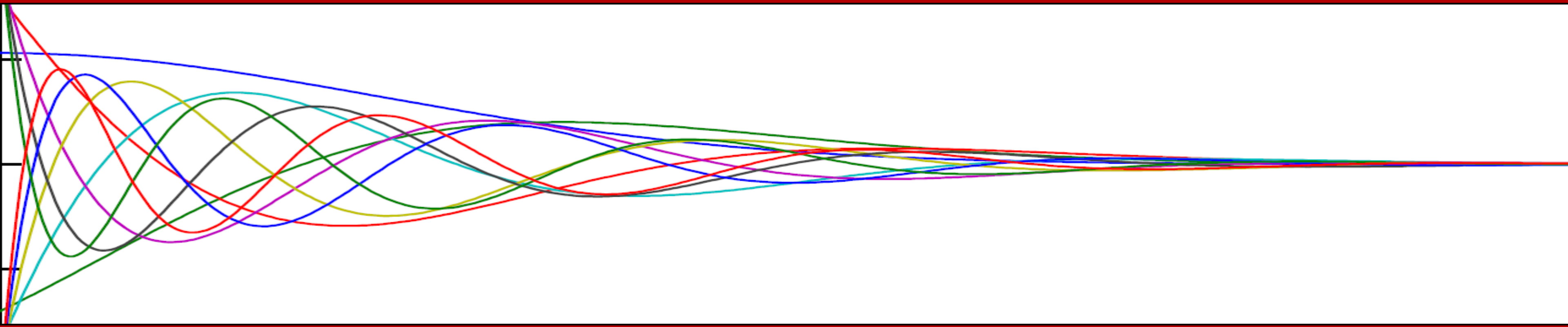
Computation of pedestal and stellarator neoclassical effects using a new spectral speed grid



Matt Landreman, MIT PSFC

Thanks to Michael Barnes, Peter Catto, Darin Ernst, Felix Parra, Istvan Pusztai

First part of work: J Comp Phys (2013) http://dx.doi.org/10.1016/j.jcp.2013.02.041

Outline

- New spectral discretization scheme for v or $v_\perp$.
- Application 1: Pedestal global Fokker-Planck code.
- Application 2: Stellarator Fokker-Planck code.

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Laguerre: $\int_0^\infty dy\, L_i^m(y) L_j^m(y)\, y^m e^{-y} \propto \delta_{i,j}$ $\quad f = \sum_{j=0}^{n} \hat{f}_j\, L_j^m\left(2\varepsilon / \upsilon_{th}^2\right) e^{-2\varepsilon/\upsilon_{th}^2}$

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New polynomials: $\int_0^\infty d\upsilon\, P_i(x) P_j(x) e^{-x^2} \propto \delta_{i,j}$ $\qquad f = \sum_{j=0}^{n} \hat{f}_j\, P_j(\upsilon / \upsilon_{th})\, e^{-(\upsilon/\upsilon_{th})^2}$

These new non-standard polynomials lead to an integration and differentiation scheme.

$$f = \sum_{j=0}^{n} \hat{f}_j \, P_j(x) \, e^{-x^2}$$

$$x = \upsilon / \sqrt{2T/m}$$

$$\int_0^\infty dx \, P_i(x) P_j(x) e^{-x^2} \propto \delta_{ij}$$

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\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x - \frac{1}{\sqrt{\pi}} \\
P_2(x) &= x^2 - \frac{\sqrt{\pi}}{\pi - 2} x + \frac{4 - \pi}{2(\pi - 2)} \\
P_3(x) &= x^3 - \frac{3\pi - 8}{2\sqrt{\pi}(\pi - 3)} x^2 + \frac{10 - 3\pi}{2(\pi - 3)} x - \frac{16 - 5\pi}{4\sqrt{\pi}(\pi - 3)}
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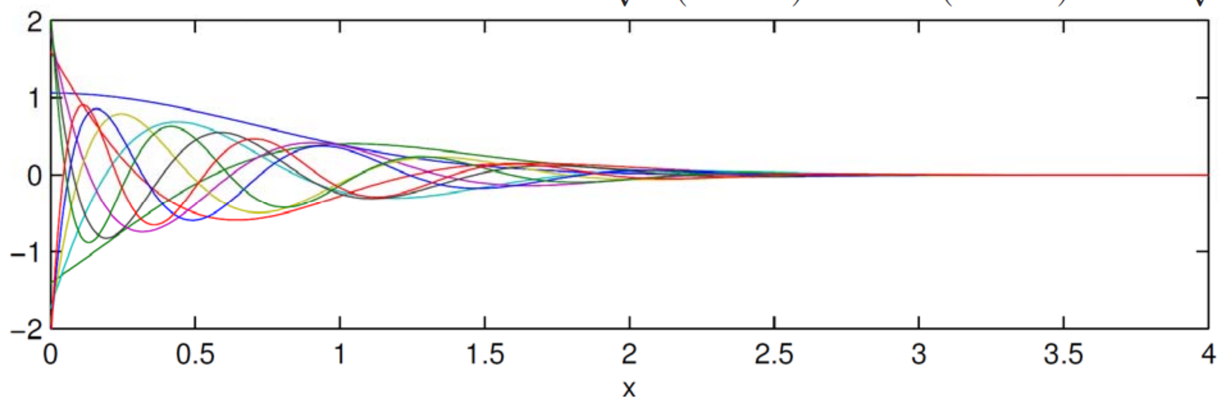
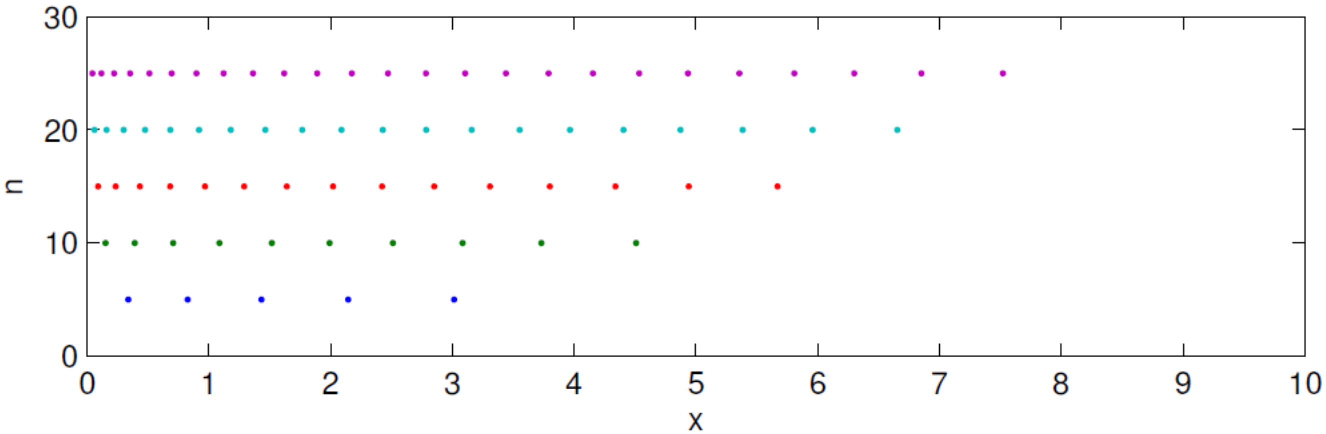
First 10 modes:



Locations of zeros: (can scale to smaller x)



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- I use collocation method, but could also use a modal approach.
- Grid points at polynomial zeros.
- Can add a point at x=0 if desired.

- Gaussian integration
- Spectral differentiation: *Weideman & Reddy, ACM Trans. Math. Software 26, 465 (2000).*

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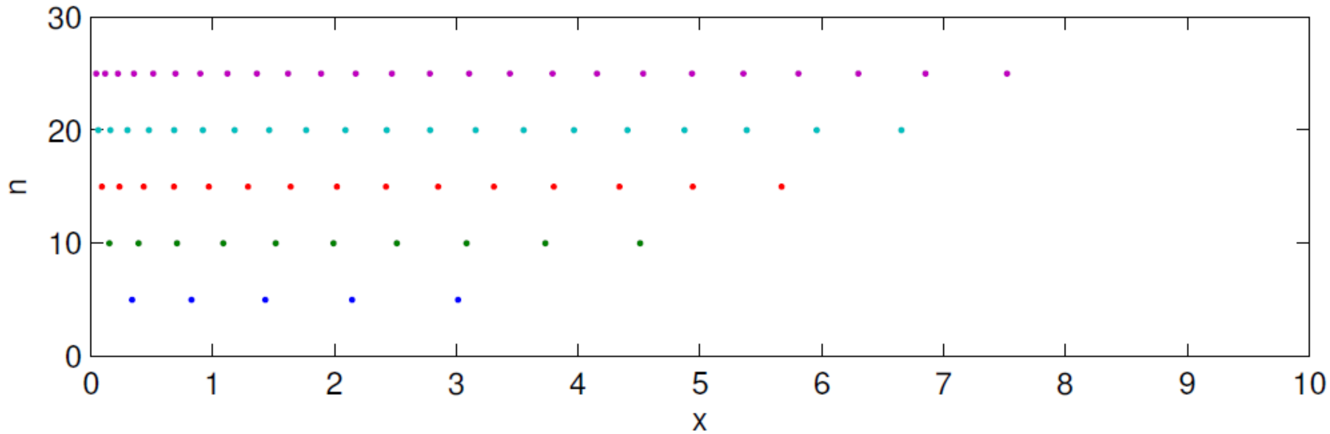
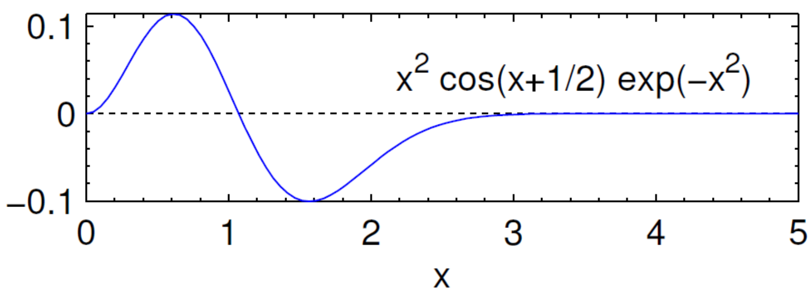
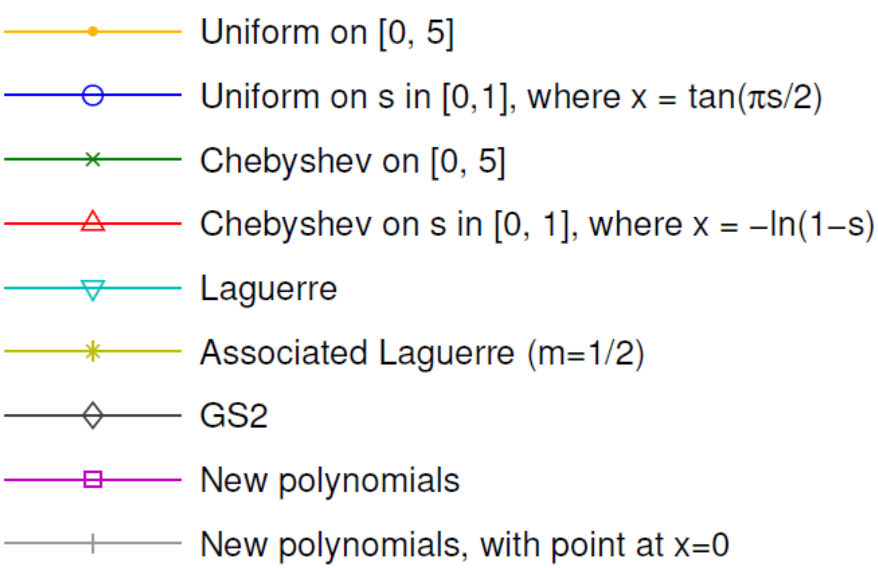
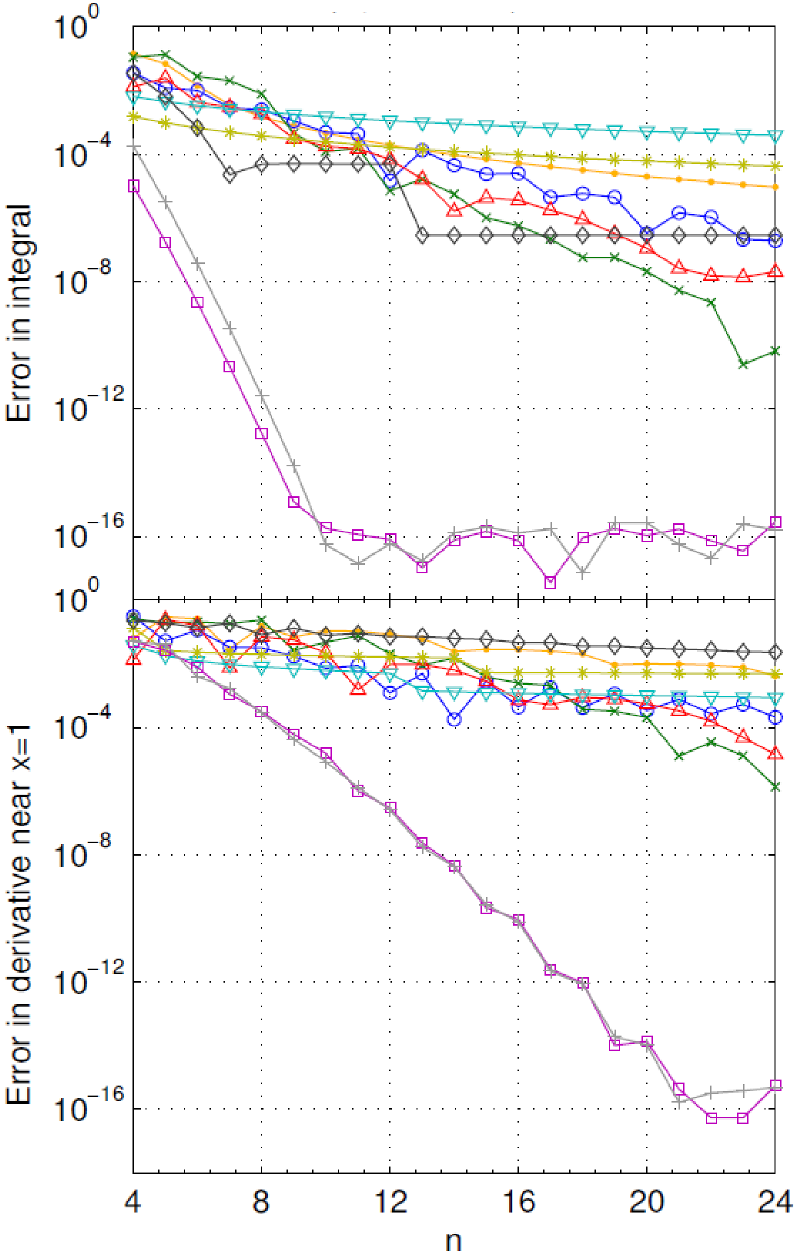


New scheme outperforms others at both integration and differentiation





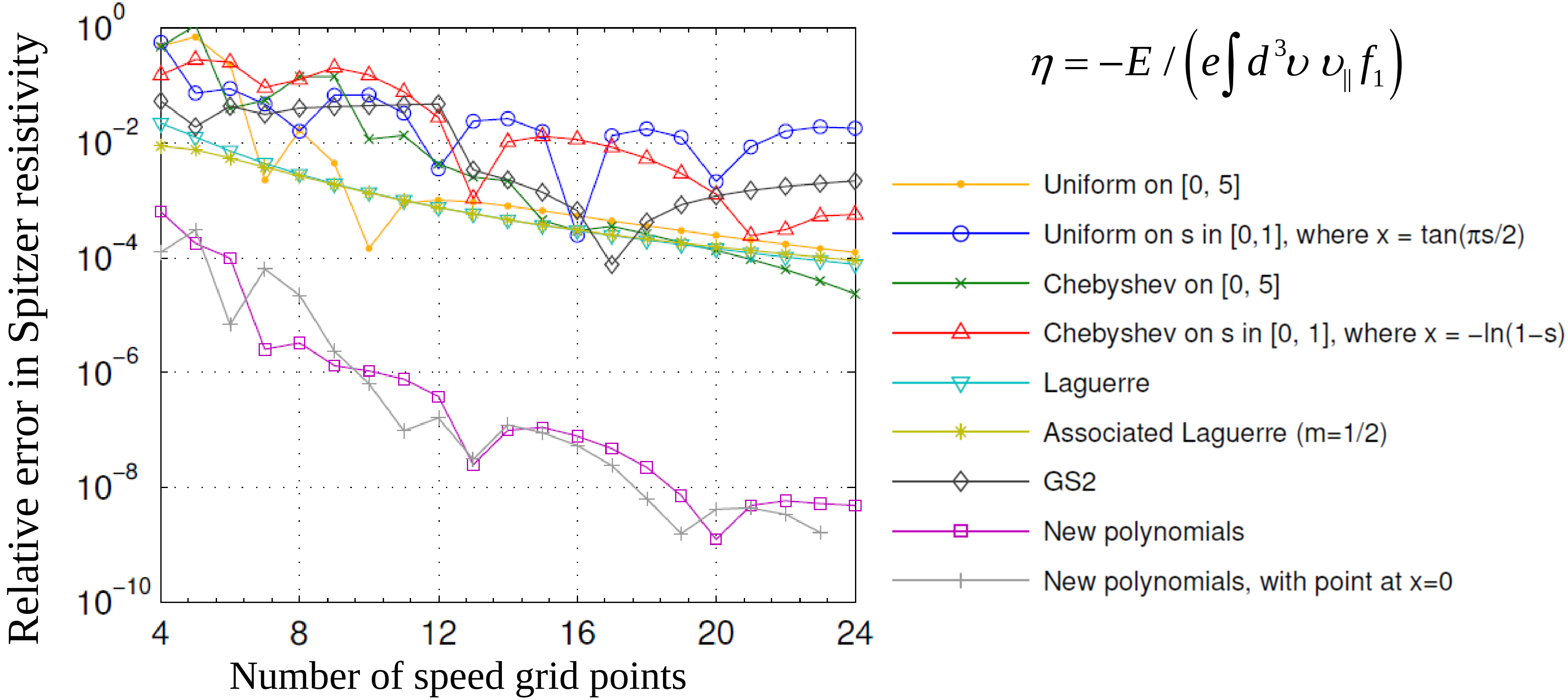


New scheme outperforms others on some physics applications

1D problem: Spitzer resistivity

$$C\{f_1\} = \frac{eE}{T} \upsilon_{\parallel} f_M$$

$$\eta = -E / \left(e \int d^3\upsilon \, \upsilon_{\parallel} f_1 \right)$$



New spectral scheme may or may not work well for your problem

- Pros:
  - Spectrally accurate integration and differentiation.
  - Very small # of points needed.
  - Can be exactly conservative: $\int_0^\infty d\upsilon \frac{\partial A}{\partial \upsilon} = -A(0)$
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- Cons:
  - Differentiation matrix is dense (though diagonal is a great preconditioner for Krylov solvers.)
  - So far, seems unstable for time-dependent problems, even with implicit time-advance!

Application 2: stellarator Fokker-Planck code SFINCS

Stellarator Fokker-Planck Iterative Neoclassical Conservative Solver

$$\left(\upsilon_{\|}\mathbf{b}+\mathbf{v}_E\right)\cdot\nabla f_1+\dot{\upsilon}\frac{\partial f_1}{\partial\upsilon}+\dot{\xi}\frac{\partial f_1}{\partial\xi}-C_{FP}\left\{f_1\right\}=-\mathbf{v}_m\cdot\nabla\psi\frac{\partial f_M}{\partial\psi}$$

$$f_1=f_1\left(\theta,\zeta,\upsilon,\xi\right)$$

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$$\begin{pmatrix} \langle \Gamma \cdot \nabla \psi \rangle \\ \langle \mathbf{q} \cdot \nabla \psi \rangle \\ \langle V_{\|} B \rangle \end{pmatrix} = \underbrace{\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}}_{\text{Transport matrix}} \begin{pmatrix} \frac{d \ln p}{d\psi} + \frac{e}{T} \frac{d\Phi}{d\psi} + \frac{d \ln T}{d\psi} \\ \frac{d \ln T}{d\psi} \\ \langle E_{\|} B \rangle \end{pmatrix}$$

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For ion neoclassical physics in LHD, momentum-conserving model collision operator compares well to full Fokker-Planck operator.

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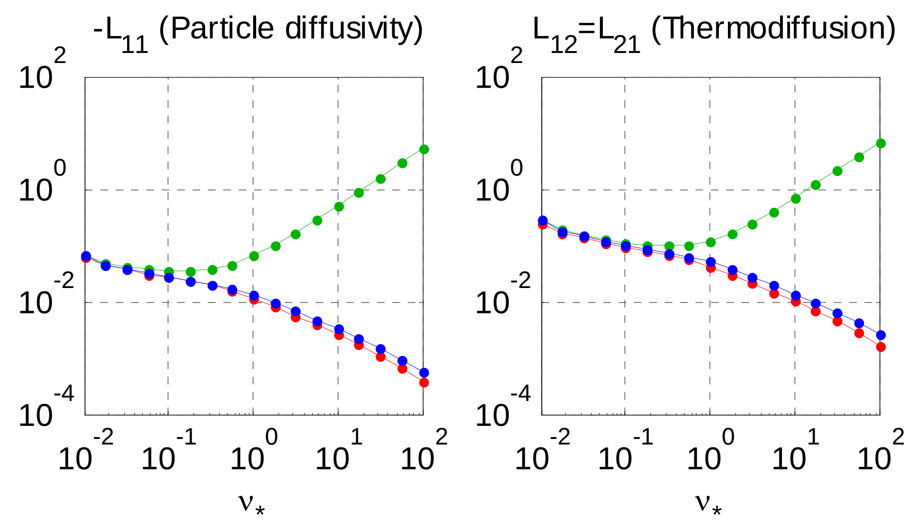
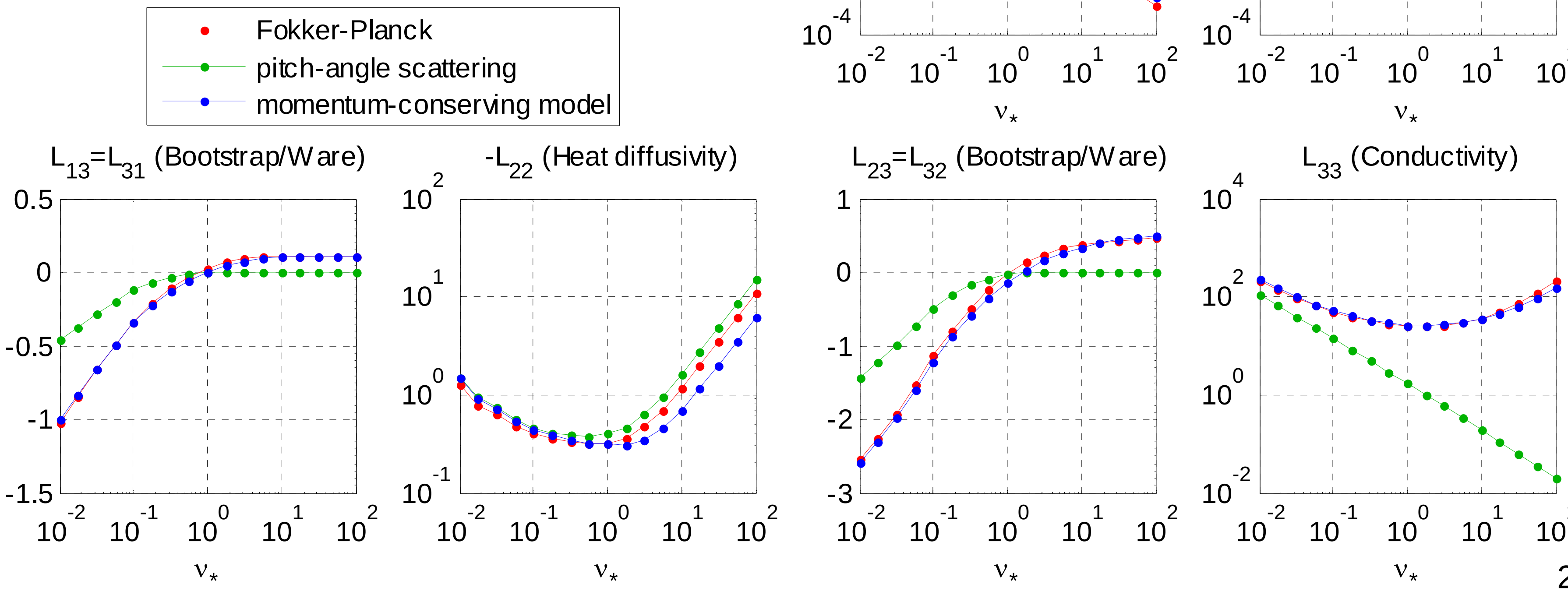


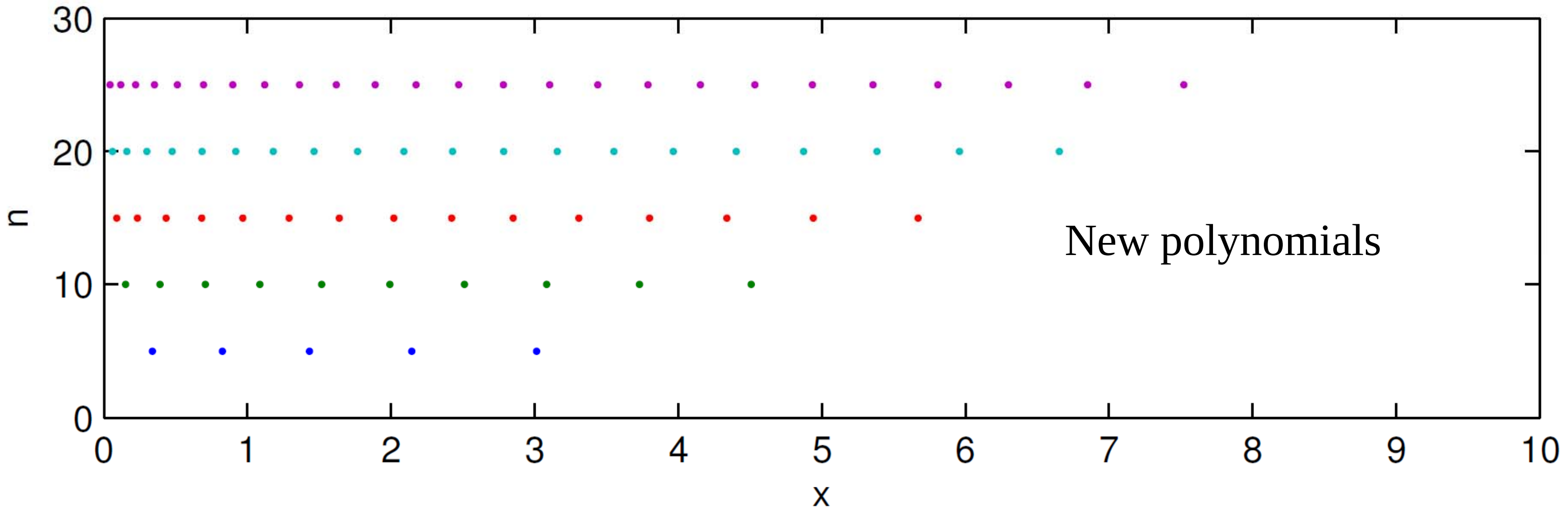


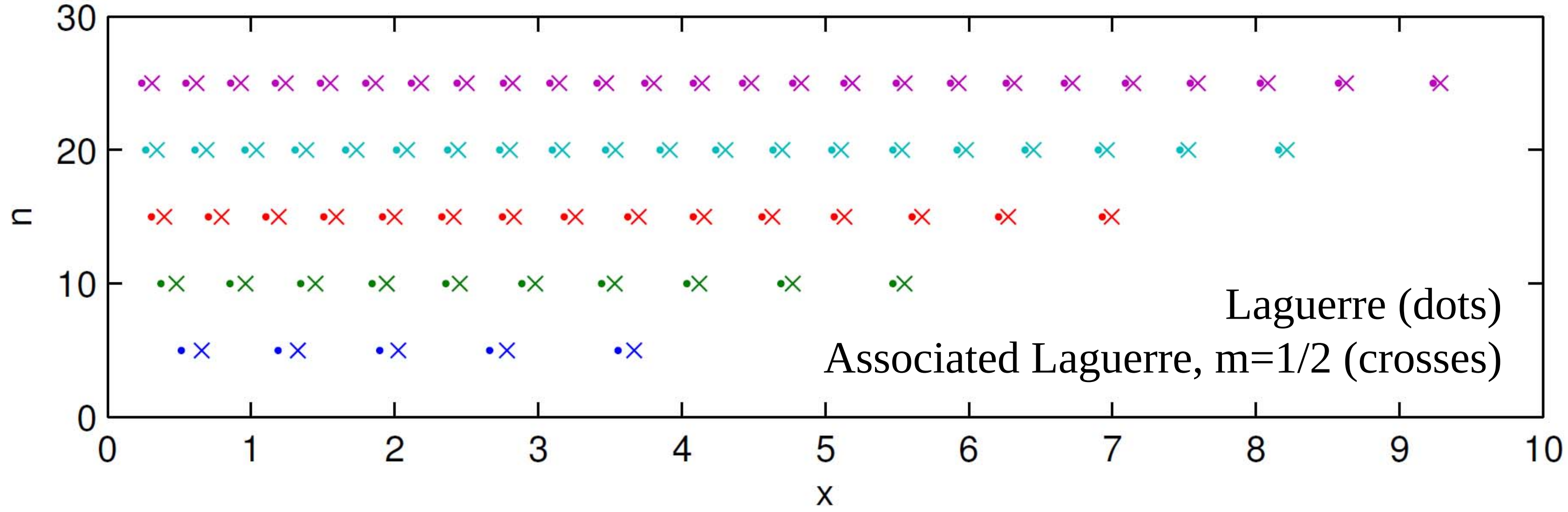
Summary

- New spectral discretization scheme for v or $v_\perp$ gives rapid convergence with # of grid points.
  - Very useful for the time-independent collisional problems I've considered. Other applications?
  - Matlab & fortran source code for generating grid, integration weights, & differentiation matrices available at http://web.mit.edu/landrema/www/software/
  - J Comp Phys (2013) http://dx.doi.org/10.1016/j.jcp.2013.02.041
- Scheme implemented in global Fokker-Planck code for tokamak pedestals.
  - Strong poloidal asymmetries arise in flow.
- Scheme implemented in stellarator Fokker-Planck code.

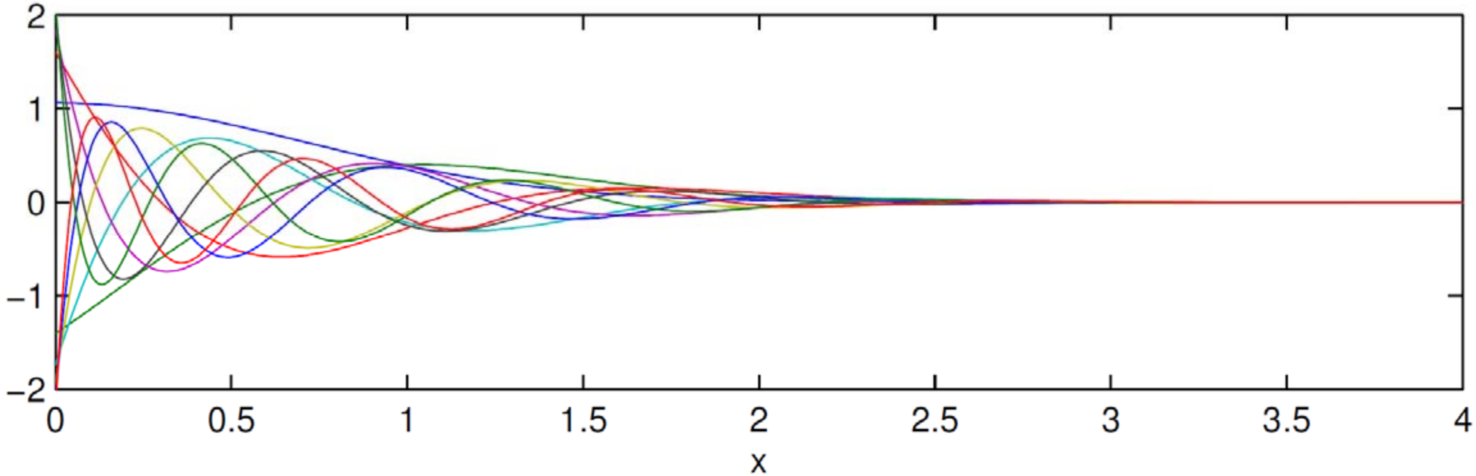
Extra slides

Zeros of polynomials = Grids for Gaussian integration





First 10 new polynomial modes: $P_j(x)e^{-x^2}$



First 10 Laguerre polynomial modes: $L_j(x^2)e^{-x^2}$