Around the quasilinear approximation of the Vlasov equation

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Contents

Vlasov–Poisson equation and weak turbulent scaling

- Vlasov–Poisson:

$$\partial_t f + v\cdot\nabla_x f + \frac{q}{m}E\cdot\nabla_v f = 0, \qquad E = -\nabla\Phi, \quad -\Delta\Phi = \frac{q}{\varepsilon_0}\left(\int_{\mathbb{R}^d} dv\, f - 1\right).$$

$$t\in\mathbb{R}, \quad x\in\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d, \quad v\in\mathbb{R}^d, \quad Q = \mathbb{T}^d\times\mathbb{R}^d = \text{phase space}$$

- Weak turbulent regime:
  - $\varepsilon\in(0,1)$ be a small dimensionless parameter
  - $\tau_L = 1/\gamma_L$ = inverse of the instantaneous growth/damping rate of electric field.
  - $\mathcal{E}_{el}$ = electric energy.
  - $\mathcal{E}_{kin}$ = kinetic energy.

$$\hat{t} := \tau_L, \quad \hat{x} := \lambda_D, \quad \hat{v} := v_{th}, \quad \frac{\mathcal{E}_{el}}{\mathcal{E}_{kin}} = \frac{\varepsilon_0|\widehat{E}|^2}{\hat{n}m\hat{v}^2} = \varepsilon, \quad \frac{1}{\omega_p\hat{t}} = \varepsilon^2.$$

- Rescaled Vlasov–Poisson equation:

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon^2}\cdot\nabla_x f^\varepsilon + \frac{E^\varepsilon}{\varepsilon}\cdot\nabla_v f^\varepsilon = 0, \quad (t,x,v)\in\mathbb{R}^+\times\mathbb{T}^d\times\mathbb{R}^d,$$

$$E^\varepsilon = -\nabla\Phi^\varepsilon, \quad -\Delta\Phi^\varepsilon = \int_{\mathbb{R}^d} dv\, f^\varepsilon - 1.$$

Necesseary condition for non-degenerate diffusion 1/2

Theorem (A)

Let $\{f_0^\varepsilon\}_{\varepsilon>0}$ be a sequence of non-negative initial data and C_0 be a positive constant such that

$$\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \leq C_0, \quad \int_Q f_0^\varepsilon |v|^2 \, dxdv \leq C_0, \quad \left\| E_0^\varepsilon := \nabla \Delta^{-1} \left(\int_{\mathbb{R}^d} f_0^\varepsilon \, dv - 1 \right) \right\|_{L^2(\mathbb{T}^d)} \leq C_0.$$

Let $(f^\varepsilon, E^\varepsilon)_{\varepsilon>0}$, be a sequence of weak solutions of the rescaled Vlasov–Poisson system with initial data $f^\varepsilon|_{t=0} = f_0^\varepsilon$, whose existence is known or proved (Arsenev 75, Diperna–Lions 88, ...) for all $\varepsilon > 0$.

Then:

i) There exists a function $f = f(t, v)$, independent of x, such that $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$, and up to subsequences one has,

$$\begin{aligned} f^\varepsilon &\rightharpoonup f \ \text{ in } L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{ weak}-*, \\ \fint dx\, f^\varepsilon &\rightharpoonup f \ \text{ in } L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \ \text{ weak}-*. \end{aligned}$$

ii) The electric field E^ε converges weakly to zero as $\varepsilon \to 0$, more precisely,

$$E^\varepsilon \rightharpoonup 0 \ \text{ in } L^\infty(\mathbb{R}^+; W^{1,1+2/d}(\mathbb{T}^d)) \ \text{ weak}-*.$$

Necessary condition for non-degenerate diffusion 2/2

Theorem (A)

iii) *The expression,*

$$\nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon},$$

is uniformly (with respect to ε*) bounded in* $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$*. Hence, up to a subsequence, it converges in* $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ *and we obtain*

$$\partial_t f + \nabla_v \cdot \overline{\fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \tag{1}$$

$$f_{|_{t=0}} = \fint dx\, f_0.$$

iv) *Let* $d \leq 4$*. Moreover, if we suppose that there exists a constant* κ*, independent of* ε *such that*

$$\|E^\varepsilon\|_{W^s_{\mathrm{loc}}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \leq \kappa, \quad \text{with } s > 0, \quad \text{and} \quad \|\partial_t \Phi^\varepsilon\|_{L^1_{\mathrm{loc}}(\mathbb{R}^+;L^1(\mathbb{T}^d))} \leq \kappa,$$

then,

$$\fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} \rightharpoonup 0 \ \text{ in } \mathcal{D}'([0,T] \times \mathbb{R}^d),$$

$$E^\varepsilon \to 0 \ \text{ in } L^1([0,T] \times \mathbb{T}^d) \ \text{ strong},$$

as $\varepsilon \to 0$*, and equation* (1) *degenerates into the following equations,*

$$\partial_t f = 0 \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}^d), \qquad f_{|_{t=0}} = \fint dx\, f_0.$$

Remarks on the previous theorem

- The proof is based on the ergodicity of the free-flow in a periodic torus, and on the spatial strong compactness or regularity of the electric field given by the Poisson equation.
- Point *iv*) of theorem (A) shows that the lack of time compactness or regularity is in fact a necessary condition for obtaining a genuine or a non-degenerate diffusion equation in the limit $\varepsilon \to 0$.

Treatment of the flux term: Duhamel formula and Fick-type law

Now, we derive formally a Fick-type law for the flux term,

$$\overline{\int dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}},$$

appearing in (1).

- Most of developments are formal.
  They point out the difficulties for showing rigorously the diffusion limit.
- This Fick-type law can be obtained from two ways.
  The first way is a **global-in-time approach**, which involves the initial condition f_0^ε, while the second one, a **local-in-time approach**, does not.
- Each approach has its advantages and drawbacks.
- For both approaches, the absence of time decorrelation properties prevent us to determine the structure and the properties of the diffusion matrix.
- Nevertheless a formal WKB approximation allows us to obtain the structure of a non-negative diffusion matrix in the non-selfconsistent case.

Global-in-time approach for the flux term

Using Duhamel formula, the solution of the rescaled Vlasov–Poisson equation is

$$f^\varepsilon(t) = S_t^\varepsilon f_0^\varepsilon - \frac{1}{\varepsilon}\int_0^t ds\, S_{t-s}^\varepsilon E^\varepsilon(s)\cdot\nabla_v f^\varepsilon(s). \tag{2}$$

where the map $t \mapsto S_t^\varepsilon$ is the group on $L^q(Q)$, $1 \le q \le \infty$, generated by the free-flow, i.e.

$$(S_t^\varepsilon g)(x,v) = \exp\Big(-\frac{t}{\varepsilon^2} v\cdot\nabla_x\Big) g(x,v) = g(x - vt/\varepsilon^2, v), \quad \forall g \in L^p(Q).$$

Substituting (2) into the weak formulation of the divergence of the flux term,

$$-\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = \frac{1}{\varepsilon}\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx\, \nabla_v\varphi \cdot E^\varepsilon f^\varepsilon, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d),$$

we obtain

$$-\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \varphi \nabla_v \cdot \fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon} = T_1^\varepsilon(\varphi) + T_2^\varepsilon(\varphi),$$

where

$$T_1^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \frac{1}{\varepsilon}\nabla_v\varphi(t,v) \cdot \fint dx\, E^\varepsilon(t,x) f_0^\varepsilon(x - vt/\varepsilon^2, v),$$

and

$$T_2^\varepsilon(\varphi) := -\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \frac{1}{\varepsilon^2}\nabla_v\varphi(t,v)\cdot$$
$$\int_0^t ds \fint dx\, E^\varepsilon(t,x) E^\varepsilon(s, x - v(t-s)/\varepsilon^2)\cdot(\nabla_v f^\varepsilon)(s, x - v(t-s)/\varepsilon^2, v).$$

Global-in-time approach for the flux term: term T_1^ε

For the term T_1^ε we have

Lemma

Assume that f_0^ε satisfies the hypotheses of Theorem (A). In addition we suppose that there exists a constant $\mathcal{C}_0$, independent of ε, such that for $|\alpha| \leq 1$, the initial condition f_0^ε satisfies

$$\sum_{k\in\mathbb{Z}^d_*} \Big(|k|^{-1} \|\partial_v^\alpha \hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)}\Big)^2 \leq \mathcal{C}_0, \qquad \text{if } d = 1, \text{ and,}$$

$$\sum_{k\in\mathbb{Z}^d_*} \Big(|k|^{-2} \|\partial_v^\alpha \hat{f}_0^\varepsilon(k)\|_{L^1(\mathbb{R}^d)}\Big)^{1+2/d} \leq \mathcal{C}_0, \qquad \text{if } d \geq 2.$$

Then

$$T_1^\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d).$$

Remark

In this lemma , the regularity assumption for f_0^ε might be refined but with the presence of the factor ε^{-1} in the term T_1^ε, some mixing-type hypotheses seem compulsory.

Global-in-time approach for the flux term: term T_2^ε

We now deal with the term T_2^ε that we can rewrite as

$$T_2^\varepsilon(\varphi) = J^\varepsilon(\varphi) + M^\varepsilon(\varphi),$$

where

$$J^\varepsilon(\varphi) = \int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv\, f(t,v)\nabla_v \cdot \left(\int_{\mathbb{R}+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t-\varepsilon^2\sigma, x) \otimes E^\varepsilon(t, x+v\sigma)\, \nabla_v\varphi(t,v)\right),$$

and

$$M^\varepsilon(\varphi) := \int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv \int_{\mathbb{R}+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma)$$
$$\Big(f^\varepsilon(t-\varepsilon^2\sigma, x, v) - f(t,v)\Big)\nabla_v \cdot \Big(E^\varepsilon(t-\varepsilon^2\sigma, x)\otimes E^\varepsilon(t, x+v\sigma)\nabla_v\varphi(t,v)\Big).$$

If we assume that

$$\lim_{\varepsilon\to 0} J^\varepsilon(\varphi) \text{ exists}, \quad \text{and} \quad \lim_{\varepsilon\to 0} M^\varepsilon(\varphi) = 0,$$

then we obtain

$$\lim_{\varepsilon\to 0} T_2^\varepsilon(\varphi) = \int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv\, f(t,v)\nabla_v \cdot \Big(\mathscr{D}(t,v)^T \nabla_v\varphi(t,v)\Big),$$

with

$$\mathscr{D}(t,v) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}+} d\sigma \fint dx\, \chi_{[0,t/\varepsilon^2]}(\sigma) E^\varepsilon(t,x) \otimes E^\varepsilon(t-\varepsilon^2\sigma, x - v\sigma).$$

Finally, putting all pieces together, we obtain the following diffusion equation,

$$\partial_t f(t,v) - \nabla_v \cdot \Big(\mathscr{D}(t,v)\nabla_v f(t,v)\Big) = 0, \quad \text{in } \mathcal{D}'([0,T]\times\mathbb{R}^d).$$

Global-in-time approach for the flux term: open issues and remarks

1. All computations involving the term T_2^ε are formal and must be justified in a convenient functional framework.
   - In order to show existence of the limit $\lim_{\varepsilon\to 0} J^\varepsilon(\varphi)$ we have to show that the term
   $$R^\varepsilon(t,\sigma,x,v) := \chi_{[0,t/\varepsilon^2]}(\sigma)\nabla_v \cdot \Big(E^\varepsilon(t-\varepsilon^2\sigma,x)\otimes E^\varepsilon(t,x+v\sigma)\nabla_v\varphi(t,v)\Big)$$
   converges weakly in $L^1(\mathbb{R}_t^+\times\mathbb{R}_\sigma^+\times Q)$.
   - In order to prove $\lim_{\varepsilon\to 0} M^\varepsilon(\varphi)=0$, and justify the diffusion equation, we have to show that
   $$R^\varepsilon \text{ converges strongly in } L^1(\mathbb{R}_t^+\times\mathbb{R}_\sigma^+\times Q),$$
   since $f^\varepsilon(t-\varepsilon^2\sigma,x,v)-f(t,v) \rightharpoonup 0$ in $L^\infty(\mathbb{R}_t^+\times\mathbb{R}_\sigma^+\times Q)$ weak–$*$.
   - We observe that a crucial point is to obtain enough integrability with respect the time variable σ, uniformly in ε.
2. Strong time compactness would help to justify the above formal computations for the term T_2^ε. However, lack of time compactness is in fact necessary if we do not want to obtain a trivial equation, as stated in point *iv*) of Theorem (A).
3. Without time compactness, the electric field E^ε always converges weakly to zero, but not the quadratic electric tensor $E^\varepsilon\otimes E^\varepsilon$ (this is a property of weak convergence), which implies a non-trivial diffusion matrix $\mathscr{D}$. Therefore, weak convergence seems mandatory to obtain a diffusion limit.
4. Fast oscillations in time should produce the diffusion, WKB expansion seems relevant for this.
5. Instead of time compactness, time decorrelation properties could help to justified rigorously above computations (cf. the non-selfconsistent stochastic part).

Global-in-time approach for the flux term: weak limit, fast-time oscillations, WKB ansatz

We can obtain an explicit form of the diffusion matrix in the non-selfconsistent deterministic case.

Using Fourier expasion,

$$E^{\varepsilon}(t,x)=\sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot x}\widehat{E}^{\varepsilon}(t,k),$$

we assume the formal WKB expansion for the Fourier mode $\widehat{E}^{\varepsilon}(t,k)$,

$$\widehat{E}^{\varepsilon}(t,k)=\sum_{j\geq 0}\varepsilon^{j}\widehat{E}_j(t,k,\Omega(t,k)/\varepsilon^2), \tag{3}$$

where complex vector-valued functions $(k,\tau)\mapsto \widehat{E}_j(t,k,\tau)$ are 2π-periodic with respect to the variable τ. As a first approximation of (3), we obtain

$$\widehat{E}^{\varepsilon}(t,k)=\widehat{E}_0(t,k)\exp\left(-\mathrm{i}\frac{\Omega(t,k)}{\varepsilon^2}\right)+\mathcal{O}(\varepsilon), \tag{4}$$

where the real vector-valued function $k\mapsto \widehat{E}_0(t,k)$ is even with respect to the variable k.

Using (4) and time Taylor expansions, we obtain from the definition of the diffusion matrix $\mathscr{D}$,

$$\mathscr{D}(t,v)=\pi\sum_{k\in\mathbb{Z}^d}\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k)\delta\Big(\partial_t\Omega(t,k)-k\cdot v\Big).$$

If we assume $\Omega(t,k)=\int_0^t d\theta\,\omega(\theta,k)$, then we obtain

$$\mathscr{D}(t,v)=\pi\sum_{k\in\mathbb{Z}^d}\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k)\delta\Big(\omega(t,k)-k\cdot v\Big).$$

Local-in-time approach for the flux term: term $\mathcal{T}_1^\varepsilon$

Integrating in time the space-averaged Vlasov equation against a test function $\varphi(t,v)$, we obtain

$$\left\langle \frac{f^\varepsilon(t+\theta)-f^\varepsilon(t)}{\theta},\varphi\right\rangle = \frac{1}{\varepsilon\theta}\int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds ⨍ dx\, f^\varepsilon(s)E^\varepsilon(s)\cdot\nabla_v\varphi(t,v).$$

Using the Duhamel representation formula for $f^\varepsilon(s)$,

$$f^\varepsilon(s) = S^\varepsilon_{s-t+\hat\theta} f^\varepsilon(t-\hat\theta) - \frac{1}{\varepsilon}\int_{t-\hat\theta}^s d\sigma\, S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma)\cdot\nabla_v f^\varepsilon(\sigma),$$

with $\hat\theta$ an arbitrary non-negative time we obtain

$$\left\langle \frac{f^\varepsilon(t+\theta)-f^\varepsilon(t)}{\theta},\varphi\right\rangle = \mathcal{T}_1^\varepsilon(\varphi)+\mathcal{T}_2^\varepsilon(\varphi),$$

where

$$\mathcal{T}_1^\varepsilon(\varphi) := \int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv \int_t^{t+\theta} ds ⨍ dx\, \frac{1}{\varepsilon\theta}E^\varepsilon(s)\cdot\nabla_v\varphi(t,v)S^\varepsilon_{s-t+\hat\theta}f^\varepsilon(t-\hat\theta),$$

and

$$\mathcal{T}_2^\varepsilon(\varphi) := \int_{\mathbb{R}+} dt \int_{\mathbb{R}^d} dv$$
$$\int_t^{t+\theta} ds \int_{t-\hat\theta}^s d\sigma ⨍ dx\, \frac{1}{\varepsilon^2\theta} S^\varepsilon_{s-\sigma}f^\varepsilon(\sigma)\nabla_v\cdot\Big(S^\varepsilon_{s-\sigma}E^\varepsilon(\sigma,x)\otimes E^\varepsilon(s,x)\nabla_v\varphi(t,v)\Big).$$

For the term $\mathcal{T}_1^\varepsilon$, we assume that

$$\mathcal{T}_1^\varepsilon \rightharpoonup 0 \text{ in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d).$$

Local-in-time approach for the flux term: term $\mathcal{T}_2^\varepsilon$

We now deal with the term $\mathcal{T}_2^\varepsilon$, which can be recast as

$$\mathcal{T}_2^\varepsilon(\varphi) = \mathcal{J}^\varepsilon(\varphi) + \mathcal{M}^\varepsilon(\varphi),$$

where, using the change of variables $\theta = \varepsilon^2\tau$ and $\hat{\theta} = \varepsilon^2\eta$,

$$\mathcal{J}^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv f(t,v)\nabla_v \cdot \Big(\frac{1}{\tau}\int_0^\tau ds \int_{-\eta}^s d\sigma \fint dx$$
$$E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta))\nabla_v\varphi(t,v)\Big),$$

and

$$\mathcal{M}^\varepsilon(\varphi) := \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \frac{1}{\tau}\int_0^\tau ds \int_{-\eta}^s d\sigma \fint dx \left(f^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x, v) - f(t,v)\right)$$
$$\nabla_v \cdot \Big(E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x) \otimes E^\varepsilon(t+s\varepsilon^2, x+v(\sigma+\eta))\nabla_v\varphi(t,v)\Big).$$

The next lemma justifies that in the case where τ and η are finite, the term $\mathcal{J}^\varepsilon(\varphi)$ has a limit as $\varepsilon \to 0$. Defining

$$\mathscr{D}^\varepsilon(t,v) = \frac{1}{\tau}\int_0^\tau ds \int_{-\eta}^s d\sigma \fint dx \, E^\varepsilon(t+s\varepsilon^2, x) \otimes E^\varepsilon(t+(s-\eta-\sigma)\varepsilon^2, x-v(\sigma+\eta))$$

we have the following lemma,

Local-in-time approach for the flux term: term $\mathcal{T}_2^\varepsilon$

Lemma

Let τ and η be finite. Then, $\mathcal{J}^\varepsilon$ has a limit in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ such that

$$\lim_{\varepsilon \to 0} \mathcal{J}^\varepsilon(\varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f \nabla_v \cdot (\mathscr{D}^T \nabla_v \varphi), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d),$$

where $\mathscr{D}$ is the weak limit of $\mathscr{D}^\varepsilon$ (up to a subsequence) in the following sense,

$$\mathscr{D}^\varepsilon \rightharpoonup \mathscr{D} \quad \text{in} \quad L^1_{\rm loc}(\mathbb{R}^+; W^{1,1}_{\rm loc}(\mathbb{R}^d)) \ \text{weak},$$

and

$$\mathscr{D}^\varepsilon \rightharpoonup \mathscr{D} \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^d)) \ \text{weak}-*.$$

Finally, if we now assume

$$\lim_{\varepsilon \to 0} \mathcal{M}^\varepsilon(\varphi) = 0,$$

putting all pieces together, we obtain the following diffusion equation

$$\partial_t f(t,v) - \nabla_v \cdot \Big(\mathscr{D}(t,v) \nabla_v f(t,v)\Big) = 0, \quad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}^d).$$

Local-in-time approach for the flux term: open issues and remarks

1. In order to justify $\lim_{\varepsilon\to 0} \mathcal{M}^\varepsilon(\varphi) = 0$, we have to prove that the term

$$\nabla_v \cdot \Big(E^\varepsilon(t + (s - \eta - \sigma)\varepsilon^2, x) \otimes E^\varepsilon(t + s\varepsilon^2, x + v(\sigma + \eta))\nabla_v \varphi(t, v)\Big),$$

converges strongly in $L^1(\mathbb{R}_t^+ \times \mathbb{R}_s^+ \times \mathbb{R}_\sigma^+ \times Q)$ as $\varepsilon \to 0$.

2. Show that $\mathcal{T}_1^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ remains an open issue.
Nevertheless, we may expect that there exist some mixing-type hypotheses, which could justify such limit.

4. The parameter τ is reminiscent of the autocorrelation time of particles τ, which will be introduced in case of the non-selfconsistent stochastic electric field.

5. As before we can use a WKB expansion to obtain the structure of the diffusion matrix in the non-selfconsistent deterministic case.

Local-in-time approach for the flux term: weak limit, fast-time oscillations, WKB ansatz

Using the following WKB ansatz for the Fourier modes of the electric field,

$$\widehat{E}^{\varepsilon}(t,k) = \widehat{E}_0(t,k)\exp\left(-\mathrm{i}\frac{\Omega(t,k)}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon),$$

in the definition of $\mathscr{D}$, we obtain

$$\mathscr{D}(t,v) = \sum_{k\in\mathbb{Z}^d} \frac{\sin(\tau\Delta\Omega/2)}{\tau\Delta\Omega/2}\,\frac{\sin((\tau/2+\eta)\Delta\Omega)}{\Delta\Omega}\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k).$$

with

$$\Delta\Omega := \partial_t\Omega(t,k) - k\cdot v.$$

1. Limit $\eta\to+\infty$:

$$\mathscr{D}(t,v) = \pi\sum_{k\in\mathbb{Z}^d}\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k)\delta\left(\partial_t\Omega(t,k)-k\cdot v\right).$$

2. Limit $\eta\to 0$:

$$\mathscr{D}(t,v) = \frac{\tau}{2}\sum_{k\in\mathbb{Z}^d}\left(\frac{\sin\left(\frac{\tau}{2}\left(\partial_t\Omega(t,k)-k\cdot v\right)\right)}{\frac{\tau}{2}\left(\partial_t\Omega(t,k)-k\cdot v\right)}\right)^2\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k).$$

3. Limit $\tau\to+\infty$ with η fixed and finite:

$$\mathscr{D}(t,v) = \pi\sum_{k\in\mathbb{Z}^d}\widehat{E}_0(t,k)\otimes\widehat{E}_0(t,k)\delta\left(\partial_t\Omega(t,k)-k\cdot v\right).$$

The turbulent electric field

Here, electrostatic turbulence is modeled through a given random vector field E^{ε}.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with $\mathbb{P}$ being a σ-finite measure. A random vector F is real vector-valued function defined on Ω. When $F : \Omega \to \mathbb{R}^d$ is an integrable random vector, its expectation is given by

$$\mathbb{E}[F] = \int_{\Omega} \mathrm{d}\mathbb{P}(\omega)\, F(\omega).$$

The turbulent electric field E^{ε} has two time scales, one slow and the other fast, and is given by

$$E^{\varepsilon}(t,x) = E(t, t/\varepsilon^2, x; \omega),$$

where, the integrable random vector field E satisfies the following "stochastic" assumptions:

(H1): The random vector field E is centered, i.e.

$$\mathbb{E}[E(t,\tau,x)] = 0, \quad \forall (t,\tau,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{T}^d.$$

(H2): There exists a constant $\uptau > 0$ such that for every $x, y \in \mathbb{R}^d$ and for every $\tau, \sigma \in \mathbb{R}^+$ the electric fields $E(t,\tau,x)$ and $E(s,\sigma,y)$ are independent random vector fields as soon as $|\tau - \sigma| \geq \uptau$.
The autocorrelation time $\uptau$ is supposed fixed and finite, hence independent of ε.

(H3): There exists a matrix-valued function $\mathcal{R}_{\uptau} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^{2d}$, called the autocorrelation matrix or the Reynolds electric stress tensor, such that

$$\mathbb{E}[E(t,\tau,x) \otimes E(s,\sigma,y)] = \mathcal{R}_{\uptau}(t, s, \tau - \sigma, x - y).$$

(H4): The regularity of E is such that

$$E \in L^{\infty}\left(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,\infty}(\mathbb{T}^d)\right), \quad \text{and} \quad \mathbb{E}\left[\|E\|^3_{L^{\infty}(\mathbb{R}^+ \times \mathbb{R}^+; W^{2,\infty}(\mathbb{T}^d))}\right] =: C_E < \infty.$$

Theorem (B)

- *Let E be an integrable random vector field satisfying assumptions* (H1)-(H4)*, and let E^ε be given by $E^\varepsilon(t,x) = E(t, t/\varepsilon^2, x; \omega)$.*
- *Let $\{f_0^\varepsilon\}_{\varepsilon>0}$ be a sequence of independent random non-negative initial data and C_0 be a positive constant such that for a.e. $\omega \in \Omega$, $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \leq C_0 < \infty$.*
- *Let $\mathscr{D}_\tau = \mathscr{D}_\tau(t,v)$ be the matrix-valued function defined by*

$$\mathscr{D}_\tau(t,v) = \int_0^\tau d\sigma\, \mathcal{R}_\tau(t,t,\sigma,\sigma v).$$

- *Let f^ε be the unique weak solution of the rescaled Vlasov equation with initial data $f^\varepsilon|_{t=0} = f_0^\varepsilon$.*

Then up to extraction of a subsequence,

$$\mathbb{E}[f_0^\varepsilon] \rightharpoonup f_0 \in L^1 \cap L^\infty(Q) \quad \text{in} \quad L^\infty(Q) \ \text{weak--}*,$$

$$\mathbb{E}[f^\varepsilon] \rightharpoonup f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)) \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{weak--}*,$$

$$\mathbb{E}\left[\fint dx\, f^\varepsilon \right] \rightharpoonup f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)) \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{weak--}*.$$

Moreover $\mathbb{E}[\fint dx\, f^\varepsilon]$ converges in $\mathscr{C}(0,T; L^p(\mathbb{R}^d) - \text{weak})$ to f, for $1 < p < \infty$ and for all $T > 0$. The limit point $f = f(t,v)$ is solution of the following diffusion equation in the sense of distributions:

$$\partial_t f - \nabla_v \cdot (\mathscr{D}_\tau \nabla_v f) = 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$

$$f|_{t=0} = \fint dx\, f_0.$$

Proposition (properties of the diffusion matrix $\mathscr{D}_\tau$)

Under assumptions (H1)-(H4), *the matrix-valued function* $\mathcal{R}_\tau : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}^{2d}$, *and the diffusion matrix* $\mathscr{D}_\tau : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{2d}$ *satisfy the following properties:*

i) $\mathcal{R}_\tau(t,t,\tau,x) = \mathcal{R}_\tau^T(t,t,-\tau,-x)$, *and* $\mathcal{R}_\tau(t,t,\tau,x+2\pi k) = \mathcal{R}_\tau(t,t,\tau,x)$, $\forall k \in \mathbb{Z}$.

ii) $\mathcal{R}_\tau \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}; W^{2,\infty}(\mathbb{T}^d))$, *and* $\mathrm{supp}(\mathcal{R}_\tau) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times [-\tau,\tau] \times \mathbb{T}^d$.

iii) $\mathscr{D}_\tau \in L^\infty(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^d))$, *and* $\mathrm{supp}(\mathscr{D}_\tau) \subset \mathbb{R}^+ \times \mathbb{R}^d$.

iv) *The symmetric part of* $\mathscr{D}_\tau$ *is non-negative, i.e.* $X^T \mathscr{D}_\tau X \geq 0$, $\forall \in X \in \mathbb{R}^d$.

Proof of Theorem (B): time decorrelation

Let us rewrite the rescaled Vlasov equation in the following form,

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L} f^\varepsilon = \mathcal{N}_t^\varepsilon f^\varepsilon,$$
$$f^\varepsilon|_{t=0} = f_0^\varepsilon,$$

where the linear operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are defined by

$$\mathcal{L} = v \cdot \nabla_x, \qquad \mathcal{N}_t^\varepsilon = -\frac{1}{\varepsilon} E^\varepsilon(t,x) \cdot \nabla_v = -\frac{1}{\varepsilon} E(t, t/\varepsilon^2, x) \cdot \nabla_v.$$

- The operators $\mathcal{L}$ and $\mathcal{N}_t^\varepsilon$ are skew-adjoint for the scalar product of $L^2(Q)$.
- The operator $\mathcal{L}$ and the deterministic group S_t^ε, generated by $\varepsilon^{-2}\mathcal{L}$ commute with the statistical average $\mathbb{E}$.
- The space and statistical averages commute.

From hypothesis (H2), the random operators $\mathcal{N}_t^\varepsilon$ and $\mathcal{N}_s^\varepsilon$ are independent as soon as $|t-s| > \varepsilon^2 \tau$.

The next useful proposition states that time decorrelation of the stochastic electric field also entails time decorrelation between the distribution function and the electric field.

Proposition (time decorrelation property between f^ε and E^ε)

Assume (H2). Suppose that the random initial data f_0^ε and the electric field E^ε are independent. Then $\mathcal{N}_s^\varepsilon$ is independent of $f^\varepsilon(t)$ as soon as $s \geq t + \varepsilon^2\tau$.

Proposition (Existence of limits in the proof of Theorem (B))

Assume (H4) *and consider a sequence* $\{f_0^\varepsilon\}_{\varepsilon>0}$ *of initial data such that*

$$f_0^\varepsilon \geq 0, \quad \text{and for a.e. } \omega \in \Omega,\ \|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \leq C_0 < \infty.$$

Then, for any $\varepsilon > 0$, *the rescaled Vlasov equation has a unique non-negative solution* $f^\varepsilon \in \mathscr{C}(\mathbb{R}^+, L^1 \cap L^\infty(Q))$, *which is given by*

$$f^\varepsilon(t,x,v) = f_0^\varepsilon(X^\varepsilon(0;t,x,v), V^\varepsilon(0;t,x,v)),$$

where the characteristic curves $(X^\varepsilon, V^\varepsilon)$ *are solutions to the ODEs,*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon^2} V^\varepsilon(t), \quad \frac{dV^\varepsilon}{dt}(t) = \frac{1}{\varepsilon} E^\varepsilon(t, X(t)), \quad X^\varepsilon(0;0,x,v) = x, \ \ V^\varepsilon(0;0,x,v) = v.$$

In addition, there exist a function $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, *and a function* $f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$, *such as, up to subsequences,*

$$\mathbb{E}[f_0^\varepsilon] \rightharpoonup f_0 \ \text{ in } \ L^\infty(Q) \ \text{ weak}-*, \qquad \text{and} \qquad \mathbb{E}[f^\varepsilon] \rightharpoonup f \ \text{ in } \ L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{ weak}-*.$$

The limit point f *is such that* $\fint dx\, f = f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d))$. *The function* $\mathbb{E}[\fint dx\, f^\varepsilon]$ *is the solution of*

$$\partial_t \mathbb{E}\left[\fint dx\, f^\varepsilon\right] + \nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon f^\varepsilon}{\varepsilon}\right] = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$

$$\mathbb{E}\left[\fint dx\, f^\varepsilon\right]_{|t=0} = \mathbb{E}\left[\fint dx\, f_0^\varepsilon\right].$$

Proof of Theorem (B): double Duhamel iteration, and decomposition of the flux term

Using the group S_t^ε and the Duhamel formula, the formal solution to the rescaled Vlasov equation is given by

$$f^\varepsilon(t) = S^\varepsilon_{t-s} f^\varepsilon(s) + \int_s^t d\tau\, S^\varepsilon_{t-\tau}\mathcal{N}^\varepsilon_\tau f^\varepsilon(\tau). \qquad (5)$$

Taking $s = t - \varepsilon^2\tau$ in (5), and making the change of variable $\tau = t - \sigma$, we obtain from (5),

$$f^\varepsilon(t) = S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau) + \int_0^{\varepsilon^2\tau} d\sigma\, S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} f^\varepsilon(t-\sigma). \qquad (6)$$

In the integral term of (6), we observe that the electric field and the distribution function are evaluated at the same time $t-\sigma$. As a consequence, if we substitute (6) to f^ε in the flux term of the space-averaged rescaled Vlasov equation we obtain a quadratic term with respect to the electric field that we cannot decorrelate in time from the distribution function. For this reason we iterate a second time the Duhamel formula. In the same way that we obtained (5), we obtain

$$f^\varepsilon(t-\sigma) = S^\varepsilon_{2\varepsilon^2\tau-\sigma} f^\varepsilon(t-2\varepsilon^2\tau) + \int_0^{2\varepsilon^2\tau-\sigma} ds\, S^\varepsilon_s \mathcal{N}^\varepsilon_{t-\sigma-s} f^\varepsilon(t-\sigma-s). \qquad (7)$$

Substituting the right-hand side of (7) to $f^\varepsilon(t-\sigma)$ in the right-hand side of (6), and using the properties of the group S_t^ε, we obtain

$$\begin{aligned} f^\varepsilon(t) = {} & S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau) + \int_0^{\varepsilon^2\tau} d\sigma\, S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma} S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau) \\ & + \int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds\, S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_s \mathcal{N}^\varepsilon_{t-\sigma-s} f^\varepsilon(t-\sigma-s). \end{aligned} \qquad (8)$$

Applying the operator $\mathcal{N}_t^\varepsilon$ to (8), and then applying successively the average in space and the expectation value, we obtain

$$-\nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] = \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau)\right] + \int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma} S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)\right] + \mu_t^\varepsilon, \quad (9)$$

with

$$\mu_t^\varepsilon = \int_0^{\varepsilon^2\tau} d\sigma \int_0^{2\varepsilon^2\tau-\sigma} ds \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_s \mathcal{N}^\varepsilon_{t-\sigma-s} f^\varepsilon(t-\sigma-s)\right].$$

Using Proposition 2, we obtain that $f^\varepsilon(t)$ is independent of $\mathcal{N}_s^\varepsilon$ as soon as $s \geq t + \varepsilon^2\tau$. Then, using hypothesis $(\mathrm{H1})$, we obtain

$$\mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_{\varepsilon^2\tau} f^\varepsilon(t-\varepsilon^2\tau)\right] = \mathbb{E}\left[\mathcal{N}_t^\varepsilon\right] S^\varepsilon_{\varepsilon^2\tau} \mathbb{E}\left[f^\varepsilon(t-\varepsilon^2\tau)\right] = 0.$$

Since Proposition 2 implies that $\mathcal{N}_t^\varepsilon$ and $\mathcal{N}_{t-\sigma}^\varepsilon$ are independent of $f^\varepsilon(t-2\varepsilon^2\tau)$, for $0 \leq \sigma \leq \varepsilon^2\tau$, we obtain from (9),

$$\begin{aligned} -\nabla_v \cdot \mathbb{E}\left[\fint dx\, \frac{E^\varepsilon(t) f^\varepsilon(t)}{\varepsilon}\right] &= \int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma}\right] \mathbb{E}\left[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau)\right] + \mu_t^\varepsilon \\ &= \int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma}\right] \mathbb{E}\left[f^\varepsilon(t)\right] + \mu_t^\varepsilon \\ &+ \int_0^{\varepsilon^2\tau} d\sigma \fint dx\, \mathbb{E}\left[\mathcal{N}_t^\varepsilon S^\varepsilon_\sigma \mathcal{N}^\varepsilon_{t-\sigma} S^\varepsilon_{-\sigma}\right] \mathbb{E}\left[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t-2\varepsilon^2\tau) - f^\varepsilon(t)\right] \end{aligned}$$

Proof of Theorem (B): limit of the flux term, the Fick-type law

Proposition

We define the differential operator Θ_t^ε as

$$\Theta_t^\varepsilon \varphi = \int_0^\tau d\sigma \, (\sigma \nabla_x \cdot + \nabla_v \cdot \,) \, \mathbb{E}\left[E^\varepsilon(t - \varepsilon^2 \sigma, x - \sigma v) \otimes E^\varepsilon(t, x) \right] \nabla_v \varphi, \quad \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d), \; (10)$$

and the bilinear form ν_t^ε as

$$\nu_t^\varepsilon(\psi, \varphi) = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \fint dx \, \psi \Theta_t^\varepsilon \varphi, \quad \forall \psi \in L^\infty(\mathbb{R}^+ \times Q), \; \forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d).$$

Then, the weak formulation of the flux term reads: $\forall \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \, \nabla_v \varphi \cdot \mathbb{E}\left[\fint dx \, \frac{f^\varepsilon(t) E^\varepsilon(t)}{\varepsilon} \right]$$
$$= \nu_t^\varepsilon\left(\mathbb{E}[f^\varepsilon(t)], \varphi \right) + \nu_t^\varepsilon\left(\mathbb{E}\left[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t) \right], \varphi \right) + \mu_t^\varepsilon(\varphi),$$

where the remainder term $\mu_t^\varepsilon(\varphi)$ is given by

$$\mu_t^\varepsilon(\varphi) = -\varepsilon^4 \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv \int_0^\tau d\sigma \int_0^{2\tau - \sigma} ds \fint dx$$
$$\mathbb{E}\left[f^\varepsilon(t - \varepsilon^2(\sigma + s)) \, \mathcal{N}^\varepsilon_{t - \varepsilon^2(\sigma + s)} S^\varepsilon_{-\varepsilon^2 s} \mathcal{N}^\varepsilon_{t - \varepsilon^2\sigma} S^\varepsilon_{-\varepsilon^2\sigma} \mathcal{N}^\varepsilon_t \varphi \right].$$

Proof of Theorem (B): some lemmas

Lemma

Under hypothesis (H3), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, *the operator* Θ_t^ε *defined by* (10) *becomes*

$$\Theta_t^\varepsilon \varphi = \nabla_v \cdot \left(\left(\int_0^\tau d\sigma\, \mathcal{R}_\tau(t - \varepsilon^2\sigma, t, -\sigma, -\sigma v) \right) \nabla_v \varphi \right),$$

and we obtain

$$\Theta_t^\varepsilon \varphi \longrightarrow \Theta_t^0 \varphi \quad \text{in } L^1(\mathbb{R}^+ \times \mathbb{R}^d) \text{ strong},$$

where the operator Θ_t^0 *is defined by*

$$\Theta_t^0 \varphi = \nabla_v \cdot \left(\left(\int_0^\tau d\sigma\, \mathcal{R}_\tau(t, t, -\sigma, -\sigma v) \right) \nabla_v \varphi \right).$$

Lemma

Under hypothesis (H3), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, *we obtain*

$$\lim_{\varepsilon \to 0} \nu_t^\varepsilon \left(\mathbb{E}[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t)], \varphi \right) = 0.$$

Lemma

Under hypothesis (H4), *for all* $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, *we obtain*

$$|\mu_t^\varepsilon(\varphi)| \leq \varepsilon \tau^4 C_0 C_E \|\varphi\|_{L^\infty(\mathbb{R}^+; W^{3,1}(\mathbb{R}^d))} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof of Theorem (B): the end.

We are now able to conclude the proof of Theorem (B). Indeed we have

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \mathbb{E}\left[\fint dx\, f^\varepsilon\right] \partial_t \varphi - \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \nabla_v \varphi \cdot \mathbb{E}\left[\fint dx\, \frac{f^\varepsilon E^\varepsilon}{\varepsilon}\right] = 0, \qquad (11)$$

with

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \nabla_v \varphi \cdot \mathbb{E}\left[\fint dx\, \frac{f^\varepsilon E^\varepsilon}{\varepsilon}\right]$$
$$= \nu_t^\varepsilon\left(\mathbb{E}[f^\varepsilon], \varphi\right) + \nu_t^\varepsilon\left(\mathbb{E}\left[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t)\right], \varphi\right) + \mu_t^\varepsilon(\varphi). \qquad (12)$$

From lemmas above we have

$$\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, \mathbb{E}\left[\fint dx\, f^\varepsilon\right] \partial_t \varphi \longrightarrow \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f \partial_t \varphi, \quad \text{as } \varepsilon \to 0, \qquad (13)$$

$$\nu_t^\varepsilon\left(\mathbb{E}[f^\varepsilon], \varphi\right) \longrightarrow \int_{\mathbb{R}^+} dt \int_{\mathbb{R}^d} dv\, f \Theta_t^0 \varphi, \quad \text{as } \varepsilon \to 0, \qquad (14)$$

and

$$\nu_t^\varepsilon\left(\mathbb{E}\left[S^\varepsilon_{2\varepsilon^2\tau} f^\varepsilon(t - 2\varepsilon^2\tau) - f^\varepsilon(t)\right], \varphi\right) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \quad \text{as } \varepsilon \to 0, \qquad (15)$$

$$\mu_t^\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \quad \text{as } \varepsilon \to 0. \qquad (16)$$

Using (13)-(16) to pass to the limit $\varepsilon \to 0$ in (12) and (11), we obtain

$$\partial_t f - \nabla_v \cdot \left(\left(\int_0^\tau d\sigma\, \mathcal{R}_\tau(t, t, \sigma, \sigma v)\right) \nabla_v f\right) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d).$$

Links with some kinetic turbulence theories of plasma physics

The diffusion matrix $\mathscr{D}_\tau$ obtained in Theorem (B), can be recast as

$$\mathscr{D}_\tau(t,v) = \int_0^\tau d\sigma\, \mathbb{E}[E(t,0,x) \otimes E(t,-\sigma,x-\sigma v)]. \tag{17}$$

Introducing the Fourier series decomposition of E,

$$E(t,\tau,x) = \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}k\cdot x}\widehat{E}(t,\tau,k),$$

we can suppose without loss of generality that

$$\widehat{E}(t,\tau,k) = e^{-\mathrm{i}\omega(k)\tau}\widetilde{E}(t,\tau,k),$$

where the real-valued function $\mathbb{Z}^d \ni k \mapsto \omega(k) \in \mathbb{R}$ is odd, i.e. $\omega(-k) = -\omega(k)$ for all $k \in \mathbb{Z}^d$.

In the same spirit as assumption (H3), we now make the following assumption:

(H3′) : There exist a non-negative real-valued function $\mathcal{E}(t,k) : \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+$, with $\mathcal{E}(t,k) = \mathcal{E}(t,-k\ )$ and $|k|^2|\mathcal{E}(t,k)|^{1/2} \in L^\infty(\mathbb{R}^+;\ell^1(\mathbb{Z}^d))$, and a bounded function $A_\tau(\tau,k) : [-\tau,\tau]\times\mathbb{Z}^d \to \mathbb{R}^+$, even and compactly supported in τ, such that

$$\mathbb{E}\left[\widetilde{E}(t,\tau,k)\otimes\widetilde{E}(t,\sigma,k')\right] = A_\tau(\tau-\sigma,k)\mathcal{E}(t,k)\,\frac{k\otimes k}{|k|^2}\,\delta(k+k').$$

Links with some kinetic turbulence theories of plasma physics

Actually, the property (H3′) implies the property (H3), in other words the property (H3′) is less general than the property (H3). Indeed, we obtain from (H3′),

$$\mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)] = \sum_{k\in\mathbb{Z}^d} A_\tau(\tau-\sigma,k)\mathcal{E}(t,k)\frac{k\otimes k}{|k|^2} e^{\mathrm{i}k\cdot(x-y)} e^{-\mathrm{i}\omega(k)(\tau-\sigma)},$$

from which we easily observe that the spatio-temporal autocorrelation function $\mathbb{E}[E(t,\tau,x)\otimes E(t,\sigma,y)]$ is invariant under time and space translations. In terms of the autocorrelation matrix $\mathcal{R}_\tau$, hypothesis (H3′) is equivalent to

$$\mathcal{R}_\tau(t,t,\sigma,x) = \sum_{k\in\mathbb{Z}^d} e^{-\mathrm{i}\omega(k)\sigma} A_\tau(\sigma,k)\mathcal{E}(t,k)\frac{k\otimes k}{|k|^2} e^{\mathrm{i}k\cdot x}.$$

Using assumption (H3′) in definition (17) of $\mathscr{D}_\tau$, we obtain

$$\mathscr{D}_\tau(t,v) = \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\frac{k\otimes k}{|k|^2}\int_0^\tau d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_\tau(\sigma,k). \tag{18}$$

The diffusion matrix (18) can be rewritten as

$$\mathscr{D}_\tau(t,v) = \sum_{k\in\mathbb{Z}^d} \mathcal{E}(t,k)\frac{k\otimes k}{|k|^2} R_\tau(\omega(k)-k\cdot v,k),$$

where the resonance function R_τ is given by

$$R_\tau(\omega(k)-k\cdot v,k) = \Re e\int_0^\tau d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_\tau(\sigma,k) = \frac{1}{2}\int_{-\infty}^{+\infty} d\sigma\, e^{-\mathrm{i}(\omega(k)-k\cdot v)\sigma} A_\tau(\sigma,k).$$

Corollary

- *Let E be an integrable random vector field satisfying the assumptions* (H1)-(H2) *and* (H3′)-(H4)*.*
- *Let E^ε be given by $E^\varepsilon(t,x) = E(t, t/\varepsilon^2, x; \omega)$.*
- *Let $\{f_0^\varepsilon\}_{\varepsilon>0}$ be a sequence of independent random non-negative initial data and C_0 be a positive constant such that for a.e. $\omega \in \Omega$, $\|f_0^\varepsilon\|_{L^1(Q)} + \|f_0^\varepsilon\|_{L^\infty(Q)} \leq C_0 < \infty$.*
- *Let $\mathscr{D}_\tau$ be the matrix-valued function defined by* (18)*.*
- *Let f^ε be the unique weak solution of the rescaled Vlasov equation , with initial data $f^\varepsilon|_{t=0} = f_0^\varepsilon$.*

Then

1. *Up to extraction of a subsequence,*

$$\mathbb{E}[f_0^\varepsilon] \rightharpoonup f_0 \in L^1 \cap L^\infty(Q) \quad \text{in} \quad L^\infty(Q) \ \text{weak--}*,$$

$$\mathbb{E}[f^\varepsilon] \rightharpoonup f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)) \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{weak--}*,$$

$$\mathbb{E}\left[\fint dx\, f^\varepsilon \right] \rightharpoonup f \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)) \quad \text{in} \quad L^\infty(\mathbb{R}^+; L^\infty(Q)) \ \text{weak--}*.$$

   The limit point $f = f(t,v)$ is solution of the following diffusion equation in the sense of distributions:

$$\partial_t f - \nabla_v \cdot (\mathscr{D}_\tau \nabla_v f) = 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d),$$
$$f_{|_{t=0}} = \fint dx\, f_0.$$

2. *The diffusion matrix $\mathscr{D}_\tau$ is symmetric, non-negative and analytic in the velocity variables.*

Remarks

- Our diffusion matrix $\mathscr{D}_\tau$ is reminiscent of the diffusion matrix $\mathscr{D}_{\mathrm{rb}}$ of the resonance broadening theory (RBT).
- **RBT: finite** τ. The important difference is that in the resonance broadening theory, τ depends on the quasilinear diffusion matrix itself (through a nonlinear integral functionnal), whereas in our derivation τ is a free parameter. Indeed for the RBT we have,

$$A_\tau(\sigma, k) = \exp(-(\sigma/\tau)^3/3), \quad \text{with} \quad \tau = (k \otimes k : \mathscr{D}_{\mathrm{rb}}(\sigma, v))^{-1/3}, \quad \text{i.e.}$$
$$A_\tau(\sigma, k) = A_{\mathrm{rb}}(\sigma, k, \mathscr{D}_{\mathrm{rb}}(\sigma, v)) := \exp\left(-\tfrac{1}{3} k \otimes k : \mathscr{D}_{\mathrm{rb}}(\sigma, v)\sigma^3\right).$$

and

$$\begin{aligned}\mathscr{D}_{\mathrm{rb}}(t, v) &= \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} \Re e \int_0^\infty d\sigma\, e^{-\mathrm{i}(\omega(k) - k \cdot v)\sigma} A_{\mathrm{rb}}(\sigma, k, \mathscr{D}_{\mathrm{rb}}(\sigma, v)) \\ &= \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} R_{\mathrm{rb}}(\omega(k) - k \cdot v, k, \mathscr{D}_{\mathrm{rb}}).\end{aligned}$$

- **QLT: infinite** τ. By taking the formal limit $\tau \to +\infty$, for an autocorrelation function A_τ, such that $A_\tau \to 1$ a.e. as $\tau \to +\infty$ (e.g. $A_\tau(\sigma, k) = \mathbb{1}_{[-\tau,\tau]}(\sigma)$), and using Plemelj formula, we obtain

$$\lim_{\tau \to +\infty} R_\tau(\omega(k) - k \cdot v, k) = \pi\delta(\omega(k) - k \cdot v), \tag{19}$$

for the resonance function. Then, we recover the QL diffusion matrix of plasma physics literature

$$\mathscr{D}_\infty(t, v) = \pi \sum_{k \in \mathbb{Z}^d} \mathcal{E}(t, k) \frac{k \otimes k}{|k|^2} \delta(\omega(k) - k \cdot v).$$

Remarks

- The limit $\tau \to +\infty$ is a singular limit from different points of view:

  1. When $\tau \to \infty$, the autocorrelation matrix $\mathcal{R}_\tau$ is no nore integrable with respect to correlation time σ, but only locally integrable.

  This loss of integrability entails a loss of regularity in the velocity variables for the diffusion matrix. This loss of regularity in velocity variables is even more striking when we observe the singular limit (19) for a smooth resonance function R_τ.

  2. When $\tau \to \infty$, hypothesis $(\mathrm{H2})$ does not hold anymore. Indeed the stochastic electric field E^ε no longer satisfies a time decorrelation property since its decorrelation time tends to infinity. It is like falling back to the deterministic case.

  When $\tau \to \infty$, the autocorrelation time of particles tends to infinity and the time decorrelation of the stochastic electric field E^ε occurs at infinite time. This can be interpreted as follows. The electric field becomes deterministic and particles trajectories are almost straight lines. This seems consistent with the original deterministic derivation of the QL theory performed by physicists.

- Finally, we note that the RBT is actually a statistical (probabilistic) theory of the Vlasov equation and does not have a deterministic counterpart in the plasma physics literature. Nevertheless, for the deterministic case above, we have been able to introduce a finite autocorrelation time of particles τ, and to derive formally a diffusion matrix that is consistent with the quasilinear one in the limit $\tau \to \infty$.

Well-prepared initial data

- Our main theorem relies on well-prepared initial data, which roughly speaking are solutions of the linearized Vlasov–Poisson system around an unstable velocity profile.

- Initial distributions f_0^ε are of the form

$$f_0^\varepsilon(x,v) = F_0(v) + \varepsilon h_0(x,v), \quad x \in \mathbb{T}^d, \ v \in \mathbb{R}^d,$$

  where $\varepsilon > 0$ is a small real parameter, and the functions (F_0, h_0) satisfy the following constraints

$$\int_{\mathbb{R}^d} F_0(v)\, dv = 1, \quad \fint h_0\, dx = 0.$$

- $F_0 \geq 0$ is a stationary solution of the Vlasov–Poisson system, which is unstable, i.e. giving birth to an electrostatic instability when it is perturbed by the initial small perturbation εh_0.

- For the regularity of F_0, we suppose

$$F_0 \in \mathcal{A}_m^{r_0,\mu}, \quad r_0 > 0, \ \mu > d, \ m > d/2.$$

  where the space $\mathcal{A}_m^{r,\mu}$ is defined below.

- $\mathcal{A}_m^{r,\mu}$ is the space of analytic functions on the phase space $\mathbb{T}^d \times \mathbb{R}^d$, with radius of analyticity r, a Sobolev correction (weight) of order μ, and a velocity (moment) weight of order m.

Well-prepared initial data

- From the spectral analysis of the linearized Vlasov–Poisson system around a velocity profile F_0 satisfying the Penrose criterion for instability, the real wavenumber $k \in \mathbb{Z}_*^d$ and the complex frequency $\lambda = \gamma - i\omega$ (with $\gamma, \omega \in \mathbb{R}$) of an unstable electric wave satisfy the dispersion equation

$$\mathbb{D}_0(k, \lambda) = 0, \tag{20}$$

where the dielectric function $(k, \lambda) \mapsto \mathbb{D}_0(k, \lambda)$ is given by

$$\mathbb{D}_0(k, \lambda) := 1 - \frac{i}{|k|^2} \int_{\mathbb{R}^d} dv \, \frac{k \cdot \nabla F_0(v)}{\lambda + ik \cdot v}. \tag{21}$$

- It can be shown that the number of solutions, called roots or zeroes, to equation (20) with $\Re\mathfrak{e}\,\lambda > \delta_0 > 0$ is finite. Let N be this finite number of roots. Without loss of generality for our purpose, we suppose that these N roots are simple, i.e. with multiplicity one. Therefore we denote by

$$\{(k_{0n}, \lambda_{0n})\}_{n=1,\dots,N}, \text{ the simple roots of } \mathbb{D}_0(k, \lambda) = 0, \text{ such that } \Re\mathfrak{e}\,\lambda_{0n} > \delta_0 > 0. \tag{22}$$

- Moreover it can be shown that there exist two positive real numbers $\delta_0 < \Lambda_0 < \infty$ and $1 \leq \kappa_0 < \infty$, such that

$$0 < \delta_0 < \Re\mathfrak{e}\,\lambda_{0n} < \Lambda_0, \quad \text{and} \quad 1 \leq |k_{0n}| < \kappa_0, \quad \forall n \in \{1, \dots, N\}.$$

Well-prepared initial data

- From the dielectric function (21), if there exists $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_*^d$ such that the dispersion equation (20) is satisfied then
$$\mathbb{D}_0(-k, \bar{\lambda}) = 0.$$
The spectrum is symmetric with respect to the real axis and the number N is always even. Physically this corresponds to the fact that an unstable plasma wave (k, λ) propagates in similar way in both directions k and $-k$. Therefore there is $N/2$ simple roots $\{(k_{0n}, \lambda_{0n})\}_{n=1,\ldots,N/2}$ going hand-in-hand with their complex conjugate $\{(-k_{0n}, \bar{\lambda}_{0n})\}_{n=1,\ldots,N/2}$.

- We now precise the choice for h_0. Let $\mathbb{E}_{\lambda_{0n}}$ be the one-dimensional eigenspace associated with the eigenvalue λ_{0n}, i.e. a simple root of the set (22). We have
$$\mathbb{E}_{\lambda_{0n}} := \mathrm{Span}\left\{ \frac{k_{0n} \cdot \nabla F_0(v)}{\lambda_{0n} + ik_{0n} \cdot v} \exp(ik_{0n} \cdot x) \right\}.$$
We choose h_0 as an element of the space $\oplus_{n=1}^{N} \mathbb{E}_{\lambda_{0n}}$, i.e.
$$h_0 := \sum_{n=1}^{N} i\widehat{\Phi}_0(k_{0n}) \frac{k_{0n} \cdot \nabla F_0(v)}{\lambda_{0n} + ik_{0n} \cdot v} \exp(ik_{0n} \cdot x),$$
where the given complex-valued function $k \mapsto \widehat{\Phi}_0(k)$ is hermitian, i.e. $\overline{\widehat{\Phi}_0(k)} = \widehat{\Phi}_0(-k)$.

- h_0 enjoys the following regularity
$$h_0 \in \mathcal{A}_m^{\varrho_0, \mu}, \quad \varrho_0 \geq \min\left\{ \frac{\delta_0}{\kappa_0}, r_0^- \right\}, \quad \mu > d, \quad m > d/2.$$

Theorem (C)

- *Let* (d, m, N) *be some integers such that* $d \geq 1$, $m > d/2$ *and* $N \geq 1$.
- *Let* $\mu > d$ *be a positive real constant.*
- *Let* $(\delta_0, \Lambda_0, \kappa_0, r_0, \varrho_0, \varepsilon)$ *be some positive real constants such that*

$$0 < \delta_0 < \Lambda_0 < \infty, \quad 1 \leq \kappa_0 < \infty, \quad 0 < r_0 < \infty, \quad 0 < \varepsilon \ll 1, \quad \varrho_0 \geq \min\left\{\frac{\delta_0}{\kappa_0}, r_0^-\right\}.$$

- *Let* $f_0^\varepsilon := F_0 + \varepsilon h_0$ *be an initial data for the Vlasov–Poisson system posed on* $\mathbb{T}^d \times \mathbb{R}^d$, *such that* F_0 *and* h_0 *are designed as above. In particular we have* $f_0^\varepsilon \in \mathcal{A}_m^{\varrho_0,\mu}$, *and there exist*

$$\left\{(k_{0n}, \lambda_{0n}) \in (\mathbb{Z}_*^d \times \mathbb{C})\right\}_{n=1,\ldots,N}, \quad \textit{simple roots of} \quad \mathbb{D}_0(k,\lambda) = 0,$$
$$\textit{such that } 0 < \delta_0 < \Re\mathfrak{e}\, \lambda_{0n} < \Lambda_0, \quad \textit{and} \quad 1 \leq |k_{0n}| < \kappa_0, \quad \forall n \in \{1, \ldots, N\},$$

where

$$\mathbb{D}_0(k,\lambda) := 1 - \frac{i}{|k|^2} \int_{\mathbb{R}^d} dv\, \frac{k \cdot \nabla F_0(v)}{\lambda + ik \cdot v}.$$

- *Recall that, by well-known results, there exists a time* $T_0 > 0$ *such that for any time* T, *with* $0 < T < T_0$, *there exists a unique solution* $f^\varepsilon(t) \in \mathcal{A}_m^{\varrho(t),\mu}$ *to the Vlasov–Poisson system for all* $t \in [0, T]$, *with* $\varrho(0) = \varrho_0$.

We define $F^\varepsilon = F^\varepsilon(t, v)$ *by*

$$F^\varepsilon(t,v) := \fint dx\, f^\varepsilon(t,x,v).$$

Theorem (C)

Then,

(*i*) *There exist a time* $T_1 > 0$ *such that* $T_1 < T_0$, *and* N *distinct complex-valued functions* $t \mapsto \lambda_n(t)$, *analytic on* $[0, T_1]$ *with* $\lambda_n(0) = \lambda_{0n}$, *and such that*

$$\lambda_n(t) = \gamma_n(t) - i\omega_n(t), \quad \omega_n \in \mathbb{R}, \quad 0 < \delta_0 < \gamma_n = \Re\mathfrak{e}\, \lambda_n < \Lambda_0, \quad \forall n \in \{1, \dots, N\}.$$

Moreover these N *distinct functions* $\lambda_n \in \mathcal{A}([0, T_1]; \mathbb{C})$ *are simple roots of the following dielectric function,*

$$\mathbb{D}_t(k, \lambda) := 1 - \frac{i}{|k|^2} \int_{\mathbb{R}^d} dv \, \frac{k \cdot \nabla_v F^\varepsilon(t, v)}{\lambda + ik \cdot v},$$

namely

$$\mathbb{D}_t(k_{0n}, \lambda_n(t)) = 0, \quad \forall t \in [0, T_1], \quad \forall n \in \{1, \dots, N\}.$$

Theorem (C)

(ii) *Let* s_0 *be a positive real constant such that* $0 < s_0 \le \varrho_1$ *with* $\varrho_1 \ge \min\{r_1^-, \delta_0/\kappa_0\}$, *and* $r_1 := \inf_{t\in[0,T_1]} \varrho(t) > 0$.

For any $s \in (0, s_0)$ *there exist a small parameter* $\varepsilon = \varepsilon(s) > 0$ *and a time* $T_* = T_*(s) > 0$ *such that for all* $t \in [0, T_*]$, *the function* $F^\varepsilon \in L^\infty([0, T_*]; \mathcal{A}_m^{\varrho_{T_*},\mu})$ *with* $\varrho_{T_*} = \inf_{t\in[0,T_*]} \varrho(t) > 0$, *satisfies, in the classical sense, the following diffusion equation with remainder,*

$$\partial_t F^\varepsilon(t,v) - \nabla_v \cdot \Big(\mathscr{D}(t,v)\nabla_v F^\varepsilon(t,v)\Big) = \mathcal{R}(t,x,v),$$

where the positive-definite-symmetric diffusion matrix $\mathscr{D}$, *of order* $\mathcal{O}(\varepsilon^2)$, *is given by*

$$\mathscr{D}(t,v) \;=\; 2\varepsilon^2 \sum_{n=1}^{N} \frac{\left|\widehat{\Phi}_0(k_{0n})\right|^2 \gamma_n(t)\, e^{2\int_0^t \gamma_n(\tau)d\tau}}{\gamma_n^2(t) + (\omega_n(t) - k_{0n}\cdot v)^2}\, k_{0n} \otimes k_{0n},$$

and where the remainder $\mathcal{R}(t,x,v)$ *satisfies the following estimate,*

$$\sup_{t\in[0,T_*]} \|\mathcal{R}(t)\|_{\mathcal{A}_m^{s,\mu}} = \mathcal{O}(\varepsilon^3), \quad 0 < s < s_0.$$

More precisely, the time T_* *is such that* $0 < T_* \le \min\{(s_0 - s)/\nu, T_1\}$, *where* ν *is a positive real constant depending on* T_1, ε, *and* $F^\varepsilon \in L^\infty([0, T_1]; \mathcal{A}_m^{s_0,\mu})$.

Proof. **The proof is based on the same spirit as Nash–Moser-type implicit-function problems.**

THANK YOU FOR YOUR ATTENTION