Results From and Plans for the DNA Code

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Outline

- Motivation – Hermite Polynomials
- Model – reduced gyrokinetics
  - Energetics
  - DNA code
- Hermite spectra
- Studying Damped Eigenmodes
  - Linear Spectra
  - Pseudo-spectra
  - Nonlinear spectra

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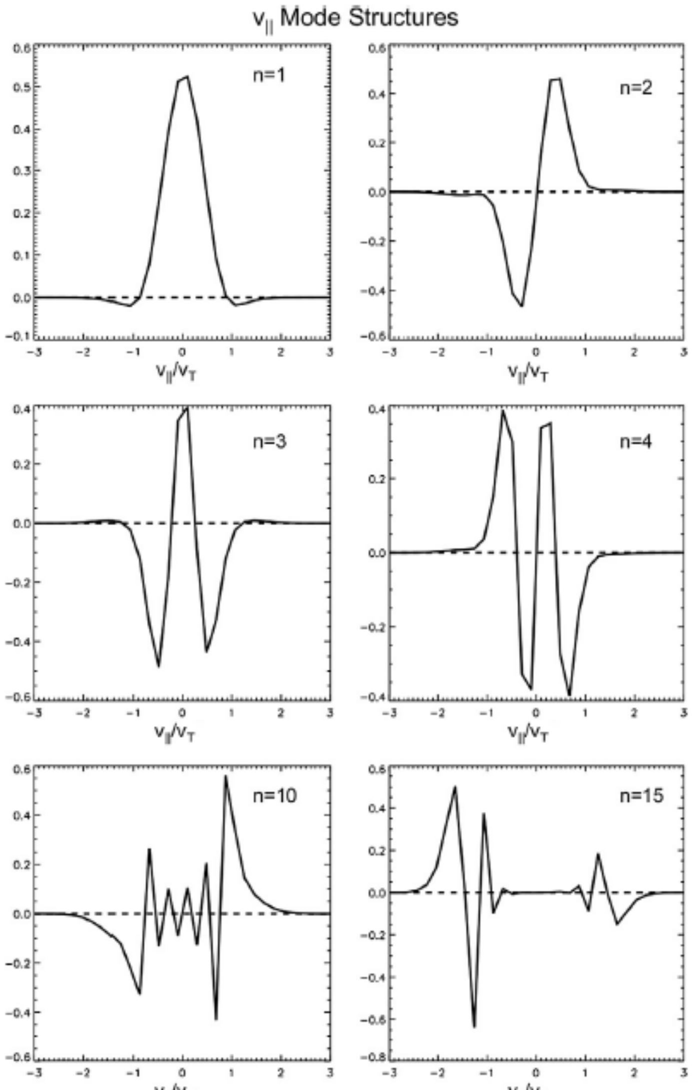
Hermite representation in GK



Higher order singular value decomposition (HOSVD) applied to 6D gyrokinetics (5D plus time)—extracts mode structures for each coordinate.

➔Hermite polynomials—natural representation of parallel velocity in GK

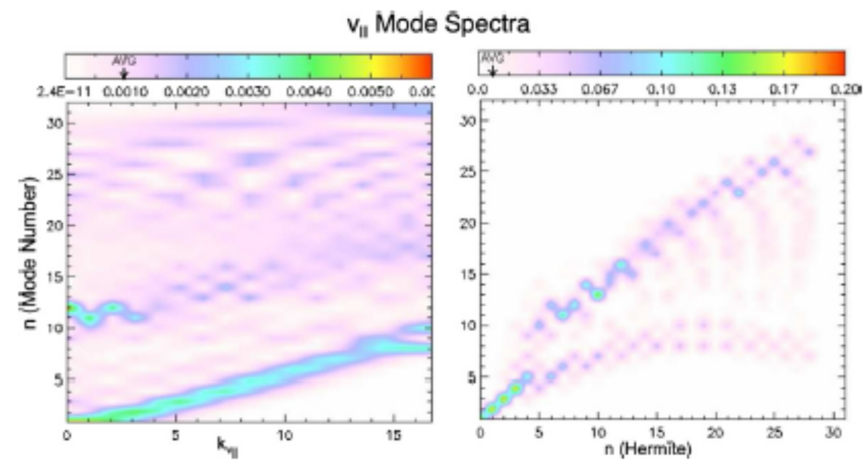


Fig. 9. Fourier (A) and Hermite (B) spectra of the HOSVD $v_{||}$-modes.

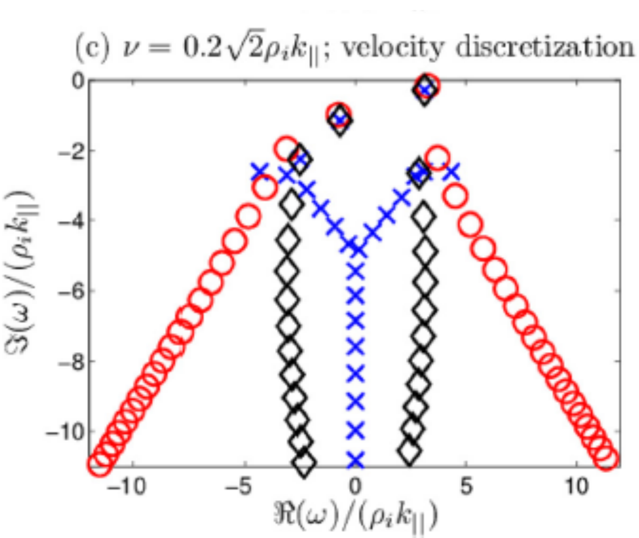
Hatch et al. JCP '12

Hermites-Effective for resolving linear eigenmodes

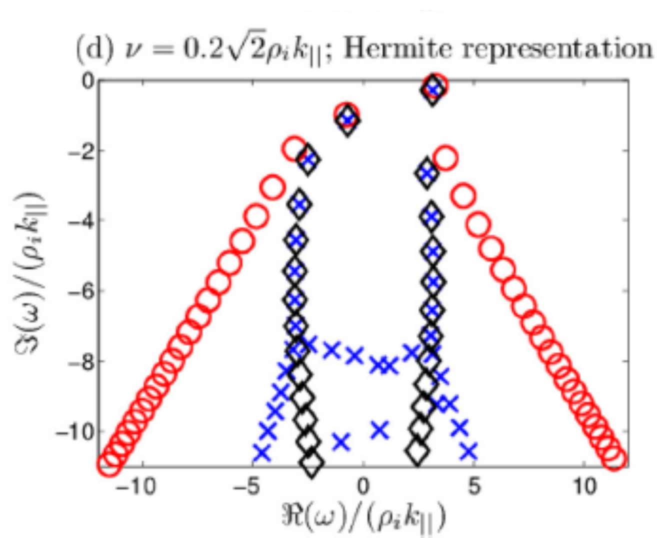
Collisionless Landau roots (analytical)

Collisional Landau roots (analytical)

Numerical Landau roots



Finite Differences



Hermite Polynomials

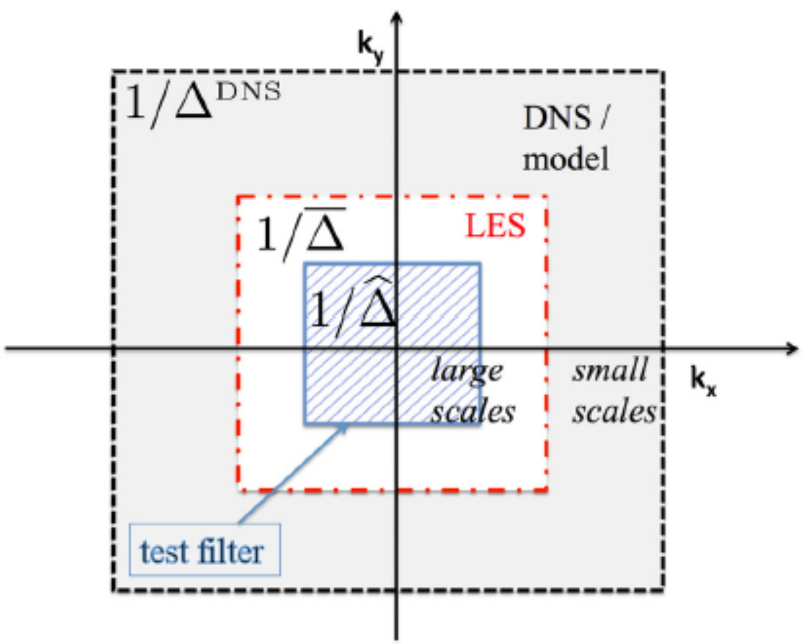
Bratanov et al. PoP ’13
(see also Skiff et al. PRL ’98, Ng et al. PRL ’99—also use Hermites)

Spectral Representation in v-space

Spectra v-space: possible application of LES* techniques.

In k-space separately for different Hermite n

Or in Hermite space.



*P. Morel, A. Bañón Navarro, M. Albrecht-Marc, D. Carati, F. Merz, T. Görler, and F. Jenko, Physics of Plasmas 19, 012311 (2012)

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Model--reduced gyrokinetics

Model (Very similar to Watanabe PoP '04):

μ -averaged gyrokinetic equations*

FLR effects: gyroaverage➔ $e^{-\boldsymbol{k}_\perp^2/2}$

Slab geometry

Adiabatic electrons

Hermite representation:

$$f(v) = \sum_{n=0}^{\infty} \hat{f}_n H_n(v) e^{-v^2}$$

$$H_n(x) \equiv \frac{(-1)^n e^{x^2}}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}} \frac{d^n}{dx^n} e^{-x^2}$$

Final Equations:

$$\begin{aligned}\frac{\partial \hat{f}_n}{\partial t} = & \frac{\eta_i i k_y}{\pi^{\frac{1}{4}}} \frac{k_\perp^2}{2} \bar{\phi} \delta_{n,0} - \frac{i k_y}{\pi^{\frac{1}{4}}} \bar{\phi} \delta_{n,0} - \frac{\eta_i i k_y}{\sqrt{2}\pi^{\frac{1}{4}}} \bar{\phi} \delta_{n,2} \\ & - \frac{i k_z}{\pi^{\frac{1}{4}}} \bar{\phi} \delta_{n,1} - i k_z \left(\sqrt{n} \hat{f}_{n-1} + \sqrt{n+1} \hat{f}_{n+1} \right) \\ & - \nu n \hat{f}_n + \sum_{\boldsymbol{k}'} \left(k_x' k_y - k_x k_y' \right) \bar{\phi}_{\boldsymbol{k}'} \hat{f}_{n,\boldsymbol{k}-\boldsymbol{k}'} .\end{aligned}$$

$$\phi_{k_x,k_y} = \frac{\pi^{\frac{1}{4}} e^{-k_\perp^2/2} \hat{f}_0 + \Delta\tau \langle \phi \rangle_{FS}}{\tau + 1 - \Gamma_0(k_\perp^2)}$$

$$\langle \phi \rangle_{FS} = \frac{\pi^{\frac{1}{4}} e^{-k_\perp^2/2} \hat{f}_0 \delta_{k_y,0} \delta_{k_z,0}}{[1 - \Gamma_0(k_x^2)]}$$

*planned extension to 5D

Free Energy

Free energy equations:

$$\varepsilon_{\boldsymbol{k},n} = \varepsilon^{(f)}_{\boldsymbol{k},n} + \varepsilon^{(\phi)}_{\boldsymbol{k}}\delta_{n,0}$$

$$\varepsilon^{(f)}_{\boldsymbol{k},n} \equiv \frac{1}{2}\pi^{\frac{1}{2}}|\hat{f}_{\boldsymbol{k},n}|^2$$

$$\varepsilon^{(\phi)}_{\boldsymbol{k}} \equiv \frac{1}{2D(k_\perp)}|\phi_{\boldsymbol{k}}|^2$$

Energy Evolution

Electrostatic part:

$$\frac{\partial \varepsilon_{\boldsymbol{k}}^{(\phi)}}{\partial t} = J_{\boldsymbol{k}}^{(\phi)} + N_{\boldsymbol{k}}^{(\phi)}$$

$$J_{\boldsymbol{k}}^{(\phi)} \equiv \Re\left[-ik_z \pi^{\frac{1}{4}} \bar{\phi}^* \hat{f}_{\boldsymbol{k},1}\right]$$

$$N_{\boldsymbol{k}}^{(\phi)} \equiv \Re\left[\sum_{\boldsymbol{k}'} T_{\boldsymbol{k},\boldsymbol{k}'}^{(\phi)}\right] \qquad T_{\boldsymbol{k},\boldsymbol{k}'}^{(\phi)} = \pi^{1/4}(k_x' k_y - k_x k_y')\bar{\phi}_{\boldsymbol{k}}^* \bar{\phi}_{\boldsymbol{k}'} \hat{f}_{0,\boldsymbol{k}-\boldsymbol{k}'}$$

Entropy part:

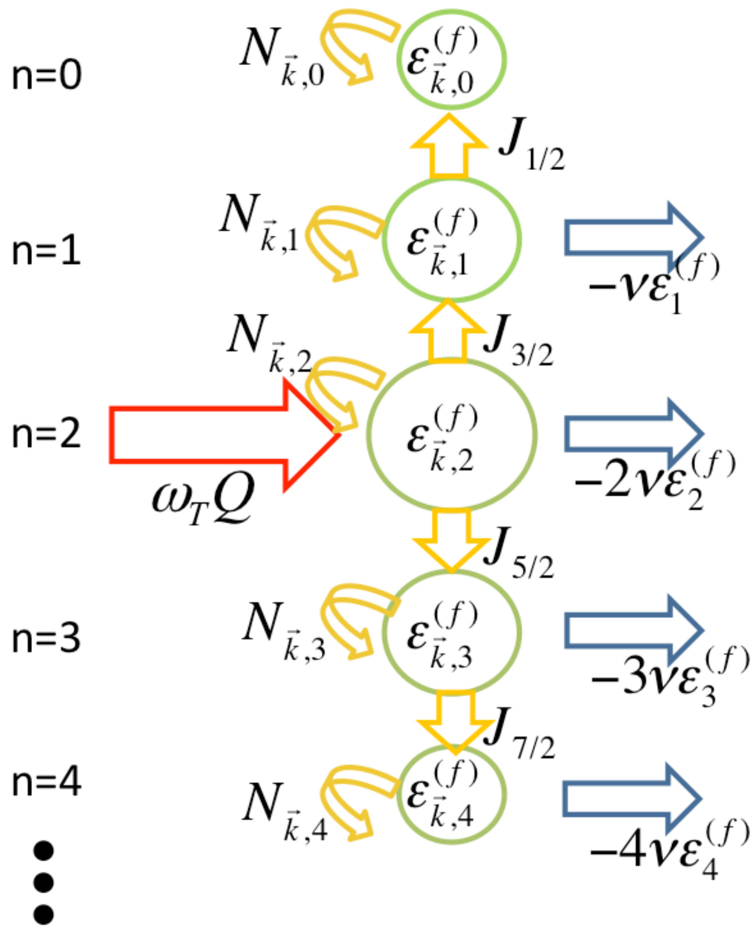
$$\frac{\partial \varepsilon_{\boldsymbol{k},n}^{(f)}}{\partial t} = \eta_i Q_{\boldsymbol{k}} \delta_{n,2} - \nu n \varepsilon_{\boldsymbol{k},n}^{(f)} - J_{\boldsymbol{k}}^{(\phi)} \delta_{n,1} + J_{\boldsymbol{k},n-1/2} - J_{\boldsymbol{k},n+1/2} + N_{\boldsymbol{k}}^{(f)}$$

$$\eta_i Q = \Re\left[-\frac{\pi^{1/4}}{\sqrt{2}} \eta_i i k_y e^{-k_\perp^2/2} \hat{f}_2^* \phi\right]$$

$$J_{\boldsymbol{k},n-1/2} \equiv \Re\left[-\pi^{\frac{1}{2}} i k_z \sqrt{n} \hat{f}_{\boldsymbol{k},n}^* \hat{f}_{\boldsymbol{k},n-1}\right] \qquad J_{\boldsymbol{k},n+1/2} \equiv \Re\left[\pi^{\frac{1}{2}} i k_z \sqrt{n+1} \hat{f}_{\boldsymbol{k},n}^* \hat{f}_{\boldsymbol{k},n+1}\right]$$

$$N_{\boldsymbol{k},n}^{(f)} \equiv \Re\left[\sum_{\boldsymbol{k}'} T_{\boldsymbol{k},\boldsymbol{k}',n}^{(f)}\right] \qquad T_{\boldsymbol{k},\boldsymbol{k}',n}^{(f)} = -\pi^{1/2}(k_x' k_y - k_x k_y')\hat{f}_{\boldsymbol{k},n}^* \bar{\phi}_{\boldsymbol{k}-\boldsymbol{k}'} \hat{f}_{\boldsymbol{k}',n}$$

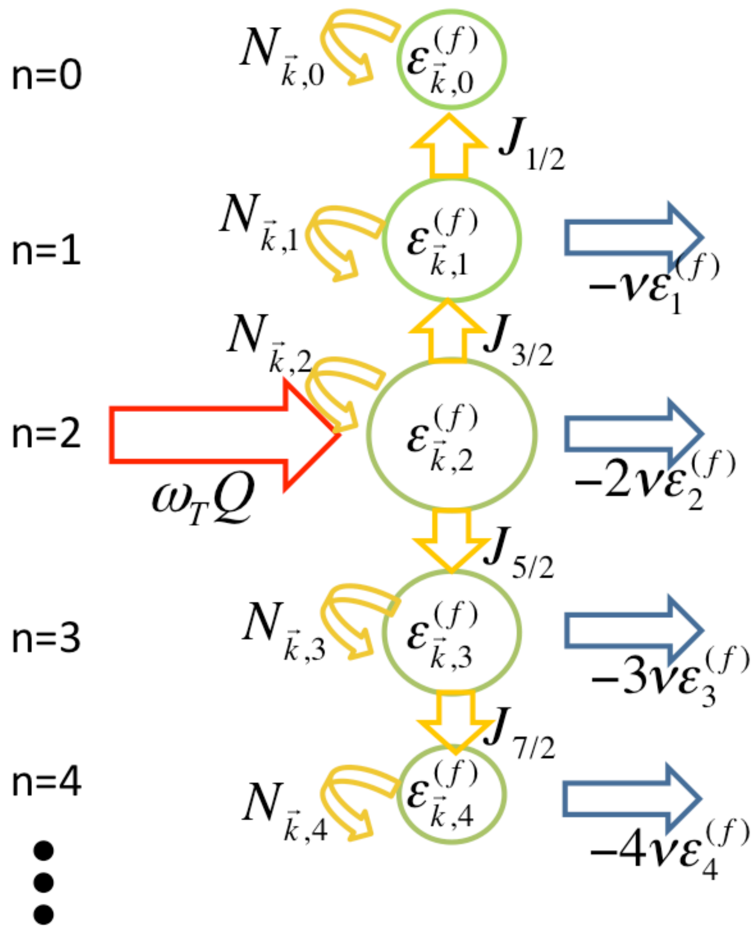
Schematic Energy Transfer



Drive:
Proportional to heat flux
Limited to n=2

$$\boxed{\eta_i Q = \Re\left[-\frac{\pi^{1/4}}{\sqrt{2}}\eta_i i k_y e^{-k_\perp^2/2}\hat{f}_2^*\phi\right]}$$

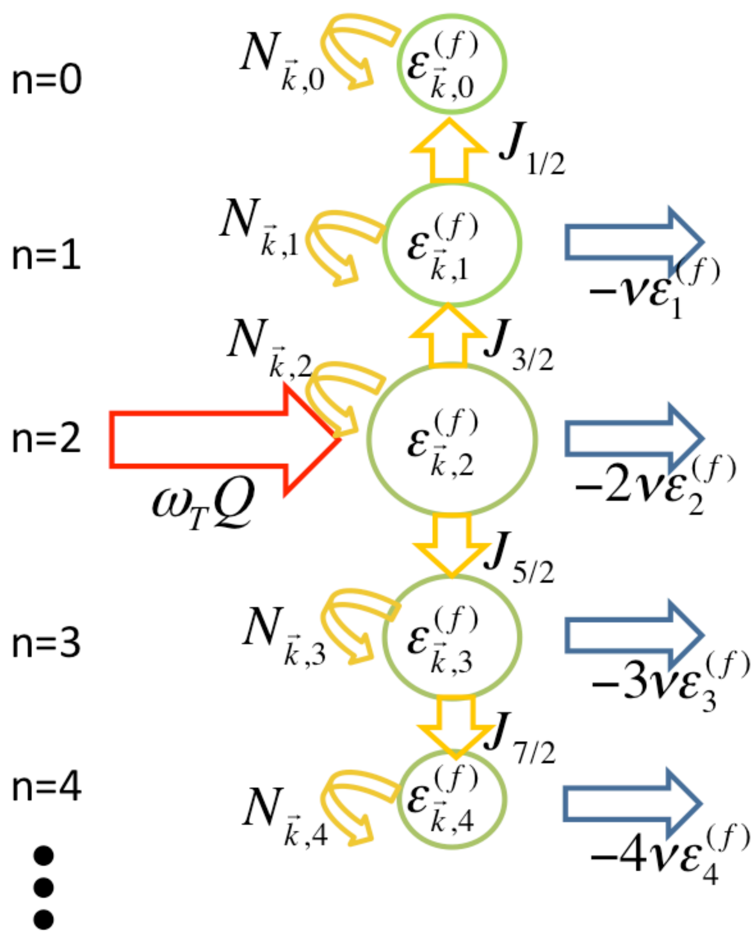
Schematic Energy Transfer



Dissipation:
Collisional dissipation

$$\boxed{-\nu n \varepsilon^{(f)}_{\boldsymbol{k},n}}$$

Schematic Energy Transfer

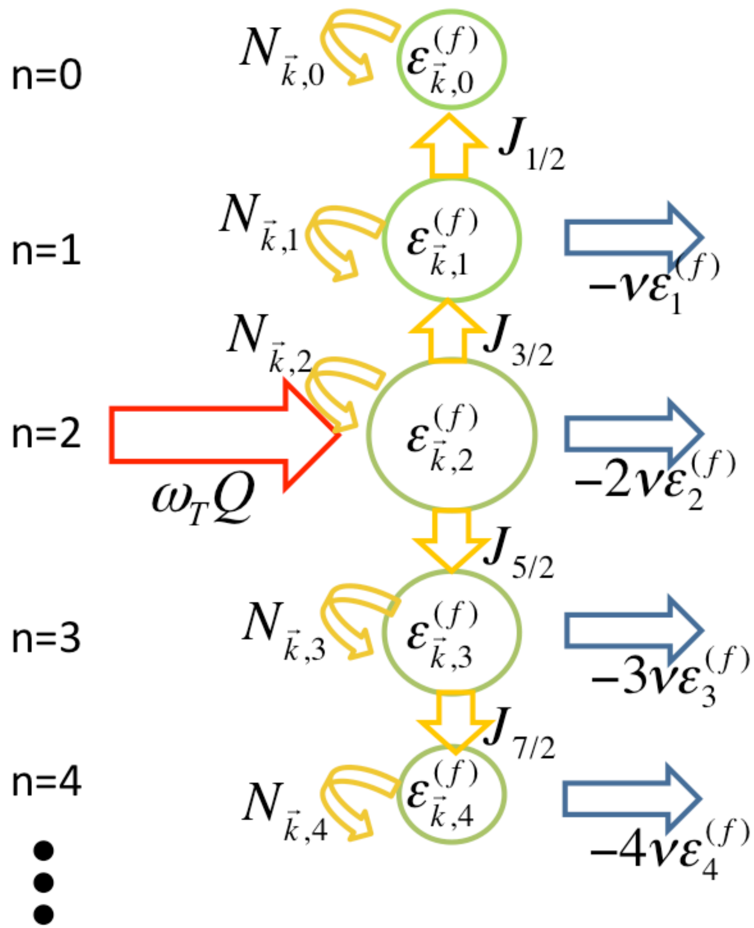


Nonlinear energy transfer:
Conservative transfer
Confined to each Hermite

$$\boxed{N_{\boldsymbol{k},n}^{(f)} \equiv \Re\left[\sum_{\boldsymbol{k}'} T_{\boldsymbol{k},\boldsymbol{k}',n}^{(f)}\right]}$$

$$\boxed{T_{\boldsymbol{k},\boldsymbol{k}',n}^{(f)} = -\pi^{1/2}(k_x' k_y - k_x k_y')\hat{f}_{\boldsymbol{k},n}^{*}\bar{\phi}_{\boldsymbol{k}-\boldsymbol{k}'}\hat{f}_{\boldsymbol{k}',n}}$$

Schematic Energy Transfer



Landau damping:
Transfer between $\varepsilon_{\boldsymbol{k}}^{(\phi)}$ and $\varepsilon_{\boldsymbol{k},n}^{(f)}$

Phase mixing:
Conservative linear local cascade in Hermite space

$$J_{\boldsymbol{k},n+1/2} \equiv \Re\left[\pi^{\frac{1}{2}} i k_z \sqrt{n+1}\hat{f}^*_{\boldsymbol{k},n}\hat{f}_{\boldsymbol{k},n+1}\right]$$

$$J_{\boldsymbol{k},n-1/2} \equiv \Re\left[-\pi^{\frac{1}{2}} i k_z \sqrt{n}\hat{f}^*_{\boldsymbol{k},n}\hat{f}_{\boldsymbol{k},n-1}\right]$$

DNA Code

DNA code:
Direct Numerical Analysis
of Fundamental Gyrokinetic Turbulence Dynamics

Fully Spectral:
Fourier in three spatial dimensions
Hermite in parallel velocity

Developed from scratch, but much of the structure and many algorithms inspired and informed by GENE.

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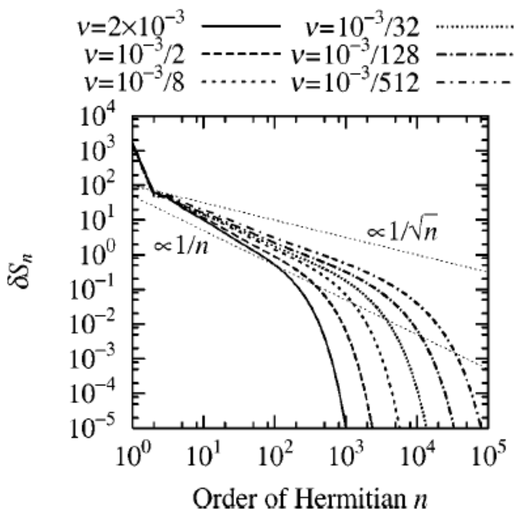
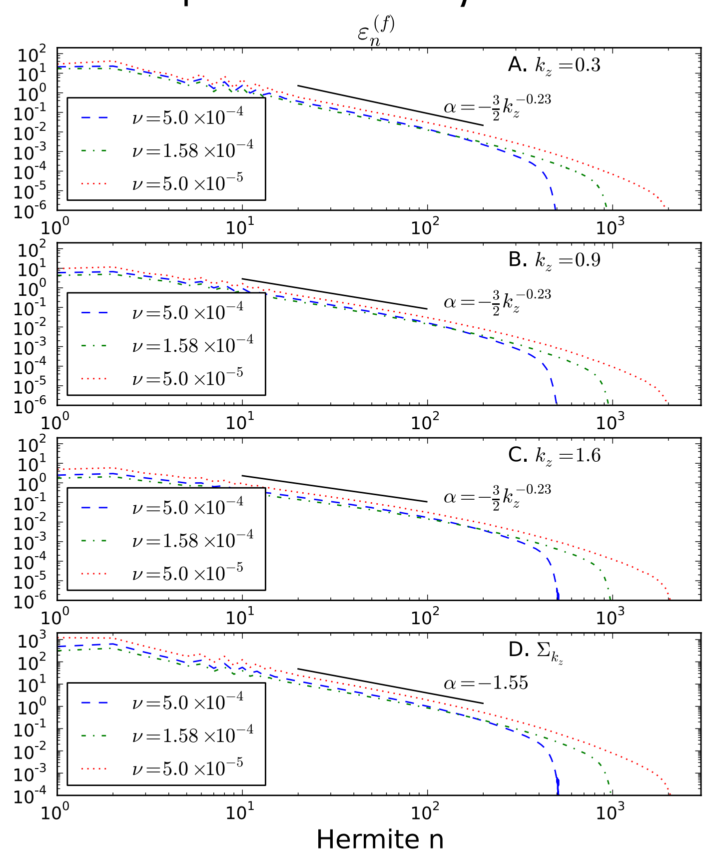
Hermite Spectra

Watanabe, Sugama '04: Collisionality-dependent Hermite spectra (note: model has kz=fixed).



DNA: kz-dependent Hermite spectra (summed over kx,ky)—independent of (or weakly dependent on) collisionality.



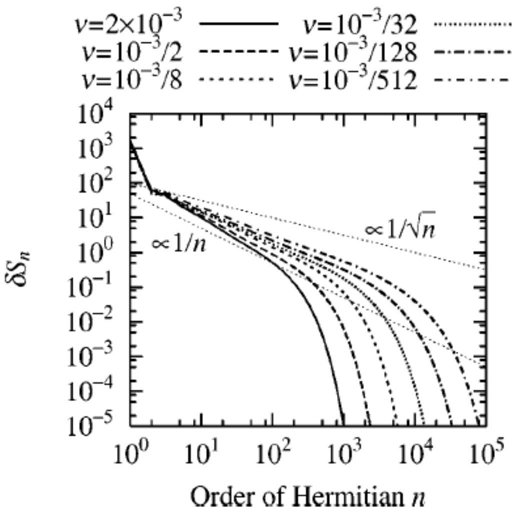
α=-1.98

α=-1.54

α=-1.33

α=-1.55

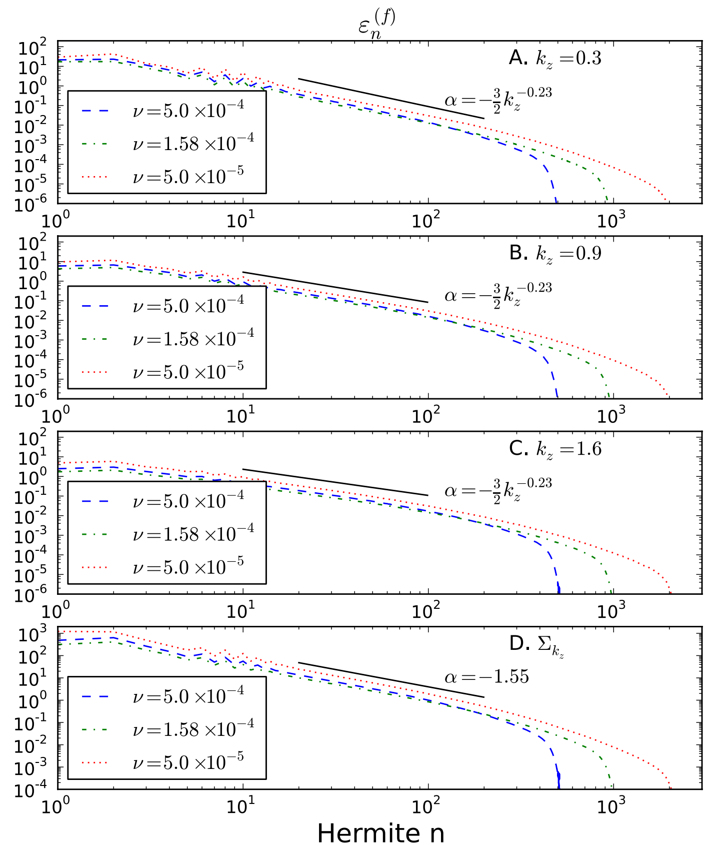
Hermite Spectra



Watanabe and Sugama
--predict $n^{-1/2}$ or n^{-1}
Zocco et al. PoP ‘11:

$$E_m = \frac{C(k_\|)}{\sqrt{m}} \exp\left[-\left(\frac{m}{m_c}\right)^{3/2}\right]$$

DNA: kz-dependent Hermite spectra—independent of (or weakly dependent on) collisionality.



α=-1.98

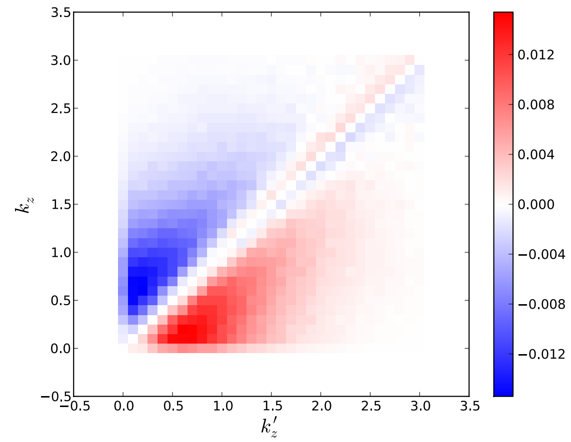
α=-1.54

α=-1.33

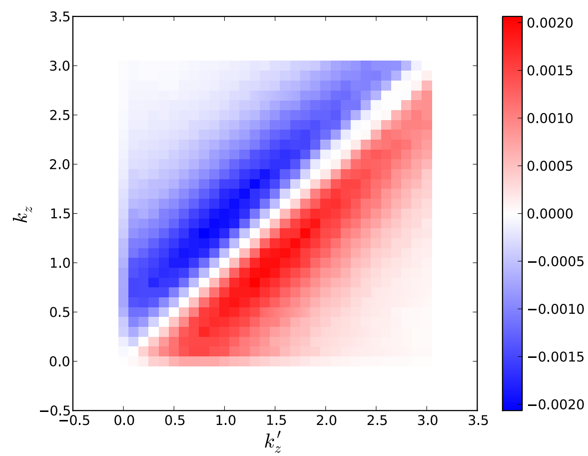
α=-1.55

Nonlinear Energy Transfer

Inverse cascade in kz for high n:



n=8



n=16

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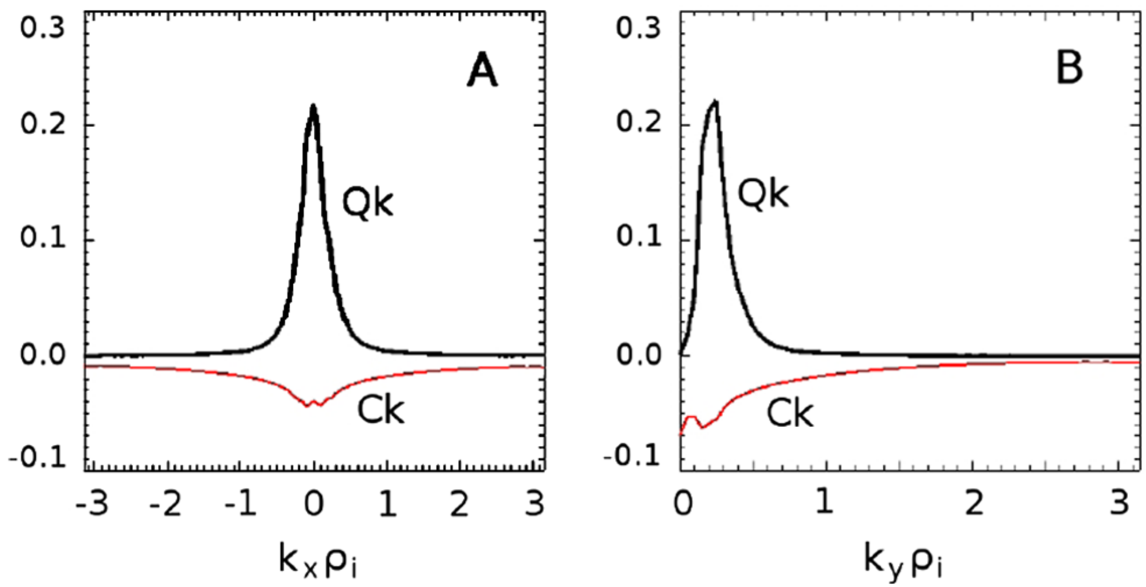
Saturation Through Damped Eigenmodes



Cyclone Base Case ITG (and everything else we've looked at): Significant dissipation in region of instability (Hatch et al. PRL 2011).
➔What modes are responsible?

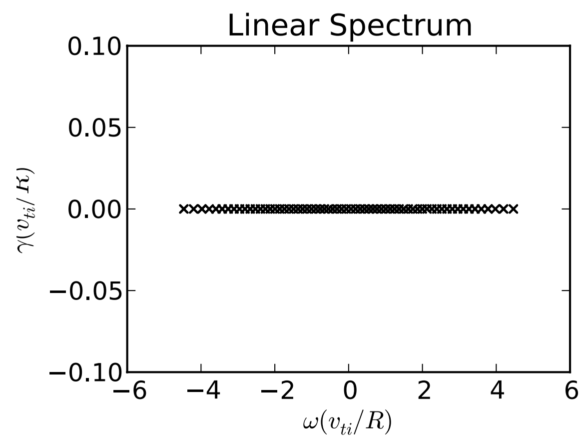
Linear Operator – Matrix Representation

$$L = -ik_z\sqrt{1/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{2}\pi^{-\frac{1}{4}}e^{-k_\perp^2}D(k_\perp) + \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & 0 & 0 & \sqrt{4} & \ddots \end{pmatrix}$$

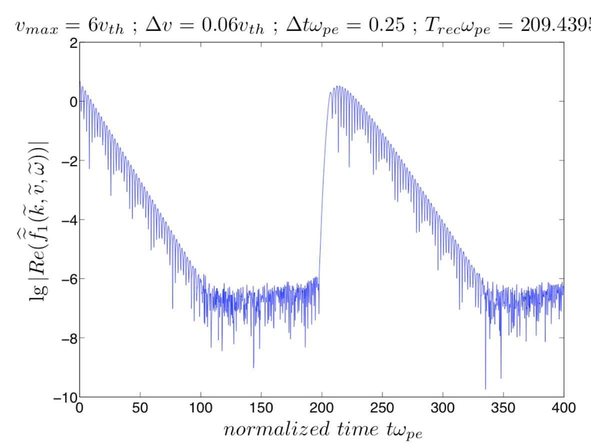
$$-\nu\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix} + ik_y e^{-k_\perp^2}\pi^{-1/4}D(k_\perp)\begin{pmatrix} \omega_T k_\perp^2/2 - \omega_n & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \omega_T/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix}$$

Linear Spectra

	No Collisions	Collisions
No Gradient	X	
Gradient		



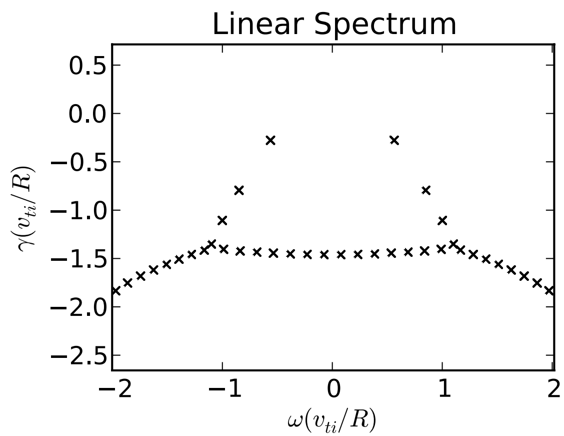
$$L = -ik_z\sqrt{1/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{2}\pi^{-\frac{1}{4}}e^{-k_\perp^2}D(k_\perp)+\sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & 0 & 0 & \sqrt{4} & \ddots \end{pmatrix}$$



Case, Van Kampen

Linear Spectra

	No Collisions	Collisions
No Gradient		X
Gradient		

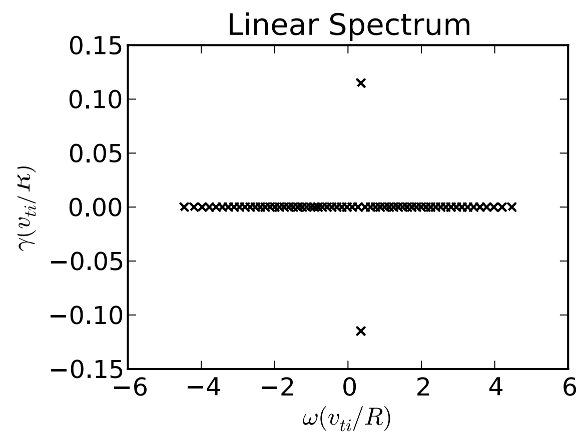


$$L = -ik_z\sqrt{1/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{2}\pi^{-\frac{1}{4}}e^{-k_\perp^2}D(k_\perp)+\sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & 0 & 0 & \sqrt{4} & \ddots \end{pmatrix} - \nu\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix}$$

(Skiff et al. PRL ’98, Ng et al. PRL ’99)

Linear Spectra

	No Collisions	Collisions
No Gradient		
Gradient	X	

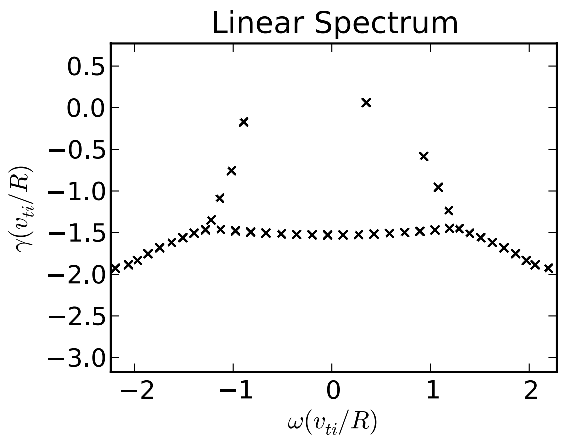


$$L = -ik_z\sqrt{1/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{2}\pi^{-\frac{1}{4}}e^{-k_\perp^2}D(k_\perp)+\sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & 0 & 0 & \sqrt{4} & \ddots \end{pmatrix}$$

$$+ik_y e^{-k_\perp^2}\pi^{-1/4}D(k_\perp)\begin{pmatrix} \omega_T k_\perp^2/2-\omega_n & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \omega_T/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix}$$

Linear Spectra

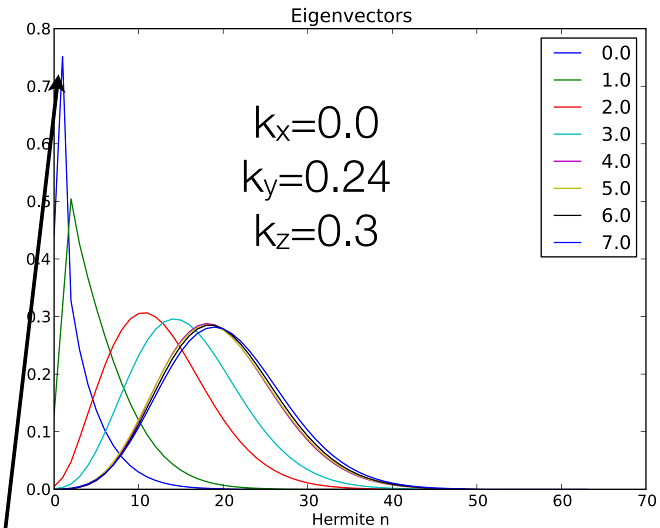
	No Collisions	Collisions
No Gradient		
Gradient		X

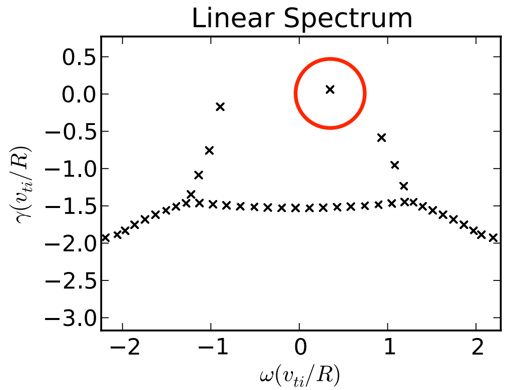


$$L = -ik_z\sqrt{1/2}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{2}\pi^{-\frac{1}{4}}e^{-k_\perp^2}D(k_\perp)+\sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & 0 & 0 & \sqrt{4} & \ddots \end{pmatrix}$$

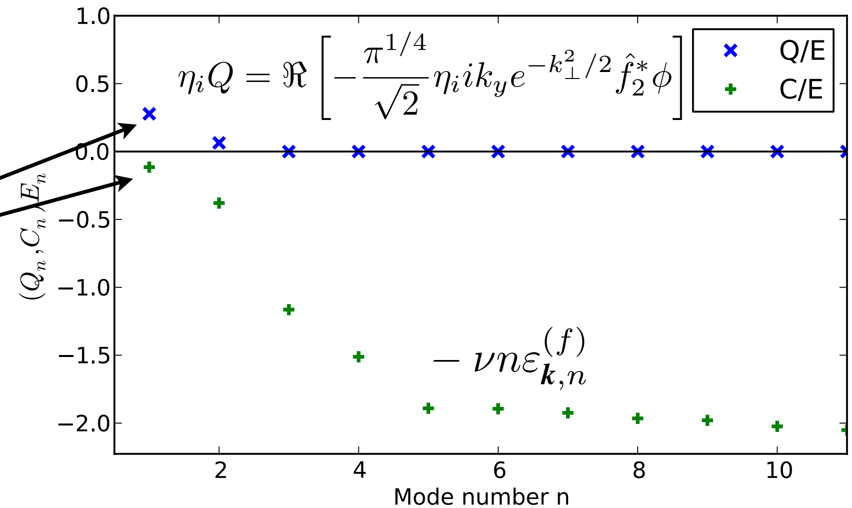
$$+ik_y e^{-k_\perp^2}\pi^{-1/4}D(k_\perp)\begin{pmatrix} \omega_T k_\perp^2/2-\omega_n & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \omega_T/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix} \quad -\nu\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ \vdots & 0 & 0 & 0 & \ddots \end{pmatrix}$$

Linear Eigenmodes - Properties

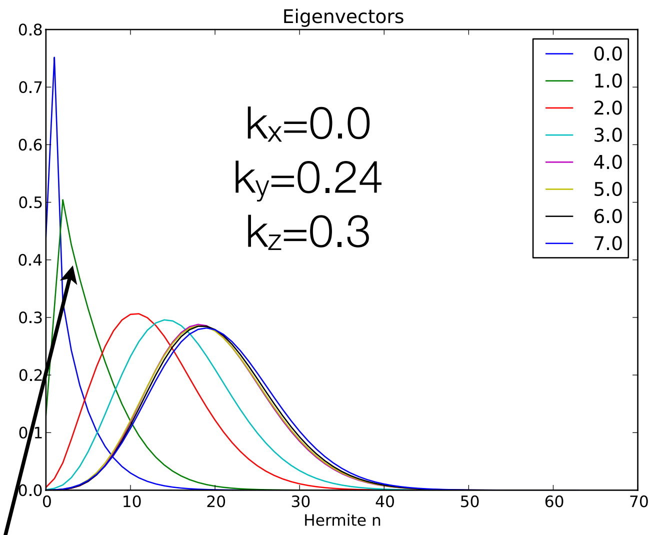


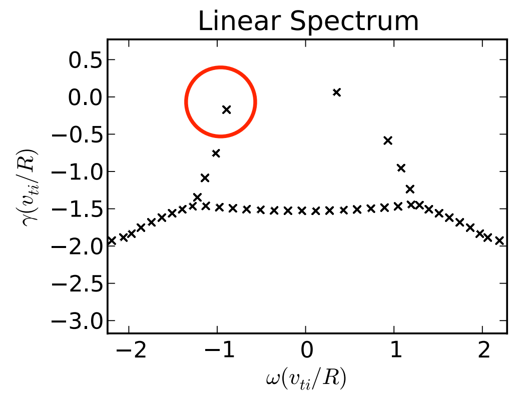


ITG mode:
Relation between f0 and f2 ==> **large drive**
smooth velocity space structure (low order Hermites):
low collisionality

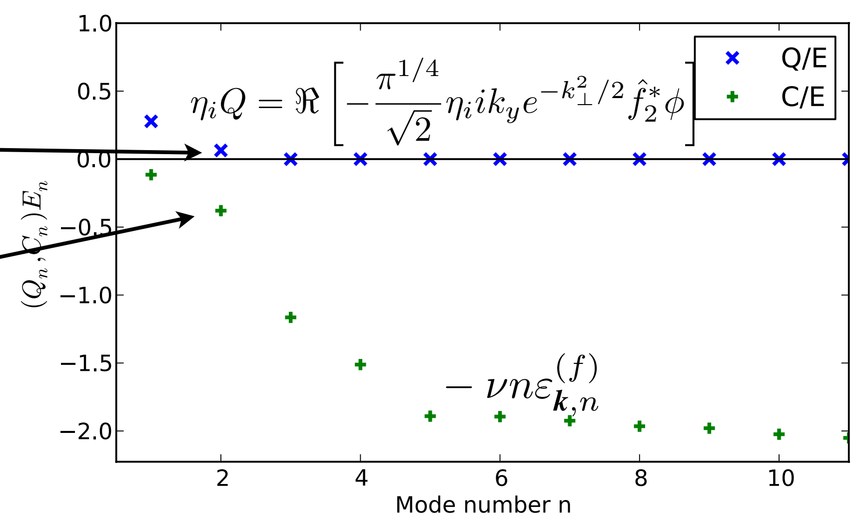


Linear Eigenmodes - Properties

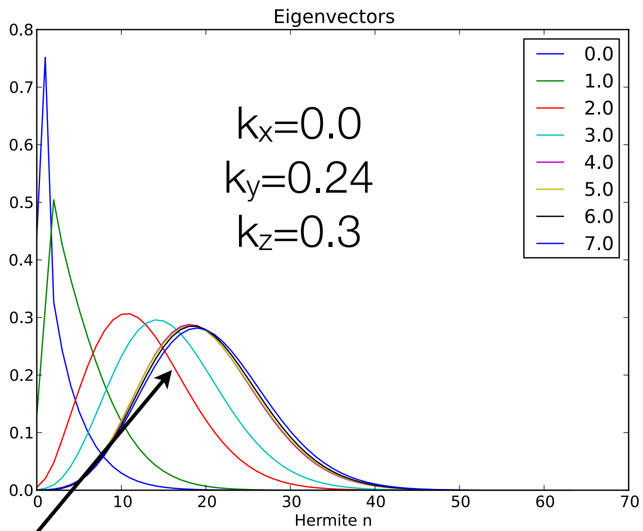


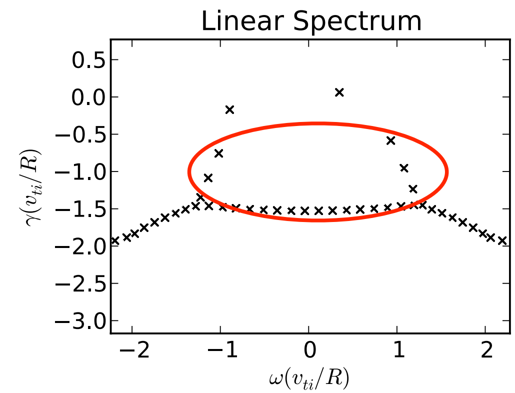


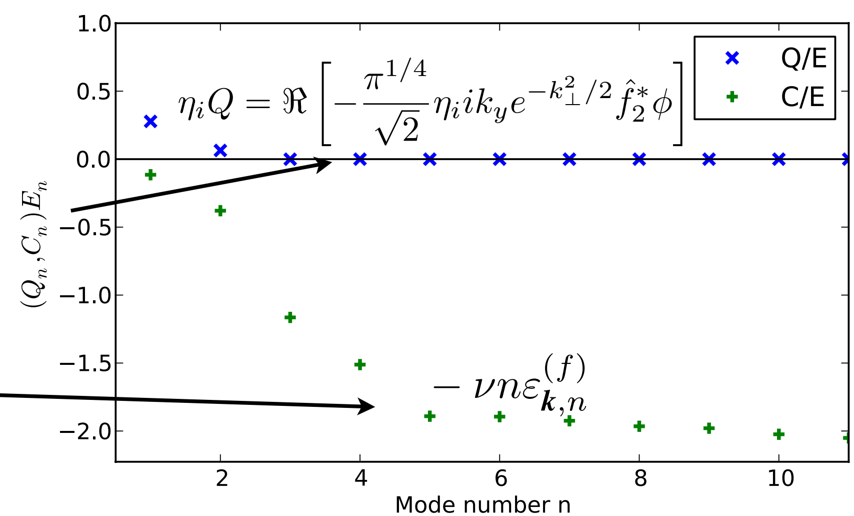
Drift Wave:
Relation between f0 and f2 ==> **moderate drive**
smooth velocity space structure (low order Hermites):
low collisionality



Linear Eigenmodes - Properties







Landau roots:
Relation between f0 and f2 ==> **virtually 0**
smooth velocity space structure (low order Hermites):
high collisionality

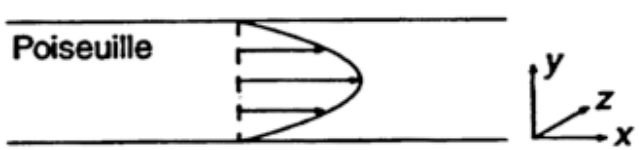
Pseudospectra

"... even if all of the eigenvalues of a linear system are distinct and lie well inside the lower half plane, inputs to that system may be amplified by arbitrarily large factors if the eigenfunctions are not orthogonal to one another."

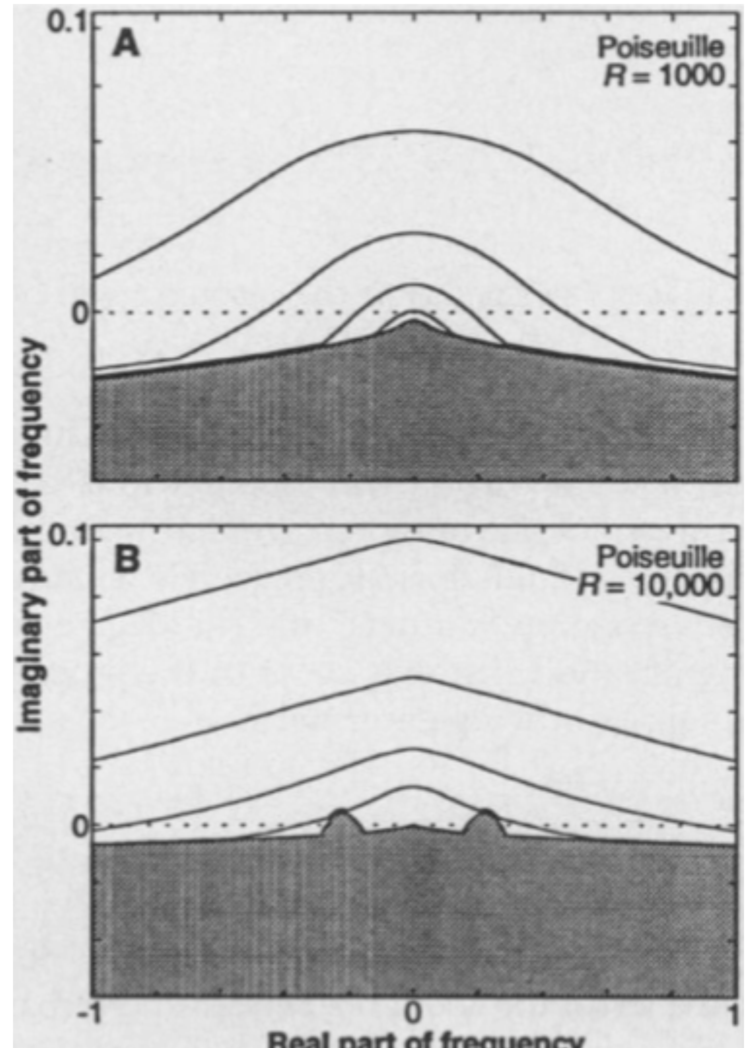
$$\Lambda_\epsilon(A) = \{ z \in \mathbf{C} : \|(zI - A)^{-1}\| \geq \epsilon^{-1} \}$$

$$\Lambda_\epsilon(A) = \{ z \in \mathbf{C} : \sigma_{\min}(zI - A) \leq \epsilon \}$$

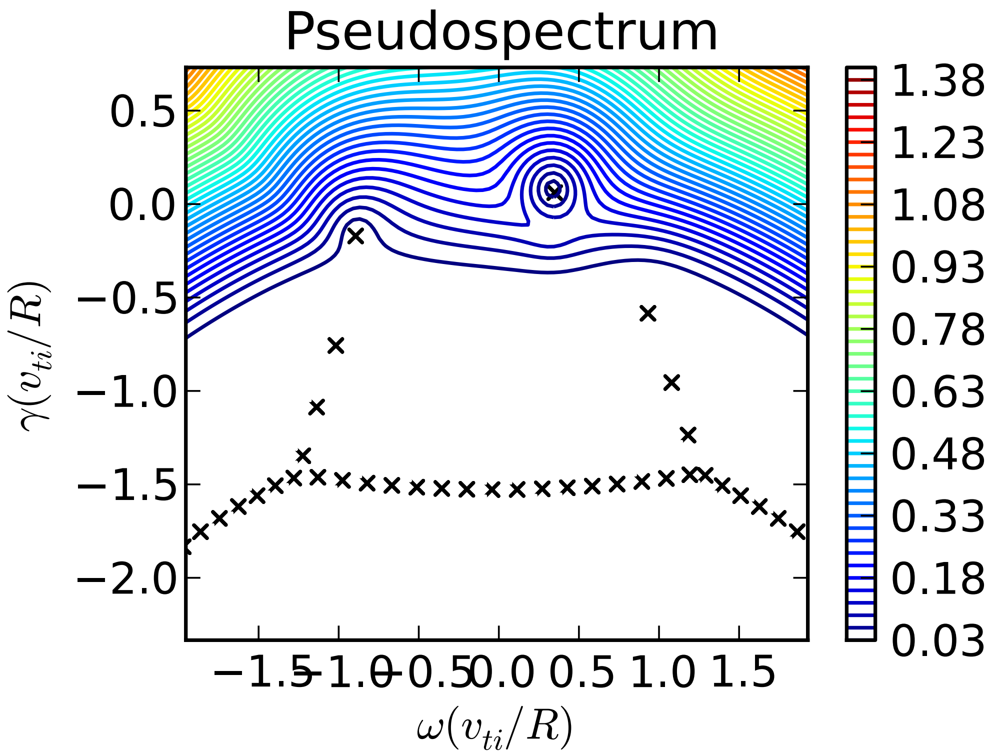
Classic Example:



Trefethen-*Science*, New Series, Vol. 261, No. 5121. (Jul. 30, 1993), pp. 578-584.
Trefethen-SIAM REV. Vol. 39, No. 3, pp. 383–406, September 1997

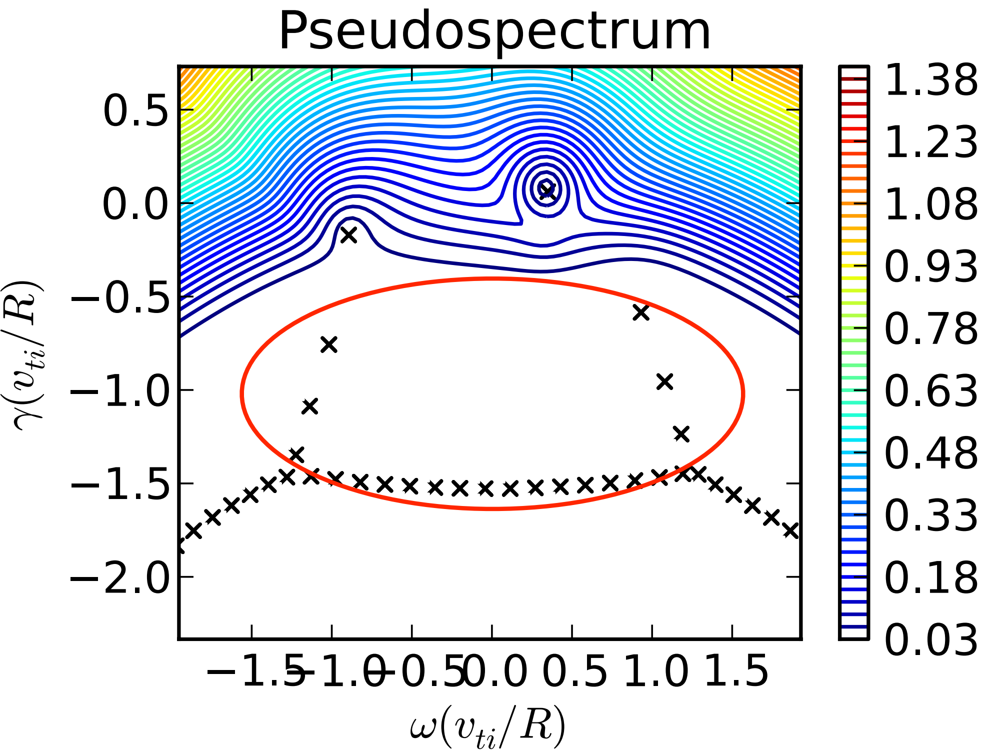


Pseudospectrum – Slab ITG



Closely nested surfaces around ITG mode (and to a lesser extent around DW.

Pseudospectrum – Slab ITG



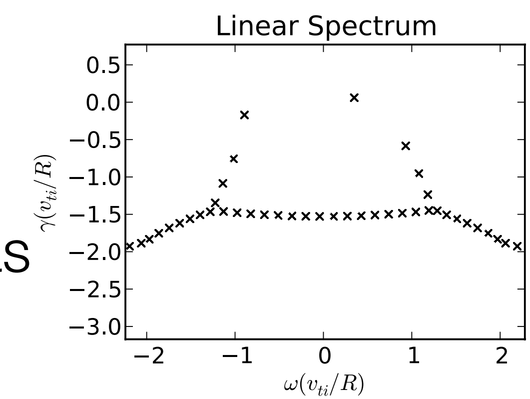
Highly non-orthogonal in this region

All frequencies in this region are highly resonant

Would not expect fluctuations to match eigenvalues here.

Direct Eigenmode Decompostion Infeasible

- Case-Van Kampen emphasized that eigenmodes form complete basis (even though they are nonorthogonal)
- Eigenvectors of adjoint operator serve as projection operators
- In practice nonorthogonality is too extreme==>cannot associate any quadratic quantity (e.g. free energy or heat flux) uniquely with individual eigenmodes.



Use SVD to Extract Optimal Modes

Take distribution (for certain kx,ky,kz) from nonlinear dataset, and construct SVD:

$$\hat{f}_n(t) = \sum_s \sigma_s \hat{g}_n^{(s)} h^{(s)}(t)$$

σ_s =singular value--average amplitude

$\hat{g}_n^{(s)}$ and $h^{(s)}(t)$ are orthonormal and 'optimal' (rigorously more efficient than any other decomposition)

Pseudo-eigenvalues From SVD Modes

Already Noted: direct projection onto linear eigenmodes is meaningless

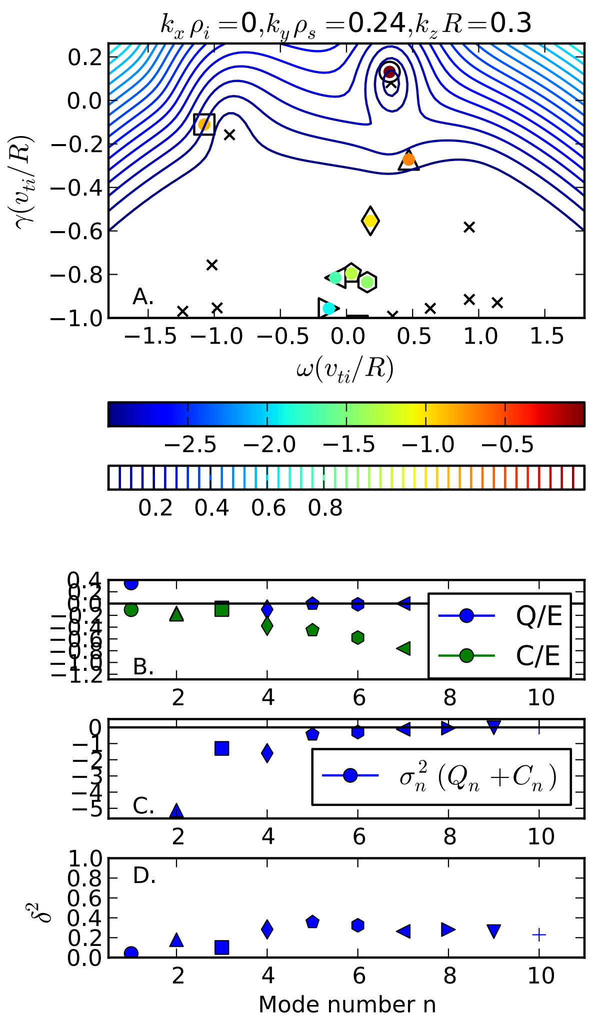
Go backwards from SVD:

Get 'pseudo-eigenvalues' by minimizing $||A\hat{g}_n - z\hat{g}_n||$ over the complex plane for a given SVD mode g_n

This is zero for an exact eigenvector

pseudo-eigenvalue is the location in the complex plane where $||A\hat{g}_n - z\hat{g}_n||$ is minimized

Nonlinear spectra



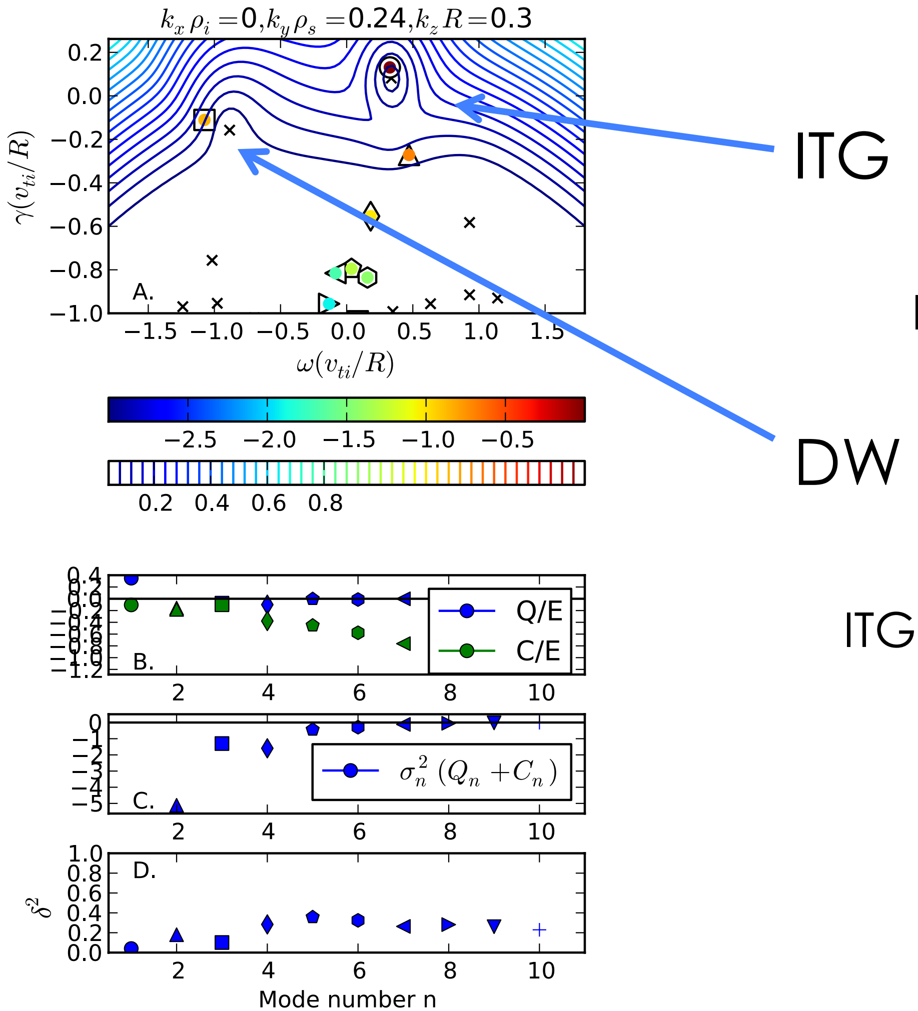
Combine information
- -linear
- -pseudospectra
- -nonlinear

Mode growth/damping rates for Q and C

Net damping rate (including amplitude)

How close to being an eigenvalue

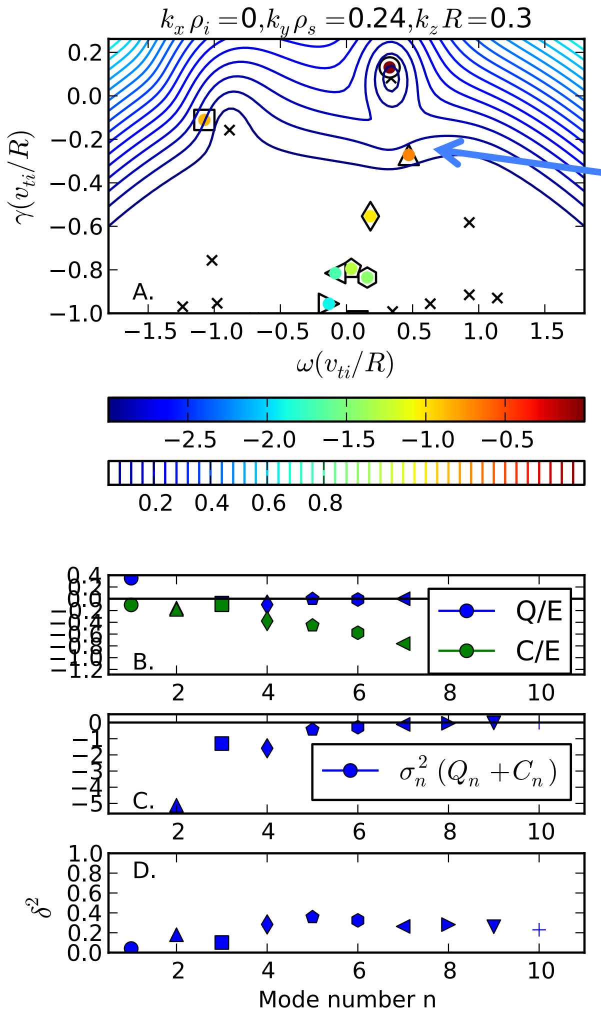
Nonlinear spectra



ITG and DW clearly identified.

ITG and DW clearly identified.

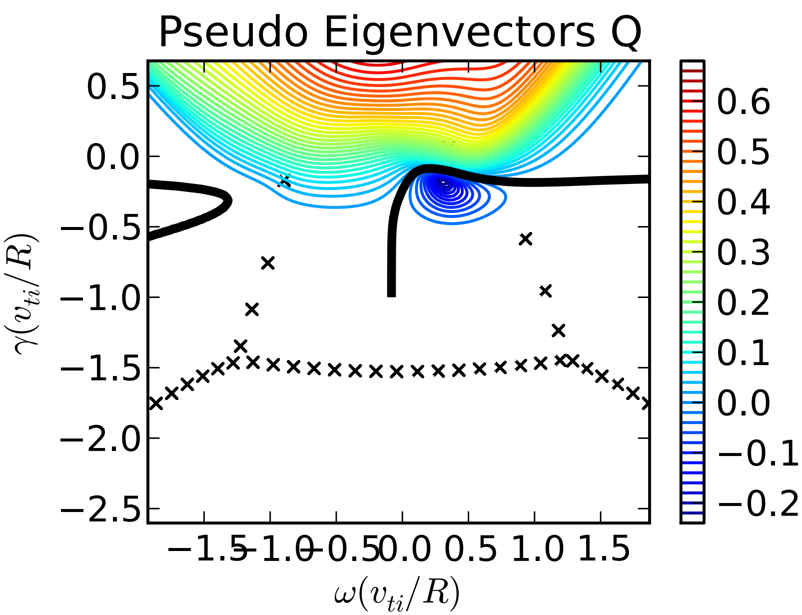
Nonlinear spectra

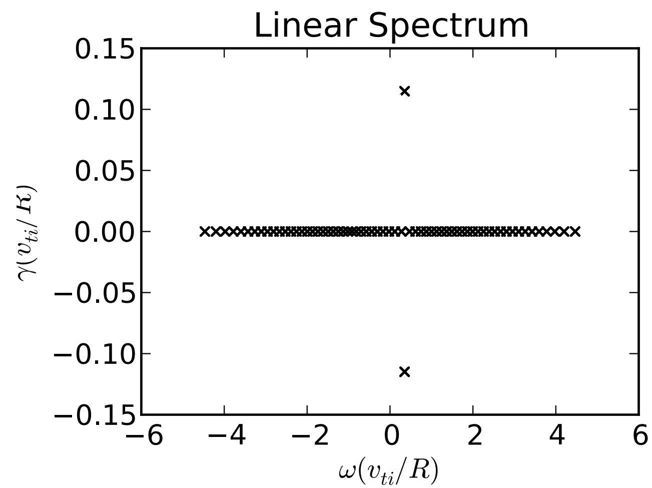


Typically one or more modes with negative Q—i.e., inward heat flux. Significant energy sink.

Dissimilar to anything in the linear spectrum.

Landau-like roots don't play big role (at low k).

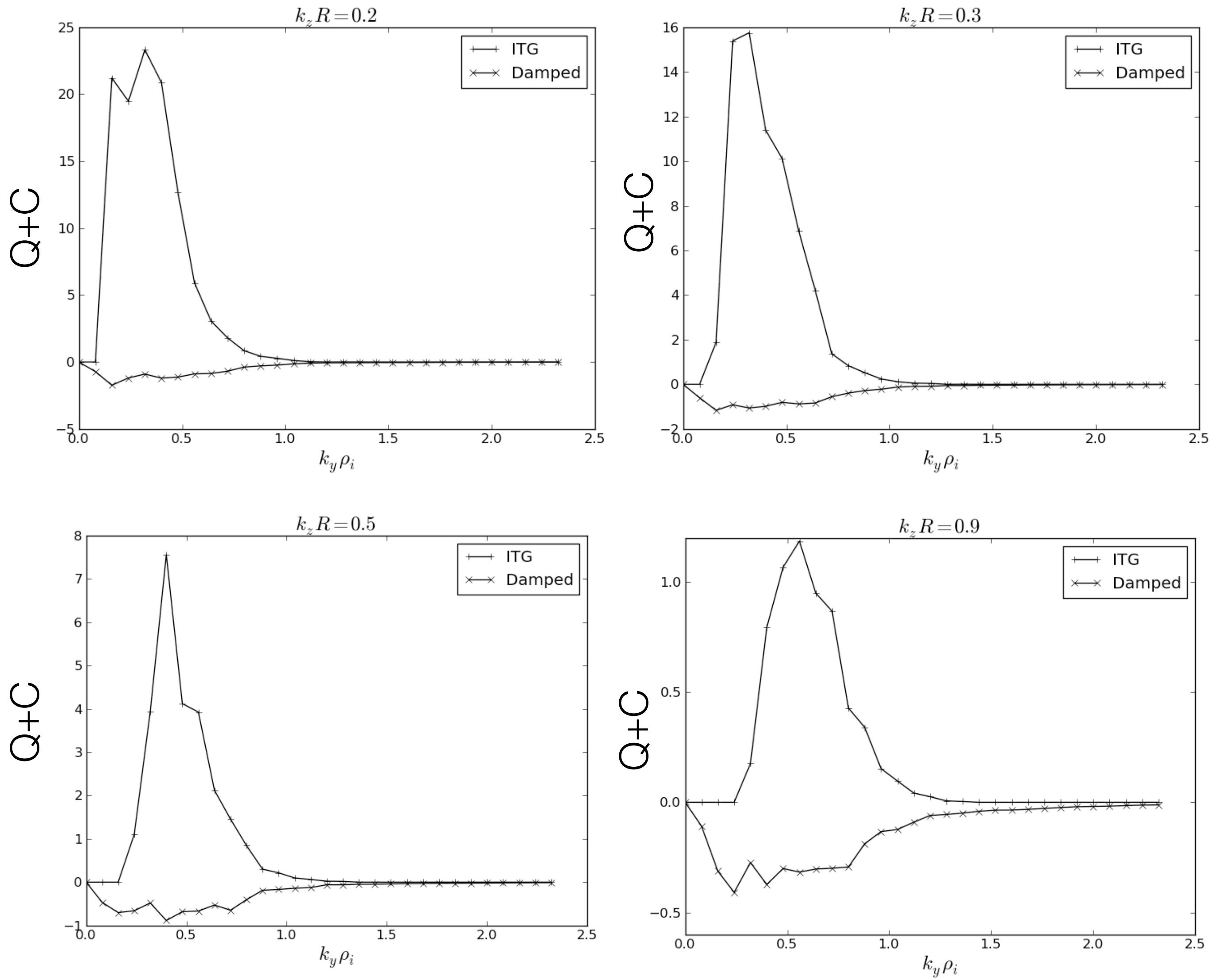




Negative Q is still observed in pseudo spectrum

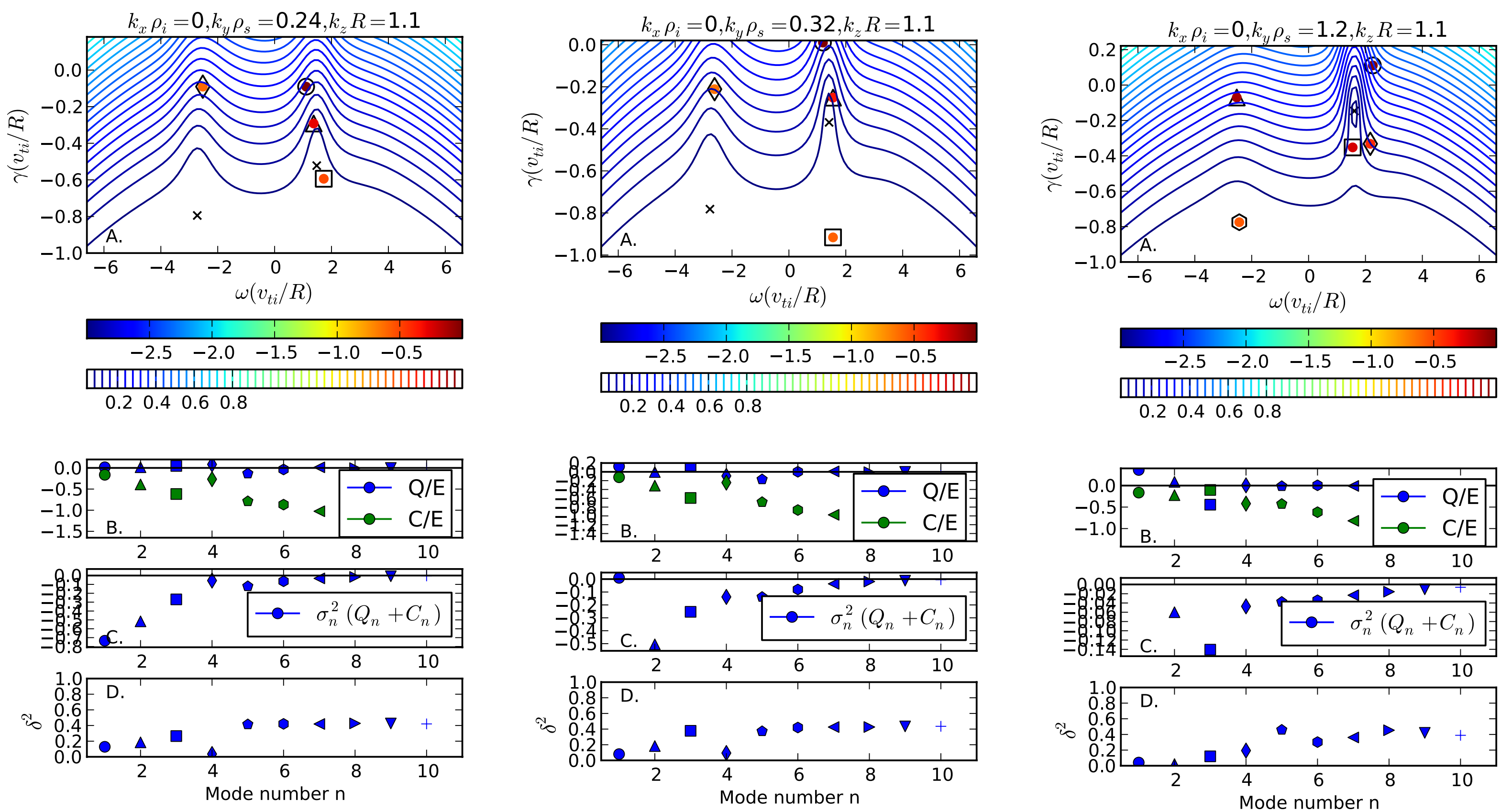
Same region as mirror mode in collisionless linear spectrum!

Damped Modes – Significant Net Energy Sink



Significant **energy sink** from damped modes ==>**associated with small scales in neither k-space nor v-space**

High k Spectra



ITG and DW significantly less damped than linear

Also manifest in deformation of pseudospectra

Relevance for cascades in presence of Landau damping

May be connection with G. Plunk--Phys. Plasmas **20**, 032304 (2013).

Summary / Conclusions

- DNA code solves reduced gyrokinetics in Hermite representation
- Hermite spectra—independent of collisionality, dependent on kz
- Damped eigenmodes
  - ITG mode and marginal DW identifiable in nonlinear spectrum
  - Also a mode with negative heat flux—large energy sink
  - Many features of nonlinear spectrum can be interpreted in light of pseudospectrum
  - High ky, kz→significantly less damping than linear

Landau Damping w/ Nonlinearity