e$

*For simple wave-electron interaction: lower hybrid current drive or whistler/helicon waves

$$Q\{f_0\} = \frac{v_\parallel}{v}\frac{\partial}{\partial v}\left(D\,\frac{v}{v_\parallel}\frac{\partial f_0}{\partial v}\right),$$

where for a cylinder $f_0 = f_0(r, v, \mu)$, $v = |\vec{v}|$, and

$$D = \frac{\pi e^2}{2m^2 v^2}\sum_k \delta(\omega - k_\parallel v_\parallel)\left|\vec{e}_k \cdot [\vec{z} v_\parallel J_0(\eta) + i\vec{z}\times\vec{k} k_\perp^{-1} v_\perp \partial J_0/\partial\eta]\right|^2,$$

with $\vec{e}_k$ the perturbed electric field and $\eta = k_\perp v_\perp/\Omega$

QL operator with collisions

*Narrow collisional boundary layer use

$$C\{h\} \rightarrow \nu v_\perp^2 \, \partial^2 h / \partial v_\parallel^2$$

*Delta function in D replaced by

$$\pi\delta(\omega - k_\parallel v_\parallel) \rightarrow \mathrm{Re}\int_0^\infty dt\, e^{-i(k_\parallel v_\parallel - \omega)t - \nu k_\parallel^2 v_\perp^2 t^3/3}$$

Weak collisional disruption unless $\nu k_\parallel^2 v_\perp^2 t^3 > 1$, allowing

$$\omega t \sim k_\parallel v_\parallel t > (k_\parallel v / \nu)^{1/3} \sim 1/w >> 1$$

Whistler or helicon wave QL operator

Perp. electric field dominates: $\omega^2 \simeq k_\perp^2 \rho_e^2 (1 + k_\parallel^2 c^2/\omega_{pi}^2)\Omega_i \Omega_e/\beta_e$

$$D_{w/h} \simeq (\pi e^2/2m^2 v^2)\sum_k \delta(\omega - k_\parallel v_\parallel)\left|\vec{e}_k \cdot \vec{z} \times \vec{k} k_\perp^{-1} v_\perp \partial J_0/\partial\eta]\right|^2$$

Lower hybrid wave QL operator

Parallel electric field dominates: $\omega^2 \simeq \dfrac{\Omega_i \Omega_e (1 + M k_\parallel^2/m k_\perp^2)}{(1 + \Omega_e^2/\omega_{pe}^2)}$

$$D_{lh} \simeq (\pi e^2/2m^2 v^2)\sum_k \delta(\omega - k_\parallel v_\parallel)\left|\vec{e}_k \cdot \vec{z} v_\parallel\right|^2$$

Tokamak requires poloidal variation: $v_{\|}(\vartheta)$ & $B(\vartheta)$

Heating & current drive: correlated poloidal transits

$$\delta(\omega - p\Omega - k_{\|}v_{\|}) \rightarrow \delta[\omega - \tau_f^{-1}\oint_f d\tau(p\Omega + k_{\|}v_{\|})]$$

where

$$\tau_f = \oint_f d\tau = \oint_f d\vartheta \,/ v_{\|}\vec{n}\cdot\nabla\vartheta$$

*Resonance transit averaged, it is not at $\omega = p\Omega + k_{\|}v_{\|}$

*Localization due to applied fields from drive term

*Can also keep drifts in resonance

Lower hybrid in tokamak geometry

$$D \simeq (\pi e^2/2m^2v^2)\sum_k \delta(\omega - \tau_f^{-1}\oint_f d\tau k_{\parallel}v_{\parallel})\left|\tau_f^{-1}\oint_f d\tau\, \vec{e}_k\cdot\vec{n}v_{\parallel}\right|^2$$

$$\tau_f = \oint_f d\tau = \oint_f d\vartheta\,/v_{\parallel}\vec{n}\cdot\nabla\vartheta$$

$$\oint_f d\tau k_{\parallel}v_{\parallel} = \begin{cases} 2\pi v_{\parallel}(qn-m)/|v_{\parallel}| & \text{passing} \\ 0 & \text{trapped} \end{cases}$$

Here $v_{\parallel}\vec{n}\cdot\nabla f_0 = C\{f_0\} + Q\{f_0\}$ and $\overline{C\{f_0\}} + \overline{Q\{f_0\}} = 0$ with

$$\tau_f^{-1}\oint_f d\tau\, Q\{f_0\} = \overline{Q\{f_0\}} = \sum_k \frac{1}{\tau_f v}\frac{\partial}{\partial v}(\tau_f v D\frac{\partial f_0}{\partial v})$$

Example: Adjoint method solution for LHCD

*Lowest order Maxwellian $v_{\parallel}\vec{n}\cdot\nabla\overline{f}_0 = C\{\overline{f}_0\} = 0$

*Next order only requires $\overline{Q\{\overline{f}_0\}}$

*Use Cordey eigenfunctions for $\overline{C\{\tilde{f}_0\}}$ where $f_0 = \overline{f}_0 + \tilde{f}_0$

*Normalize LH current driven $J_{\parallel}$ by rf power density P_{cd} to get LHCD efficiency with aspect ratio =1/ε modifications

$$\frac{J_{\parallel}/en_e v_e}{P_{cd}/n_e m_e v_e^2 \nu_{ee}} \equiv \eta = \frac{4(1+0.62\sqrt{\varepsilon})\omega^2}{\sqrt{\pi}[(Z+1)(1+2.06\sqrt{\varepsilon})+4]k_{\parallel}^2 v_e^2}$$

*Fisch: $\varepsilon = 0$, but ε reduces η slightly

Summary for magnetic fusion

*A resonant plateau interpretation of QL theory seems sensible - it collisionally resolves the delta functions

*QL theory is likely beginning to fail once it leads to significant departures from Maxwellian

*Tokamak resonances are transit averaged

*The velocity dependent resonance can be a magnetic drift with a radial gradient drive as for TAE or NTM modes

Replacement check

Of course, for $\xi = v_{\parallel}/v$, in $\int_{-\infty}^{\infty} dv_{\parallel}(\ldots)$ gives

$$\pi\int_{-1}^{1} d\xi\,\delta(\omega - k_{\parallel}v_{\parallel}) = \frac{\pi}{k_{\parallel}v}$$

while for $k_{\parallel}vt > \omega t >> 1$ "phase mixing" leads to

$$\mathrm{Re}\int_{0}^{\infty} dt\, e^{-\nu k_{\parallel}^{2}v_{\perp}^{2}t^{3}/3 + i\omega t}\int_{-1}^{1} d\xi\, e^{-i\xi k_{\parallel}vt}$$

$$= \frac{2}{k_{\parallel}v}\int_{0}^{\infty} dt\,\frac{\sin(k_{\parallel}vt)\cos(\omega t)}{t}\, e^{-\nu k_{\parallel}^{2}v_{\perp}^{2}t^{3}/3} \simeq \frac{\pi}{k_{\parallel}v}$$