Nonlinear theory of the parallel firehose instability in a weakly collisional plasma with FLR effects

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Abstract

Understanding small-scale turbulence in magnetized plasmas and how it affects the transport of quantities such as heat or momentum is crucial to explain the observed large-scale properties of many objects in the Universe. Astrophysical plasmas such as the solar wind, the intracluster medium or AGN plasmas are however only weakly collisional, which prohibits the use of fluid theories to describe turbulence in these systems. One must in general instead resort to a kinetic description that notably incorporates finite Larmor radius effects if turbulent fluctuations develop at scales down to the ion gyroscale. In this study, we consider the astrophysically relevant case of a weakly collisional high-β plasma, in which magnetized turbulence naturally generates pressure anisotropies that drive mirror and firehose instabilities at scales comparable to the ion gyroscale. As a first step towards understanding the dynamical macroscopic consequences of these instabilities, we present a kinetic theory of the nonlinear development of the parallel firehose instability with finite Larmor radius effects. We show that infinitesimal magnetic fluctuations reach amplitudes comparable to that of the background field on a turbulent turnover timescale and that the original pressure anisotropy evolves nonlinearly towards its critical value for instability.

Kinetic turbulence in astrophysical plasmas

Observations usually only reveal the large-scale properties of distant astrophysical objects, such as their global magnetic field, temperature or luminosity. Somehow paradoxically, explaining the origin of these structures requires to understand the details of the small-scale, unresolved dynamics within the plasma. The reason is that most astrophysical plasmas are in a turbulent state, which drastically affects the transport processes that regulate their thermodynamics and dynamics on global scales. For instance, the X-ray emission of compact objects is tightly related to turbulent angular momentum transport associated with their hot accretion flow (Quataert et al. 2002). In galaxy cluster physics, determining if and how a temperature gradient can be maintained between the inner and outer regions of a cluster requires to understand how heat gets transported in the intracluster medium (Voigt et al. 2004).

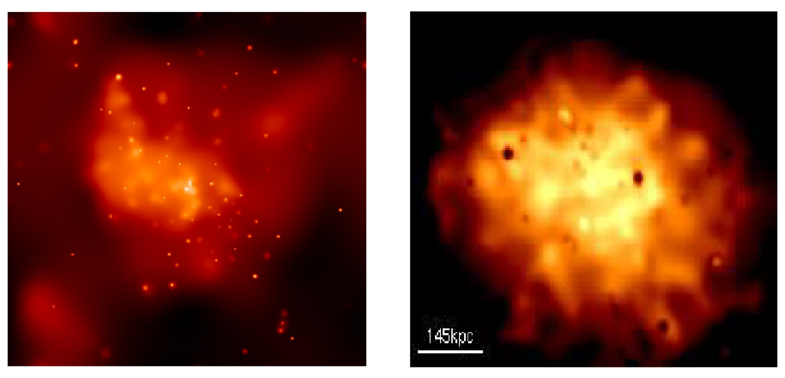

Fig. 1. Left: X-ray emission of the hot gas in the Sgr A* region obtained by Chandra (for the central accreting region of size $\sim 10^8$ km, $T_i \sim 10^{12}$ K, $\lambda_{\mathrm{mfp}i} \sim 10^9$ km, $\rho_i = 10^{-1}$ km, $\beta_i \sim 4$. The field-of-view here is $\sim 20-30$ pc). Right: pressure map (Schueker et al. 2004) of the intracluster medium of the Coma cluster ($T_i \sim 10^7$ K, $\lambda_{\mathrm{mfp}i} \sim 10^{15}$ km, $\rho_i = 10^5$ km, $\beta_i \sim 10$). 1 pc $\sim 3.08\ 10^{13}$ km.

Many astrophysical plasmas are not only turbulent, they are also magnetized, weakly collisional and have $\beta \equiv 8\pi p/B_0^2 \geq 1$. In such plasmas (see Fig. 1), the turbulent fluctuation scales and the ion Larmor radius (or gyroscale) $\rho_i = v_{\mathrm{th}i}/\Omega_i$ are both far smaller than the ion mean free path $\lambda_{\mathrm{mfp}i} = v_{\mathrm{th}i}/\nu_{ii}$ (the ion-ion collision frequency is ν_{ii}, the ion thermal speed is $v_{\mathrm{th}i} = \sqrt{2T_i/m_i}$, and the ion cyclotron frequency is $\Omega_i = e_iB/m_ic$). This turbulence cannot be described by fluid theories such as isotropic MHD or Braginskii MHD which are not valid in this parameter regime. Instead, one often has to resort to a kinetic description, which usually increases theoretical and numerical complexity rather drastically.

Anisotropy and high-β kinetic instabilities

Pressure anisotropy generation in magnetized plasmas. In magnetized, high-β weakly collisional plasmas, pressure anisotropy (with respect to the magnetic field orientation $\hat{\mathbf{b}}_0$) is an important element of the plasma dynamics. The first adiabatic invariant $\langle\mu\rangle = p_\perp/\rho B_0$ is almost conserved in such a system, meaning that any change in magnetic field immediately translates into a change in perpendicular pressure and in the build-up of a pressure anisotropy $\Delta \equiv (p_\perp - p_\parallel)/p$. Changes in magnetic field can be due to a spatially decaying field experienced by particles moving with respect to it (as in the solar wind) or to magnetic field induction by a velocity field $\mathbf{u}_0$, following $\mathrm{d}\ln B_0/\mathrm{d}t = \hat{\mathbf{b}}_0\hat{\mathbf{b}}_0 : \nabla\mathbf{u}_0$. Such anisotropic, high-β configurations are unstable to either the mirror or the firehose instability (depending on the sign of Δ), which both develop at scales as small as a few ρ_i. Consequently, in the presence of any velocity field (even a large-scale one), these instabilities will always kick in and inject energy down to the ion gyroscale. Since ρ_i can be very small in magnetized plasmas, the problem contains extremely disparate scales and is not directly manageable numerically. Other techniques based for instance on asymptotic scale separation have to be used to solve the problem.

The linear parallel firehose instability. Hereafter, we focus on the parallel firehose problem, for which both velocity and magnetic field perturbations are perpendicular to $\hat{\mathbf{b}}_0$ and have $k_\perp = 0$. In the absence of FLR effects, the linear dispersion relation is an anisotropic Alfvénic dispersion relation

$$\omega^2 = \frac{k_\parallel^2 v_{\mathrm{th}i}^2}{2}\left(\Delta + \frac{2}{\beta}\right). \tag{1}$$

For $\Delta = 0$, we recover a standard Alfvén wave. For $\Delta < -2/\beta$, the wave turns into an exponentially growing unstable mode. This occurs when parallel pressure is larger than the perpendicular pressure, i.e. in regions of decreasing magnetic field (in a tangled field, these are also the regions of high field curvature). Note that in the unstable case, the growth rate should become infinite for $k_\parallel \to \infty$, which is of course not physical. In practice, the dispersion relation must be regularized using FLR corrections.

Asymptotic nonlinear theory with FLR

Asymptotic expansion. We consider the nonlinear evolution of a firehose-unstable system in which anisotropy is consistently driven by a flow $\mathbf{u}_0$ at scales far larger than the instability scale. To close the kinetic equations, we seek an asymptotic expansion based on this scale separation. For a turbulent system of size L and corresponding large-scale velocity U such that the Mach number $M = U/v_{\mathrm{th}i} \ll 1$, we define a Reynolds number $Re = UL/\nu$, where $\nu \equiv v_{\mathrm{th}i}\lambda_{\mathrm{mfp}i}$ is a fluid estimate of the plasma viscosity. Based on Kolmogorov arguments, the largest field stretching rate γ_0 comes from the viscous scales $\ell_0 \sim LRe^{-3/4}$ which have a velocity $u_0 \sim URe^{-1/4}$. We introduce a small parameter

$$\varepsilon \equiv MRe^{-1/4} = u_0/v_{\mathrm{th}i} = \lambda_{\mathrm{mfp}i}/\ell_0 \ , \tag{2}$$

which provides a first scale separation between the subsonic stirring scales and the ion mean free path. We further take $v_{\mathrm{th}i}$ as unit of velocity, so that $u_0 \sim \varepsilon$ in nondimensional form, and a typical firehose fluctuation scale as unit of length. The fluctuation scales lie below the ion mean free path, so we order $\lambda_{\mathrm{mfp}i} \sim \varepsilon^{-1}$ and $\nu_{ii} \sim \varepsilon$, leading to $\ell_0 \sim \varepsilon^{-2}$ and $\gamma_0 \sim \nabla u_0 \sim \varepsilon^3$ for the slow scales. Now, the pressure anisotropy that builds up before the firehose saturates results from a competition between magnetic induction by $\mathbf{u}_0$ and relaxation by weak collisions, $\Delta \sim \gamma_0/\nu_{ii} \sim \varepsilon^2$. From Eq. (1), we must similarly order $\beta \sim \varepsilon^{-2}$ to obtain a firehose instability at $k_\parallel \sim 1$ with growth rate $\gamma \sim \varepsilon$. Finally, we know from the hot plasma theory (Stix 1992) that the FLR correction is proportional to $(\rho_i v_{\mathrm{th}i})^2$ in the actual firehose dispersion relation, so we order $\rho_i \sim \varepsilon$ for the FLR term to be comparable to the Δ term in Eq. (1). This means that we are doing a low frequency approximation, i.e. $\Omega_i \sim \varepsilon^{-1} \gg \gamma, \gamma_0$.

$1/\Omega_i \sim \varepsilon$	$1/\gamma, 1/\nu_{ii} \sim \varepsilon^{-1}$		$1/\gamma_0 \sim \varepsilon^{-3}$	Time
$\rho_i \sim \varepsilon$	$1/k_\parallel \sim 1$	$\lambda_{\mathrm{mfp}i} \sim \varepsilon^{-1}$	$\ell_0 \sim \varepsilon^{-2}$	Space

We first consider the nondimensional Vlasov equation for the electrons and expand it in terms of the electron to ion mass ratio m_e/m_i. At order -1/2, we find that the electron distribution function is a Maxwellian, and derive a simple Ohm's law. We then expand $\mathbf{u}$, $\mathbf{B}$ and the ion distribution function f_i in integer powers of ε starting from $\mathbf{u}_0 \sim \varepsilon$, $\mathbf{B}_0 \sim \varepsilon$ and $f_{0i} \sim 1$ and plug the ε ordering in the Vlasov equation for ions and Ohm's law, systematically expanding derivatives in terms of slow and fast variables. At orders -1, 0, 1 and 2, we most notably find that the zeroth and first order ion distribution functions are Maxwellian and obtain the gyrophase dependence of higher and higher order bits of f_i. We need to proceed to order 3 to obtain information on the time evolution of the unstable mode $\mathbf{u}_{1i}$, $\mathbf{B}_1$. Pressure anisotropy notably kicks in consistently at this order in f_{2i}, as a consequence of our ordering for γ_0. We obtain an evolution equation for the anisotropic part h_{2i} of f_{2i}

$$\frac{\partial h_{2i}}{\partial t} + \frac{2f_{0i}}{v_{\mathrm{th}i}^2}\left(v_\parallel^2 - \frac{v_\perp^2}{2}\right)\left[\hat{\mathbf{b}}_0\hat{\mathbf{b}}_0 : \nabla\mathbf{u}_0 + \frac{\mathbf{B}_1}{B_0}\hat{\mathbf{b}}_0 : \nabla\mathbf{u}_{1i}\right] = \left.\frac{\partial h_{2i}}{\partial t}\right|_c \ . \tag{3}$$

This equation contains an anisotropy source $\sim \hat{\mathbf{b}}_0\hat{\mathbf{b}}_0 : \nabla\mathbf{u}_0$ due to magnetic induction by $\mathbf{u}_0$, a nonlinear term $\sim (\mathbf{B}_1/B_0)\,\hat{\mathbf{b}}_0 : \nabla\mathbf{u}_{1i}$ which becomes important once the firehose fluctuations saturate, and a collisional relaxation term. The induction and momentum equations for $\mathbf{B}_1$ and $\mathbf{u}_{1i}$ read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{B}_1}{B_0} = \hat{\mathbf{b}}_0 \cdot \nabla\mathbf{u}_{1i} \ , \tag{4}$$

$$\frac{\mathrm{d}\mathbf{u}_{1i}}{\mathrm{d}t} = \hat{\mathbf{b}}_0 \cdot \nabla\left[\frac{v_{\mathrm{th}i}^2}{2}\left(\Delta(t) + \frac{2}{\beta_{0i}}\right)\frac{\mathbf{B}_1}{B_0} + \frac{\rho_i v_{\mathrm{th}i}}{2}\left(\hat{\mathbf{b}}_0 \cdot \nabla\mathbf{u}_{1i}\right) \times \hat{\mathbf{b}}_0\right] \ , \tag{5}$$

where the second order (negative) ion pressure anisotropy $\Delta(t) = (p_{\perp 2i} - p_{\parallel 2i})/p_{0i}$ has been introduced. Equations (4)-(5) can be combined into the following wave-like equation for $\mathbf{B}_1$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\mathbf{B}_1}{B_0} = \nabla_\parallel^2\left[\frac{v_{\mathrm{th}i}^2}{2}\left(\Delta(t) + \frac{2}{\beta_{0i}}\right)\frac{\mathbf{B}_1}{B_0} + \frac{\rho_i v_{\mathrm{th}i}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{B}_1}{B_0} \times \hat{\mathbf{b}}_0\right] \ . \tag{6}$$

$\Delta(t)$ can be computed consistently from h_{2i}, thus Eqs. (3)-(6) form a closed system (for a given $\mathbf{u}_0$). We then solve equation (3) using a simple pitch-angle scattering collision operator. Denoting an average over small scales along field lines by an overbar, we obtain

$$\Delta(t) = 3\int_0^t \mathrm{d}t' e^{-3\nu_{ii}(t-t')}\overline{\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u}}(t') = -\frac{|\gamma_0|}{\nu_{ii}}\left(1 - e^{-3\nu_{ii}t}\right) + \frac{3}{2}\int_0^t \mathrm{d}t' e^{-3\nu_{ii}(t-t')}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\overline{B_1^2(t')}}{B_0^2} \ . \tag{7}$$

Numerical results. Equations (4)-(5)-(7) can be solved numerically, starting from random initial conditions for $\mathbf{B}_1$ and $\mathbf{u}_{1i}$. After a linear firehose growth phase triggered by the build-up of Δ to an amplitude $\Delta_0 \sim -|\gamma_0|/\nu_{ii}$, fluctuations grow secularly in the nonlinear regime, feeding back on Δ and maintaining it close to its critical value $-2/\beta_{0i}$. Fluctuations ultimately reach $B_1/B_0 \sim 1$ after a time $\sim 1/\gamma_0$, at which point the asymptotic theory breaks down. There are no interactions between different $k \neq 0$ modes here, but a full spectrum of modes builds up with time as a consequence of the time-modulated linear firehose instability, ultimately producing a spectral index close to -3.

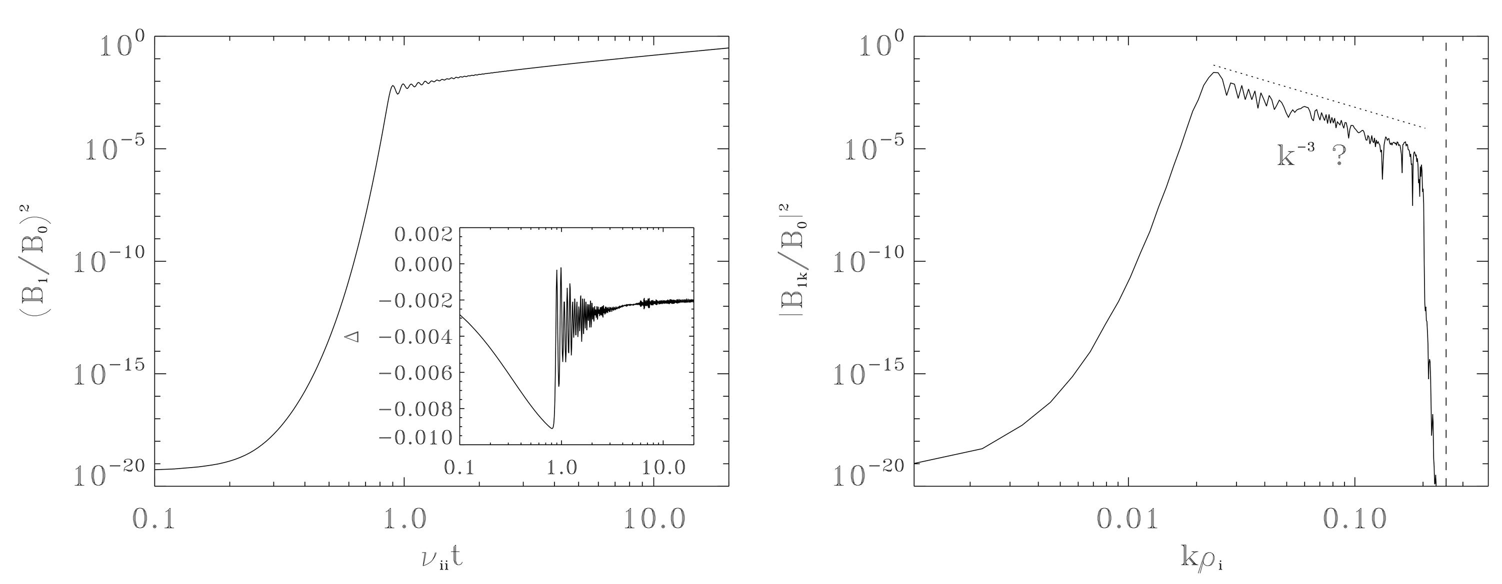

Fig. 2. Left: Ensemble average of the time evolution of $(B_1/B_0)^2$ and anisotropy Δ (inset) using a box size $L \equiv 1$, $\nu_{ii} \equiv 1$, $\lambda_{\mathrm{mfp}i} = L$, $-|\gamma_0|/\nu_{ii} = 0.01$, $2/\beta_{0i} = 0.002$ and $\rho_i = 0.00018$. Right: magnetic energy spectrum at $\nu_{ii}t = 20$. The firehose FLR cut-off scale is indicated by a vertical dashed line in the spectrum.

Discussion

Our theory, by assuming small perturbations, is a form of quasilinear theory, but our results demonstrate that a fully nonlinear theory is required at times larger than a turbulent turnover time. As transport of $\mathbf{u}_0$ takes place on a comparable timescale, we have not attempted to solve that problem in detail. It is however worth noting that the viscosity in our theory, which can be obtained at order ε^4, is proportional to $(\Delta(t) + 2/\beta)$ (this is simply Braginskii viscosity at $\nu_{ii}t \ll 1$). As this quantity tends to zero on a timescale $1/\nu_{ii} \ll 1/\gamma_0$, we suspect that the actual plasma viscosity could be far smaller than the fluid estimate as a consequence of the presence of the firehose. However, before any firm conclusion can be drawn, a more general theory including notably mirror modes should of course be elaborated.