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Lagrangian Formulation of Gyrokinetic Vlasov-Poisson-Ampere Systems

Entropy Balance in Neoclassical and Turbulent Transport

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Part I

Lagrangian Formulation of Gyrokinetic Vlasov-Poisson-Ampere Systems

- **Variational Principle and Noether's Theorem**
- **Vlasov-Poisson-Ampere System**
- **Lie Transformation of phase-space coordinates**
  Gyrocenter coodinates
- **Gyrokinetic Vlasov-Poisson-Ampere System**
  Conservation of total energy

Foundation of Gyrokinetic Theory

Gyrokinetic ordering $\frac{\delta f}{f} \sim \frac{e\delta\phi}{T} \sim \frac{\delta B}{B} \sim \frac{k_{\parallel}}{k_{\perp}} \sim \frac{\omega}{\Omega} \sim \frac{\rho}{L} << 1$

Recursive formulation

Perturbative expansion in ρ/L, Ballooning representation

Equation for δf

Lagrangian/Hamiltonian formulation

Lie transformation of phase-space coordinates

Equation for $F = F_0 + \delta f$

Exact conservation of μ and phase space volume

Lagrangian for electromagnetic fields … Sugama, "Gyrokinetic field theory", PoP(2000)

Equations for electromagnetic fields ϕ, A

Exact conservation of the total (kinetic + field) energy, Noether's theorem

Review by Brizard & Hahm, RMP(2007)

Variational Principle and Noether's Theorem

Field variables $\eta_\alpha(\mathbf{x}_\alpha, t)$

Action $I = \int_{t_1}^{t_2} L\,dt$

Part of Lagrangian associated with η_α **and** $\dot{\eta}_\alpha$

$$L_\alpha(\eta_\alpha, \dot{\eta}_\alpha) = \int d^{l_\alpha}\mathbf{x}_\alpha\, \mathfrak{L}_\alpha[\eta_\alpha(\mathbf{x}_\alpha, t),\, \dot{\eta}_\alpha(\mathbf{x}_\alpha, t), \nabla_\alpha \eta_\alpha(\mathbf{x}_\alpha, t), \cdots]$$

Variational principle $\delta I = 0$ $\Longrightarrow$ **Euler-Lagrange equations**

$$\frac{\delta I}{\delta \eta_\alpha} \equiv \frac{\partial \mathfrak{L}_\alpha}{\partial \eta_\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}_\alpha}{\partial \dot{\eta}_\alpha}\right) - \nabla_\alpha \cdot \left(\frac{\partial \mathfrak{L}_\alpha}{\partial \nabla_\alpha \eta_\alpha}\right) = 0$$

Infinitesimal transformations

$$t \to t' = t + \delta t, \quad \mathbf{x}_\alpha \to \mathbf{x}'_\alpha = \mathbf{x}_\alpha + \delta \mathbf{x}_\alpha, \quad \eta_\alpha(\mathbf{x}_\alpha, t) \to \eta'_\alpha(\mathbf{x}'_\alpha, t') = \eta_\alpha(\mathbf{x}_\alpha, t) + \delta \eta_\alpha(\mathbf{x}_\alpha, t)$$

Variation of action

$$\delta I = I' - I = -\int_{t_1}^{t_2} dt \left[\frac{dG}{dt} + \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha\, \nabla_\alpha \cdot \mathbf{J}_\alpha\right]$$

Noether's theorem

Invariance $\delta I = 0$ $\Longrightarrow$ **Conservation of** G $dG/dt = 0$

$$G = \delta t\left(\sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha\, \dot{\eta}_\alpha \frac{\partial \mathfrak{L}_\alpha}{\partial \dot{\eta}_\alpha} - L\right) + \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha \left(\delta \mathbf{x}_\alpha \cdot \nabla_\alpha \eta_\alpha \frac{\partial \mathfrak{L}_\alpha}{\partial \dot{\eta}_\alpha} - \delta \eta_\alpha \frac{\partial \mathfrak{L}_\alpha}{\partial \dot{\eta}_\alpha}\right)$$

Lagrangian with no explicit time dependence $\delta I / \delta t = 0$ $\Longrightarrow$ **Conservation of total energy**

$$E_{\text{tot}} = \sum_\alpha \int d^{l_\alpha}\mathbf{x}_\alpha\, \dot{\eta}_\alpha \frac{\partial \mathfrak{L}_\alpha}{\partial \dot{\eta}_\alpha} - L = \text{const}$$

Lagrangian Formulation of the Vlasov-Poisson-Ampre System

Variational principle $\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$

Total Lagrangian

$$L \equiv \sum_a \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0\, f_a(\mathbf{x}_0, \mathbf{v}_0, t) L_a\left[\mathbf{x}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t), \mathbf{v}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t), \dot{\mathbf{x}}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t), t\right] + L_f$$

Single-partilce Lagrangian

$$L_a(\mathbf{x}_a, \mathbf{v}_a, \dot{\mathbf{x}}_a) \equiv \left(m_a \mathbf{v}_a + \frac{e_a}{c} \mathbf{A}(\mathbf{x}_a, t) \right) \cdot \dot{\mathbf{x}}_a - \left(\frac{1}{2} m \mid \mathbf{v}_a \mid^2 + e_a \phi(\mathbf{x}_a, t) \right)$$

$$\equiv \mathbf{p}_a \cdot \dot{\mathbf{x}}_a - H_a$$

Field part

$$L_f \equiv \int d^3\mathbf{x}_f\, \mathcal{L}_f \equiv \frac{1}{8\pi} \int d^3\mathbf{x} \left(\left|\nabla\phi(\mathbf{x}, t)\right|^2 - \left|\nabla \times \mathbf{A}(\mathbf{x}, t)\right|^2 + \frac{2}{c} \lambda(\mathbf{x}, t) \nabla \cdot \mathbf{A}(\mathbf{x}, t) \right)$$

L_f **does not contain** $\partial \mathbf{A} / \partial t$

⟹ **Electromagnetic waves with the speed of light are not described.**

Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ **is derived from** $\delta I / \delta \lambda = 0$

$$\delta I/\delta \mathbf{x}_a = \delta I/\delta \mathbf{v}_a = 0$$

$\Longrightarrow$ **Nonrelativistic Newton's paricle motion equations**

$$\dot{\mathbf{x}}_a = \mathbf{v}_a, \quad m_a \dot{\mathbf{v}}_a = e_a \left[\mathbf{E}(\mathbf{x}_a, t) + \frac{\mathbf{v}_a}{c} \times \mathbf{B}(\mathbf{x}_a, t) \right]$$

where $\mathbf{E} = -\nabla\phi - c^{-1}\partial \mathbf{A}/\partial t$ **and** $\mathbf{B} = \nabla \times \mathbf{A}$

Distribution function f_a at time t

$$f_a(\mathbf{x}, \mathbf{v}, t) = \int d^3\mathbf{x}_0 \int d^3\mathbf{v}_0\, \delta^3[\mathbf{x} - \mathbf{x}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)]\delta^3[\mathbf{v} - \mathbf{v}_a(\mathbf{x}_0, \mathbf{v}_0, t_0; t)]\, f_a(\mathbf{x}_0, \mathbf{v}_0, t)$$

Vlasov equation $$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e_a}{m_a} \left\{ \mathbf{E}(\mathbf{x}, t) + \frac{1}{c} \mathbf{v} \times \mathbf{B}(\mathbf{x}, t) \right\} \cdot \frac{\partial}{\partial \mathbf{v}} \right] f_a(\mathbf{x}, \mathbf{v}, t) = 0$$

$\delta I/\delta \phi = 0 \Longrightarrow$ **Poisson's quation**

$$\nabla^2 \phi(\mathbf{x}, t) = -4\pi \sum_a e_a \int d^3\mathbf{v}\, f_a(\mathbf{x}, \mathbf{v}, t) \equiv -4\pi \sum_a e_a n_a$$

$$\delta I/\delta \mathbf{A} = 0$$

$$\Longrightarrow \quad \nabla^2 \mathbf{A}(\mathbf{x}, t) - \frac{1}{c} \nabla \lambda(\mathbf{x}, t) = -\frac{4\pi}{c} \sum_a e_a \int d^3\mathbf{v}\, f_a(\mathbf{x}, \mathbf{v}, t)\, \mathbf{v} \equiv -\frac{4\pi}{c} \mathbf{j}$$

Current density $j = j_L + j_T$

Longitudinal (or irrotational) part $\mathbf{j}_L(\mathbf{x},t) \equiv -(4\pi)^{-1}\nabla \int d^3\mathbf{x}' (\nabla' \cdot \mathbf{j}(\mathbf{x}',t)) \big/ |\mathbf{x}-\mathbf{x}'|$

Transverse (or solenoidal) part $\mathbf{j}_T(\mathbf{x},t) \equiv (4\pi)^{-1}\nabla \times \left(\nabla \times \int d^3\mathbf{x}' \, \mathbf{j}(\mathbf{x}',t) \big/ |\mathbf{x}-\mathbf{x}'|\right)$

$$\delta I/\delta \mathbf{A} = 0 \quad : \quad \nabla^2 \mathbf{A}(\mathbf{x},t) - \frac{1}{c}\nabla\lambda(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{j}$$

$\Longrightarrow$ **Longitudinal part** $\nabla^2 \mathbf{A}(\mathbf{x},t) = -\dfrac{4\pi}{c}\mathbf{j}_T$ **(Ampere's law)**

$\Longrightarrow$ **Transverse part** $-\nabla\lambda(\mathbf{x},t) = -4\pi\,\mathbf{j}_L = \partial \mathbf{E}_L / \partial t$

Darwin Model
Kaufman & Rostler, PoF (1971)

Noether's theorem $\Longrightarrow$ **conservation of total energy** $\mathrm{d}E_{\mathrm{tot}}/\mathrm{d}t = 0$

Total energy

$$E_{tot} = \sum_a \int d^3\mathbf{x} \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\left[\frac{1}{2}m_a\,|\,\mathbf{v}\,|^2 + e_a\phi(\mathbf{x},t)\right] - L_f$$

$$= \sum_a \int d^3\mathbf{x} \int d^3\mathbf{v}\, f_a(\mathbf{x},\mathbf{v},t)\frac{1}{2}m_a\,|\,\mathbf{v}\,|^2$$

$$+ \frac{1}{8}\int d^3\mathbf{x}\left(\left|\nabla\phi(\mathbf{x},t)\right|^2 + \left|\nabla\times\mathbf{A}(\mathbf{x},t)\right|^2\right)$$

Perturbation Expansion of Single-Particle Lagrangian

Electromagnetic fields

$$\mathbf{E} = \varepsilon \mathbf{E}_1(\mathbf{x},t)$$
$$= -\varepsilon\left(\nabla\phi(\mathbf{x},t) + c^{-1}\partial_t \mathbf{A}_1(\mathbf{x},t)\right)$$

$$\mathbf{B} = \mathbf{B}_0(\mathbf{x}) + \varepsilon \mathbf{B}_1(\mathbf{x},t)$$
$$= \nabla \times [\mathbf{A}_0(\mathbf{x}) + \varepsilon \mathbf{A}_1(\mathbf{x},t)]$$

$\boldsymbol{\varepsilon}$ **: ordering parameter for perturbation**

Single-particle canonical momentum

$$\mathbf{p} \equiv m\,\mathbf{v} + \frac{e}{c}(\mathbf{A}_0 + \varepsilon \mathbf{A}_1) \equiv m\,\mathbf{v}_0 + \frac{e}{c}\mathbf{A}_0$$

where

$$m\,\mathbf{v}_0 \equiv m\,\mathbf{v} + \varepsilon\frac{e}{c}\mathbf{A}_1$$

Single-particle Lagrangian

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 = \mathbf{p}\cdot\dot{\mathbf{x}} - H$$

Hamiltonian

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2$$

0 th order

$$L_0 \equiv \mathbf{p}\cdot\dot{\mathbf{x}} - H_0 \equiv \left(m\,\mathbf{v}_0 + \frac{e}{c}\mathbf{A}_0\right)\cdot\dot{\mathbf{x}} - \frac{1}{2}m\,|\,\mathbf{v}_0\,|^2$$

1 st order

$$L_1 \equiv -H_1 \equiv -e\psi \equiv -e\left(\phi - \frac{\mathbf{v}_0}{c}\cdot\mathbf{A}_1\right)$$

2 nd order

$$L_2 \equiv -H_2 \equiv -\frac{e^2}{2mc^2}\left|\mathbf{A}_1\right|^2$$

Lie Transformation

Phase-space coordinates : $\mathbf{z} = (z^i)$

Hamiltonian mechanics :

Motion equations are derived from variational principle $\delta \int \gamma = 0$

Differential 1-form : $\gamma = L\,dt = \mathbf{p} \cdot d\mathbf{q} - H(\mathbf{q},\mathbf{p})dt = \gamma_i(\mathbf{z})\,dz^i - H(\mathbf{z})dt$

determines Lagrangian *L*, Hamiltonian *H*, and Poisson brackets {*f*, *g*}

Lie transformation : $T = \cdots T_3 T_2 T_1$ $T_n = \mathrm{Exp}(\lambda^n L_n)$

Mapping on the phase space

λ **: Expansion parameter** L_n **: Differential operator**

Transformation of coordinates : $\mathbf{z} \rightarrow \mathbf{Z} = T^* \mathbf{z}$

Transformation of 1-form : $\gamma \rightarrow \Gamma = (T^{-1})^* \gamma + dS$

Construct *T* such that *Γ* (or Lagrangian / Hamiltonian) takes a simpler or desired form.

Single-particle phase-space coordinates

Position and velocity : $\left(\mathbf{x}, \mathbf{v}\right)$

Zeroth-order guiding-center coordinates : $\mathbf{z} = \left(\mathbf{x}, v_{0\|}, \mu_0, \xi_0\right), \quad \mu_0 = \dfrac{m v_{0\perp}^2}{2B_0}$

μ_0 **is not conserved exactly in inhomogeneous fields.**

Guiding-center (GC) transformation : $T^{GC} = \cdots T_3^{GC} T_2^{GC} T_1^{GC}$ **Littlejohn, PoF(1981)**

$T_n^{GC} = \mathrm{Exp}(\delta L_n^{GC}), \quad \delta \approx \rho / L$ **(drift ordering parameter)**

Guiding-center (GC) coordinates : $\mathbf{Z} = T_{GC}^* \mathbf{z} = \left(\mathbf{X}, U, \mu, \xi\right)$

μ **is conserved in equilibrium fields.**

μ **is *not* conserved in perturbed fields.**

Gyrocenter (GY) transformation : $T^{GY} = \cdots T_3^{GY} T_2^{GY} T_1^{GY}$ **Brizard & Hahm, RMP(2007)**

$T_n^{GY} = \mathrm{Exp}(\varepsilon^n L_n^{GY}), \quad \varepsilon \approx e\phi / (mv^2 / 2)$

Gyrocenter (GY) coordinates : $\overline{\mathbf{Z}} = T_{GY}^* \mathbf{Z} = \left(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, \overline{\xi}\right)$

$\overline{\mu}$ **is conserved in perturbed fields.**

Gyrocener Coordinates

Gyrocenter coordinates

$$\overline{\mathbf{Z}} = T^{*}_{GY}\mathbf{Z} = \left(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, \overline{\xi}\right)$$

$$= \mathbf{Z} + \varepsilon\left\{\tilde{S}_1, \mathbf{Z}\right\} + O(\varepsilon^2)$$

Gyrocenter Lagrangian

$$L(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, \dot{\overline{\mathbf{X}}}, \dot{\overline{\xi}}, t) = \frac{e}{c}\mathbf{A}^{*}(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}) \cdot \dot{\overline{\mathbf{X}}} + \frac{mc}{e}\overline{\mu}\,\dot{\overline{\xi}} - \overline{H}(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, t) \quad \Longrightarrow$$

independent of gyrophase $\overline{\xi}$

$\Downarrow$

Conservation of magnetic moment $\overline{\mu}$

where $$\mathbf{A}^{*} \equiv \mathbf{A}_0 + \frac{mc}{e}\overline{U}\,\mathbf{b} - \frac{mc^2}{e^2}\overline{\mu}\,\mathbf{W}$$

Gyrocenter Hamiltonian

$$\overline{H}(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, t) = \frac{1}{2}m\overline{U}^{2} + \overline{\mu}\,B_0(\overline{\mathbf{X}}) + e\left\langle \psi(\overline{\mathbf{Z}}, t)\right\rangle_{\overline{\xi}}$$

$$+ \frac{e^2}{2mc^2}\left\langle \left|\mathbf{A}_1(\overline{\mathbf{X}} + \overline{\rho}, t)\right|^2 \right\rangle_{\overline{\xi}} - \frac{e}{2}\left\langle \left\{\tilde{S}_1(\overline{\mathbf{Z}}, t), \psi(\overline{\mathbf{Z}}, t)\right\}\right\rangle_{\overline{\xi}}$$

Electromagnetic fluctuation $$\psi = \phi - \frac{\mathbf{v}_0}{c}\cdot\mathbf{A}_1$$

Gyrophase average $\left\langle \psi \right\rangle_{\overline{\xi}} \equiv \oint \psi \, d\overline{\xi}$

Gyrophase-dependent part $\tilde{\psi} \equiv \psi - \left\langle \psi \right\rangle_{\overline{\xi}}$

Generating function for gyrocenter transformation $$\tilde{S}_1 = \frac{e}{\Omega}\int^{\overline{\xi}} \tilde{\psi}\, d\overline{\xi}$$

Poisson Brackets

Nonvanishing Poisson brackets between gyrocenter coordinates $\overline{\mathbf{Z}} = \left(\overline{\mathbf{X}}, \overline{U}, \overline{\mu}, \overline{\xi}\right)$

$$\left\{\overline{\mathbf{X}}, \overline{\mathbf{X}}\right\} = \frac{c}{eB_{\|}^{*}}\mathbf{b}\times\mathbf{I} \qquad \left\{\overline{\mathbf{X}}, \overline{U}\right\} = \frac{\mathbf{B}^{*}}{mB_{\|}^{*}}$$

$$\left\{\overline{\mathbf{X}}, \overline{\xi}\right\} = \frac{c}{eB_{\|}^{*}}\mathbf{b}\times\mathbf{W} \qquad \left\{\overline{U}, \overline{\xi}\right\} = -\frac{\mathbf{B}^{*}\cdot\mathbf{W}}{mB_{\|}^{*}} \qquad \left\{\overline{\xi}, \overline{\mu}\right\} = \frac{e}{mc}$$

where

$$\mathbf{B}^{*} \equiv \nabla\times\mathbf{A}^{*} \qquad B_{\|}^{*} \equiv \mathbf{b}\cdot\mathbf{B}^{*}$$

$$\mathbf{A}^{*} \equiv \mathbf{A}_{0} + \frac{mc}{e}\overline{U}\,\mathbf{b} - \frac{mc^{2}}{e^{2}}\overline{\mu}\,\mathbf{W}$$

Gyrocenter Motion Equations

Euler-Lagrange equations $$\frac{\delta I}{\delta \overline{\mathbf{Z}}} \equiv \frac{\partial L(\overline{\mathbf{Z}},\dot{\overline{\mathbf{Z}}},t)}{\partial \overline{\mathbf{Z}}} - \frac{d}{dt}\frac{\partial L(\overline{\mathbf{Z}},\dot{\overline{\mathbf{Z}}},t)}{\partial \dot{\overline{\mathbf{Z}}}} = 0$$

are rewritten as Hamiltonian equations $$\frac{d\overline{\mathbf{Z}}}{dt} = \left\{\overline{\mathbf{Z}}, H(\overline{\mathbf{Z}},t)\right\}$$

Gyrocenter motion equations

$$\frac{d\overline{\mathbf{X}}}{dt} = \frac{1}{B_{\parallel}^{*}}\left[\left(\overline{U} + \frac{e}{m}\frac{\partial \Psi(\overline{\mathbf{Z}},t)}{\partial U}\right)\mathbf{B}^{*} + c\times\left(\frac{\overline{\mu}}{e}\nabla B_0 + \nabla\Psi(\overline{\mathbf{Z}},t)\right)\right] \qquad \frac{d\overline{\mu}}{dt} = 0$$

$$\frac{d\overline{U}}{dt} = -\frac{\mathbf{B}^{*}}{B_{\parallel}^{*}}\cdot\left[\overline{\mu}\nabla B_0 + e\nabla\Psi(\overline{\mathbf{Z}},t)\right] \qquad \frac{d\overline{\xi}}{dt} = \Omega + \mathbf{W}\cdot\frac{d\overline{\mathbf{X}}}{dt} + \frac{e^2}{mc}\frac{\partial \Psi(\overline{\mathbf{Z}},t)}{\partial \overline{\mu}}$$

Potential for electromagnetic fluctuations

$$\Psi(\overline{\mathbf{Z}},t) = \left\langle \psi(\overline{\mathbf{Z}},t)\right\rangle_{\overline{\xi}} + \left[\frac{e}{2mc^2}\left\langle\left|\mathbf{A}_1(\overline{\mathbf{X}}+\overline{\rho},t)\right|^2\right\rangle_{\overline{\xi}} - \frac{1}{2}\left\langle\left\{\tilde{S}_1(\overline{\mathbf{Z}},t),\tilde{\psi}(\overline{\mathbf{Z}},t)\right\}\right\rangle_{\overline{\xi}}\right]$$

Gyrokinetic Vlasov-Poisson-Ampere System

Lagrangian

$$L = \sum_a e_a \int d^6\overline{\mathbf{Z}}_0 \, D_a(\overline{\mathbf{Z}}_0) \, F_a(\overline{\mathbf{Z}}_0, t_0) \, L_a\left[\overline{\mathbf{Z}}_a(\overline{\mathbf{Z}}_0, t_0; t), \dot{\overline{\mathbf{Z}}}_a(\overline{\mathbf{Z}}_0, t_0; t), t\right]$$

$$+ \frac{1}{8\pi} \int d^3\mathbf{x} \left(\left| \nabla \phi(\mathbf{x}, t) \right|^2 - \left| \nabla \times \left[\mathbf{A}_0(\mathbf{x}) + \mathbf{A}_1(\mathbf{x}, t) \right] \right|^2 + \frac{2}{c} \lambda(\mathbf{x}, t) \nabla \cdot \mathbf{A}_1(\mathbf{x}, t) \right)$$

$F_a(\overline{\mathbf{Z}}_0, t_0)$ **Initial distribution function**

$D_a(\overline{\mathbf{Z}}_0)$ **Jacobian**

$L_a\left[\overline{\mathbf{Z}}_a(\overline{\mathbf{Z}}_0, t_0; t), \dot{\overline{\mathbf{Z}}}_a(\overline{\mathbf{Z}}_0, t_0; t), t\right]$ **Single-particle Lagrangian**

Governing equations for gyrokinetic Vlasov-Poisson-Ampere system are derived from variational principle $\delta I = \delta \int_{t_1}^{t_2} L \, dt = 0$

$\delta I / \delta \lambda = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{A}_1 = 0$ **(Coulomb gauge)**

Gyrokinetic Vlasov-Poisson-Ampere Equations

Gyrokinetic Vlasov Equation : $\delta I / \delta \overline{\mathbf{Z}}_a = 0$

$$\left[\frac{\partial}{\partial t} + \left\{\overline{\mathbf{Z}}, \overline{H}_a(\overline{\mathbf{Z}},t)\right\} \cdot \frac{\partial}{\partial \overline{\mathbf{Z}}}\right] F_a(\overline{\mathbf{Z}},t) = 0$$

Gyrokinetic Poisson's Equation : $\delta I / \delta \phi = 0$

$$\nabla^2 \phi(\mathbf{x},t) = -4\pi \sum_a e_a \int d^6\overline{\mathbf{Z}}\, D_a(\overline{\mathbf{Z}})\, \delta\left(\mathbf{X} + \overline{\rho}_{a0}(\overline{\mathbf{Z}}) - \mathbf{x}\right)\left[F_a(\overline{\mathbf{Z}},t) + \left\{S_{a1}(\overline{\mathbf{Z}},t), F_a(\overline{\mathbf{Z}},t)\right\}\right]$$

Gyrokinetic Ampere's Law : $\delta I / \delta \mathbf{A}_1 = 0$

$$\nabla^2 \mathbf{A}_1(\mathbf{x},t) = -\frac{4\pi}{c}\left[\mathbf{j}_T(\mathbf{x},t) - \mathbf{j}_0(\mathbf{x},t)\right]$$

Equilibrium current density $\mathbf{j}_0 = -\frac{c}{4\pi}\nabla^2 \mathbf{A}_0$

Transverse part of total current density $\mathbf{j}_T(\mathbf{x},t) \equiv \frac{1}{4\pi}\nabla \times \left(\nabla \times \int d^3\mathbf{x}'\, \frac{\mathbf{j}(\mathbf{x}',t)}{|\mathbf{x} - \mathbf{x}'|}\right)$

Total current density

$$\mathbf{j}(\mathbf{x},t) = \sum_a e_a \int d^6\overline{\mathbf{Z}}\, D_a(\overline{\mathbf{Z}})\, \delta\left(\mathbf{X} + \overline{\rho}_{a0}(\overline{\mathbf{Z}}) - \mathbf{x}\right)$$

$$\times \left(\left[\mathbf{v}_{a0}(\overline{\mathbf{Z}}) - \frac{e_a}{m_a c}\mathbf{A}_1(\overline{\mathbf{X}} + \overline{\rho}_{a0}(\overline{\mathbf{Z}}),t)\right] F_a(\overline{\mathbf{Z}},t) + v_{a0}(\overline{\mathbf{Z}})\left\{S_{a1}(\overline{\mathbf{Z}},t), F_a(\overline{\mathbf{Z}},t)\right\}\right)$$

Energy Conservation

Total Lagrangian L does not depend on t explicitly.

⇩

Noether's theorem ensures conservation of energy E_G^{tot} of the whole system

$$
\begin{aligned}
E_G^{tot} &= \sum_a \int d^6\overline{\mathbf{Z}}_0 D_a(\overline{\mathbf{Z}}_0) F_a(\overline{\mathbf{Z}}_0, t_0) \dot{\overline{\mathbf{Z}}}_a \cdot \frac{\partial L_a(\overline{\mathbf{Z}}_a, \dot{\overline{\mathbf{Z}}}_a, t)}{\partial \dot{\overline{\mathbf{Z}}}_a} - L \\
&= \sum_a \int d^6\overline{\mathbf{Z}}\, D_a(\overline{\mathbf{Z}}) F_a(\overline{\mathbf{Z}}, t) \overline{H}_a(\overline{\mathbf{Z}}, t) - L_f \\
&= \sum_a \int d^6\overline{\mathbf{Z}}\, D_a(\overline{\mathbf{Z}}) F_a(\overline{\mathbf{Z}}, t) \left(\frac{1}{2} m_a \left[\mathbf{v}_{a0}(\overline{\mathbf{Z}}) - \frac{e_a}{m_a c} \mathbf{A}_1(\overline{\mathbf{X}} + \overline{\rho}_{a0}(\overline{\mathbf{Z}}), t) \right]^2 \right. \\
&\qquad \left. + \frac{e_a^2}{2\Omega_a(\overline{\mathbf{X}})} \left[\left\{ \int \tilde{\phi}_a d\overline{\xi}, \tilde{\phi}_a \right\} - \frac{1}{c^2} \left\{ \int \widetilde{(\mathbf{v}_0 \cdot \mathbf{A}_1)} d\overline{\xi}, \widetilde{(\mathbf{v}_0 \cdot \mathbf{A}_1)} \right\} \right] \right) \\
&\qquad + \frac{1}{8\pi} \int d^3\mathbf{x} \left(\left| \nabla \phi(\mathbf{x}, t) \right|^2 + \left| \nabla \times \left[\mathbf{A}_0(\mathbf{x}, t) + \mathbf{A}_1(\mathbf{x}, t) \right] \right|^2 \right)
\end{aligned}
$$

Summary of Part I

- Gyrokinetic Vlasov-Poisson-Ampere equations are all derived from the Lagragian for the whole system.
- Total energy conservation is shown directly from Noether's theorem.
- Simplified gyrokinetic system of equations, which satisfy total energy conservation, can be obtained by simplified Lagrangian in limiting cases.

Examples) small electron gyroradius
quasineutrality
linear polarization-magnetization

References

Sugama, Phys. Plasmas, 7, 466 (2000)

Brizard & Hahm, Rev. Mod. Phys. 79, 421 (2007)

Part II

Entropy Balance in Neoclassical and Turbulent Transport

- **Gyrokinetic Equation with Collision Term**
- **Particle and Energy Balance Equations**
  - **Anomalous particle and energy fluxes**
- **Entropy Balance for Toroidal Plasmas**
  - **Entropy associated with turbulent fluctuations**
- **Slab ITG Turbulence**
  - **Kinetic and fluid simulations**
  - **Entropy transfer from macro to microscopic scales in velocity space**
- **Toroidal ITG Turbulence**

Gyrokinetic Equation with Collision Term
(Boltzmann Equation)

Boltzmann eq. $\boxed{\dfrac{dF}{dt} = C(F)}$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$
$$= \frac{\partial F}{\partial t} + \mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial F}{\partial \mathbf{v}}$$
$$= \frac{\partial F}{\partial t} + \frac{d\bar{\mathbf{X}}}{dt}\cdot\frac{\partial F}{\partial \bar{\mathbf{X}}} + \frac{d\bar{U}}{dt}\cdot\frac{\partial F}{\partial \bar{U}}$$

stationary part & fluctuation part

$$H = H_0 + H_1$$
$$F = F_0 + F_1$$

Stationary part of Boltzmann equation

$$\boxed{\{F_0, H_0\} = C(F_0)} \qquad F_0 = F_M + F_{01} \qquad F_M = F_M(\bar{\mathbf{X}}, H_0)$$

Local Maxwellian

→ **Drift-kinetic eq.** → **Neoclassical transport is derived from** F_{01}

Gyrokinetic eq. for fluctuation part of distribution function

$$\boxed{\frac{\partial F_1}{\partial t} + \{F_1, H_0 + H_1\} + \{F_0, H_1\} = C(F_1)}$$

$$F_1 = H_1\frac{\partial F_0}{\partial H_0} + h$$

↳ **Nonadiabatic part**

$$\boxed{\frac{\partial h}{\partial t} + \{h, H_0 + H_1\} = C(F_1) - \frac{\partial H_1}{\partial t}\frac{\partial F_0}{\partial H_0} - \{\bar{\mathbf{X}}, H_1\}\cdot\frac{\partial F_0}{\partial \bar{\mathbf{X}}}}$$

Anomalous transport is derived from h

Gyrokinetic Equation for Nonadiabatic Part of Perturbed Distribution Function

$$F = f(\mathbf{x},\mathbf{v},t) + \delta f(\mathbf{x},\mathbf{v},t) \longrightarrow$$

function of particle coordinates

$$= F(\overline{\mathbf{X}},\overline{\varepsilon},\overline{\mu},t) = F_0(\overline{\mathbf{X}},\overline{\varepsilon},\overline{\mu},t) + F_1(\overline{\mathbf{X}},\overline{\varepsilon},\overline{\mu},t) \longrightarrow$$

function of gyrocenter coordinates

$$\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}$$

Perturbed particle distribution function

$$\delta f = F_1 + \left[e\phi_1(\mathbf{x},t) - H_1\right]\frac{\partial F_M}{\partial H_0} = \underset{\substack{\big\downarrow \\ \textbf{adiabatic} \\ \textbf{part}}}{e\phi_1(\mathbf{x},t)\frac{\partial F_M}{\partial H_0}} + \underset{\substack{\big\downarrow \\ \textbf{nonadiabatic} \\ \textbf{part}}}{h(\mathbf{X},\varepsilon,\mu,t)}$$

WKB (or Eikonal) representation $\phi_1(\mathbf{x}) = \sum \phi(\mathbf{k}_\perp)\exp\left[i\,S(\mathbf{x})\right]$ $\qquad \mathbf{k}_\perp = \nabla S$

Gyrokinetic equation for $h(\mathbf{X}) = \sum h(\mathbf{k}_\perp)\exp\left[i\,S(\mathbf{X})\right]$

$$\boxed{\begin{aligned}&\left(\frac{\partial}{\partial t} + i\mathbf{k}_\perp\cdot\mathbf{v}_D + v_\parallel\mathbf{b}\cdot\nabla_\parallel\right)h(\mathbf{k}_\perp) - \left\langle e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}C\left[h(\mathbf{k}_\perp)e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}\right]\right\rangle_\varphi \\ &= \frac{e}{T}F_M\left(\frac{\partial}{\partial t} + i\omega_*^T + v_\parallel\mathbf{b}\cdot\nabla_\parallel\right)\psi(\mathbf{k}_\perp) + \frac{c}{B}\sum_{k'+k''=k}\left[\mathbf{b}\cdot\left(\mathbf{k}'_\perp\times\mathbf{k}''_\perp\right)\right]\psi(\mathbf{k}'_\perp)\,h(\mathbf{k}''_\perp)\end{aligned}}$$

Gyrophase-averaged potential of electromagnetic field

$$\psi(\mathbf{k}_\perp) = J_0(k_\perp\rho)\left\{\phi(\mathbf{k}_\perp) - \frac{v_\parallel}{c}A_\parallel(\mathbf{k}_\perp)\right\} + J_1(k_\perp\rho)\frac{v_\perp}{c}\frac{B_\parallel(\mathbf{k}_\perp)}{k_\perp}$$

Equations for Electromagnetic Fields

Poisson's equation

$$\left(k_\perp^2 + \lambda_D^{-2}\right)\phi(\mathbf{k}_\perp) = 4\pi \sum_a e_a \int d^3v\, h_a(\mathbf{k}_\perp) J_0(k_\perp v_\perp / \Omega_a)$$

Debye length $\lambda_D \equiv \left(4\pi \sum_a n_a e_a^2 / T_a\right)^{-1/2}$

Ampere's law

$$k_\perp^2 A_\parallel(\mathbf{k}_\perp) = \frac{4\pi}{c} \sum_a e_a \int d^3v\, v_\parallel h_a(\mathbf{k}_\perp) J_0(k_\perp v_\perp / \Omega_a)$$

$$-k_\perp B_\parallel(\mathbf{k}_\perp) = \frac{4\pi}{c} \sum_a e_a \int d^3v\, v_\perp h_a(\mathbf{k}_\perp) J_1(k_\perp v_\perp / \Omega_a)$$

Anomalous Transport Fluxes of Particles and Heat

Anomalous particle flux
(in the radial direction)

$$J_{a1}^{A} = \left\langle\left\langle \sum_{\mathbf{k}_\perp} \int d^3v\, h_a^*(\mathbf{k}_\perp)\, \mathbf{v}_{da}(\mathbf{k}_\perp)\cdot\nabla r \right\rangle\right\rangle$$

Anomalous heat flux

$$J_{a2}^{A} = \left\langle\left\langle \sum_{\mathbf{k}_\perp} \int d^3v \left(\frac{m_a v^2}{2} - \frac{5}{2}\right) h_a^*(\mathbf{k}_\perp)\, \mathbf{v}_{da}(\mathbf{k}_\perp)\cdot\nabla r \right\rangle\right\rangle$$

Nonadiabatic part of distribution function $h_a(\mathbf{k}_\perp)$

Gyrocenter velocity due to electromagnetic fluctuations

$$\begin{aligned}\mathbf{v}_{da}(\mathbf{k}_\perp) &= -i\frac{c}{B}\left(\mathbf{k}_\perp\times\mathbf{b}\right)\psi(\mathbf{k}_\perp)\\ &= -i\frac{c}{B}\left(\mathbf{k}_\perp\times\mathbf{b}\right)\left[J_0(k_\perp\rho)\left\{\phi(\mathbf{k}_\perp) - \frac{v_\parallel}{c}A_\parallel(\mathbf{k}_\perp)\right\} + J_1(k_\perp\rho)\frac{v_\perp}{c}\frac{B_\parallel(\mathbf{k}_\perp)}{k_\perp}\right]\end{aligned}$$

Particle and Energy Balance Equations for Toroidal Plasmas

$$\frac{dF}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]F = C(F)$$

Ensemble average

$$F = f + \delta f \qquad f = \langle F\rangle_{\mathrm{ens}} \qquad \mathbf{E} = \langle \mathbf{E}\rangle_{\mathrm{ens}} + \delta\mathbf{E} \qquad \mathbf{B} = \langle \mathbf{B}\rangle_{\mathrm{ens}} + \delta\mathbf{B}$$

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}\left(\langle\mathbf{E}\rangle_{\mathrm{ens}} + \frac{1}{c}\mathbf{v}\times\langle\mathbf{B}\rangle_{\mathrm{ens}}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = C + D \qquad D = -\frac{e}{m}\left\langle\left(\delta\mathbf{E} + \frac{1}{c}\mathbf{v}\times\delta\mathbf{B}\right)\cdot\frac{\partial \delta f}{\partial \mathbf{v}}\right\rangle_{\mathrm{ens}}$$

Particle balance

$$\frac{\partial n_a}{\partial t} + \frac{1}{V'}\frac{\partial\left(V'J_{a1}\right)}{\partial r} = 0$$

Particle density $n_a = \langle n_a\rangle$

Temperature $T_a = \langle T_a\rangle$

Pressure $p_a = n_a T_a$

Energy balance

$$\frac{3}{2}\frac{\partial p_a}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial r}\left[V'T_a\left(J_{a1} + \frac{5}{2}J_{a2}\right)\right]$$

$$= \frac{J_{a1}}{n_a}\frac{\partial p_a}{\partial r} + \left\langle \mathbf{u}_a\cdot\left(\nabla\cdot\pi_a\right)\right\rangle + \left\langle \int d^3v\,\frac{1}{2}m_a(v-u_a)^2\left(C+D\right)\right\rangle$$

Particle flux

$$J_{a1} = n_a\langle\mathbf{u}_a\cdot\nabla r\rangle = J_{a1}^{\mathrm{cl}} + J_{a1}^{\mathrm{ncl}} + J_{a1}^{A}$$

Heat flux

$$J_{a2} = \frac{1}{T_a}\langle\mathbf{q}_a\cdot\nabla r\rangle = J_{a2}^{\mathrm{cl}} + J_{a2}^{\mathrm{ncl}} + J_{a2}^{A}$$

Entropy associated with turbulent fluctuations

Microscopic entropy per unit volume is defined in terms of $F = f + \delta f$ **by**

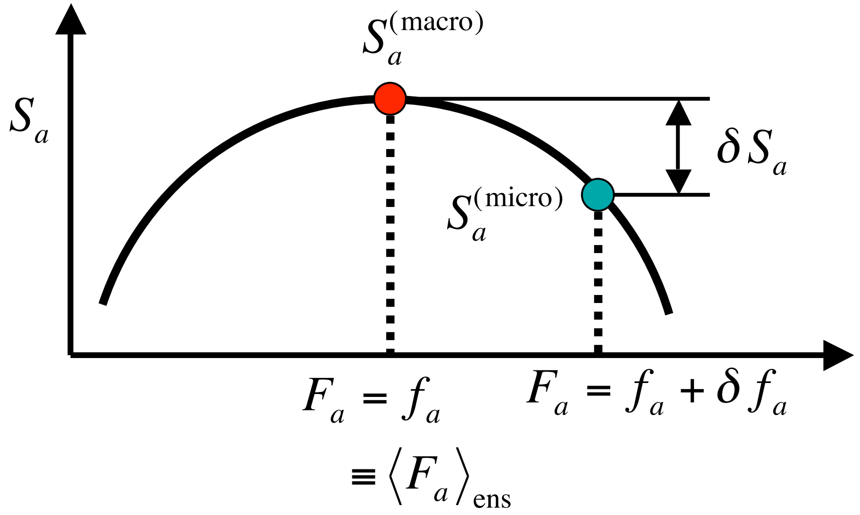
$$S_a^{(\mathrm{micro})} = -\int d^3v\ F_a \ln F_a = -\int d^3v\ (f_a + \delta f_a) \ln(f_a + \delta f_a)$$

Macroscopic entropy per unit volume is defined in terms of $f = \langle F \rangle_{\mathrm{ens}}$ **by**

$$S_a^{(\mathrm{macro})} = -\int d^3v\ f_a \ln f_a$$

Entropy associated with turbulent fluctuations is defined by

$$\delta S_a = S_a^{(\mathrm{macro})} - \left\langle S_a^{(\mathrm{micro})} \right\rangle_{\mathrm{ens}} \approx \frac{1}{2} \left\langle \int d^3v \frac{(\delta f_a)^2}{f_a} \right\rangle_{\mathrm{ens}} \approx \frac{1}{2} \left\langle \int d^3v \frac{(\delta f_a)^2}{f_{aM}} \right\rangle_{\mathrm{ens}}$$



Entropy Balance for Toroidal Plasmas

Flux-surface-averaged entropy balance equation for macroscopic entropy $S_a^{(\mathrm{macro})}$

$$\frac{\partial\left\langle S_a^{(\mathrm{macro})}\right\rangle}{\partial t} + \frac{1}{V'}\frac{\partial\left(V' J_{Sa}\right)}{\partial r} = \sigma_a$$

σ_a **Entropy production rate**

Radial transport flux of entropy $J_{Sa} = S_a u_a + \frac{q_a}{T_a} = \frac{S_a}{n_a}\left(J_{a1}^{\mathrm{cl}} + J_{a1}^{\mathrm{ncl}} + J_{a1}^{A}\right) + \left(J_{a2}^{\mathrm{cl}} + J_{a2}^{\mathrm{ncl}} + J_{a2}^{A}\right)$

Product of gradient forces $(X_{a1}, X_{a2}, X_E) = \left(-n_a^{-1}(\partial p_a/\partial r), -(\partial T_a/\partial r), \langle BE_{\parallel}\rangle/\langle B^2\rangle^{1/2}\right)$

and transport fluxes $(J_{a1}, J_{a2}, J_E) = \left(\langle n_a \mathbf{u}_a \cdot \nabla r\rangle, \langle n_a \mathbf{q}_a \cdot \nabla r\rangle/T_a, \langle BJ_{\parallel}\rangle/\langle B^2\rangle^{1/2}\right)$

yields entropy. $\sum_a T_a\sigma_a = \sum_a\left[\left(J_{a1}^{\mathrm{cl}} + J_{a1}^{\mathrm{ncl}} + J_{a1}^{A}\right)X_{a1} + \left(J_{a2}^{\mathrm{cl}} + J_{a2}^{\mathrm{ncl}} + J_{a2}^{A}\right)X_{a2}\right] + J_E X_E$

Balance equation for entropy associated with turbulent fluctuations

$$\frac{\partial}{\partial t}\left\langle \sum_a T_a \delta S_a + \underbrace{\frac{1}{8\pi}\left|\nabla_{\perp}\phi\right|^2}_{\text{This vanishes when using quasineutrality condition}}\right\rangle = \underbrace{\sum_a\left(J_{a1}^{A}X_{a1} + J_{a2}^{A}X_{a2}\right)}_{\text{Production due to anomalous particle and heat transport}} + \underbrace{\sum_a T_a\left\langle\left\langle \int d^3v\, \frac{\delta f_a}{f_{aM}} C_a(\delta f_a)\right\rangle\right\rangle}_{\text{Dissipation due to collisions}}$$

Relation between perturbed particle and gyrocenter distribution functions

Perturbed particle distribution function $\delta f^{(p)}$ **is related to**

perturbed gyrocenter distrbution function $\delta f^{(g)}$ **by**

$$\delta f^{(p)}(\mathbf{x}=\mathbf{X}+\rho, v_{\parallel}, \mu, \varphi) = \delta f^{(g)}(\mathbf{X}, v_{\parallel}, \mu) - f_M \frac{e}{T}\underbrace{\left[\phi(\mathbf{X}+\rho) - \langle \phi(\mathbf{X}+\rho)\rangle_{\varphi}\right]}_{\text{polarization}}$$

Entropy associated with perturbed particle and gyrocenter distribution functions

$$\boxed{\delta S_a^{(p)} = \frac{1}{2}\left\langle \int d^3v \frac{\left(\delta f_a^{(p)}\right)^2}{f_{aM}} \right\rangle_{\text{ens}}} \qquad \boxed{\delta S_a^{(g)} = \frac{1}{2}\left\langle \int d^3v \frac{\left(\delta f_a^{(g)}\right)^2}{f_{aM}} \right\rangle_{\text{ens}}}$$

$\delta S_a^{(p)}$ **and** $\delta S_a^{(g)}$ **are related with each other by**

$$\boxed{\sum_a T_a \delta S_a^{(p)} = \sum_a T_a \delta S_a^{(g)} + W^{(\text{pol})}}$$

where

$$\boxed{W^{(\text{pol})} = \sum_a \frac{e_a^2}{T_a} \int d^3v\, F_M \left\langle \left[\phi(\mathbf{X}+\rho) - \langle \phi(\mathbf{X}+\rho)\rangle_{\varphi}\right]^2 \right\rangle_{\text{ens}}}$$

energy density due to polarization

…… redudces to ExB kinetic energy in the low $k\rho$ limit.

Basic Equations of 2D Slab ITG Turbulence

Ion gyrokinetic equation

$$\partial_t \tilde{f}_{\mathbf{k}}(v_\parallel) + ik_y \Theta v_\parallel \tilde{f}_{\mathbf{k}}(v_\parallel) + \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \left(k'_y k''_x - k'_x k''_y\right) \Psi_{\mathbf{k}'} \tilde{f}_{\mathbf{k}''}(v_\parallel)$$
$$= -ik_y \Psi_{\mathbf{k}} F_M(v_\parallel) \left[1 + \frac{\eta_i}{2}\left(v_\parallel^2 - 1 - k^2\right) + \Theta v_\parallel\right] + C\left[\tilde{f}_{\mathbf{k}}(v_\parallel)\right]$$

2D real space (symmetry in *z*-direction),
1D velocity $v_\parallel$ space (Maxwellian assumed for $v_\perp$ space)

Model collision operator

$$C\left[\tilde{f}_{\mathbf{k}}(v_\parallel)\right] = \nu \frac{\partial}{\partial v_\parallel}\left[\frac{\partial \tilde{f}_{\mathbf{k}}(v_\parallel)}{\partial v_\parallel} + v_\parallel \tilde{f}_{\mathbf{k}}(v_\parallel)\right]$$

Quasineutrality condition and adiabatic electron response

$$\exp\left(-b_{\mathbf{k}}/2\right) n_{\mathbf{k}} - n_0 \frac{e\phi_{\mathbf{k}}}{T_i}\left[1 - \Gamma_0(b_{\mathbf{k}})\right] = \frac{e\phi_{\mathbf{k}}}{T_e} \quad \text{for} \quad k_\parallel \neq 0$$

Zonal-flow components neglected $f_{\mathbf{k}} = \phi_{\mathbf{k}} = 0 \quad \text{for} \quad k_\parallel = 0$

Slab ITG Modes

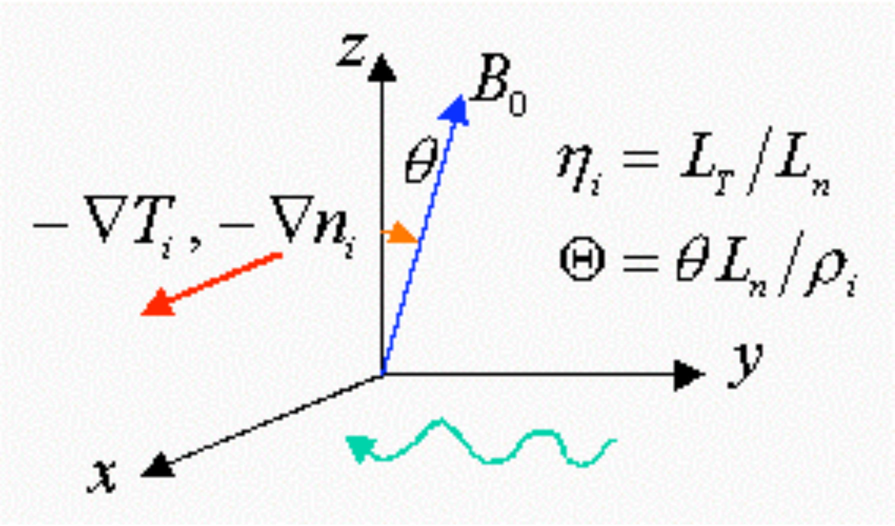
Linear Kinetic Dispersion Relation of Slab ITG Modes

$$D_{\mathbf{k}}(\omega) = 1 + \frac{T_i}{T_e} - \frac{1}{n_0}\int_L dv_\parallel F_M \frac{\omega - \omega_{*i}\left\{1 + \eta_i\left(m_i v_\parallel^2/2T_i - 1/2 - b + b I_1(b)/I_0(b)\right)\right\}}{\omega - k_\parallel v_\parallel} = 0$$

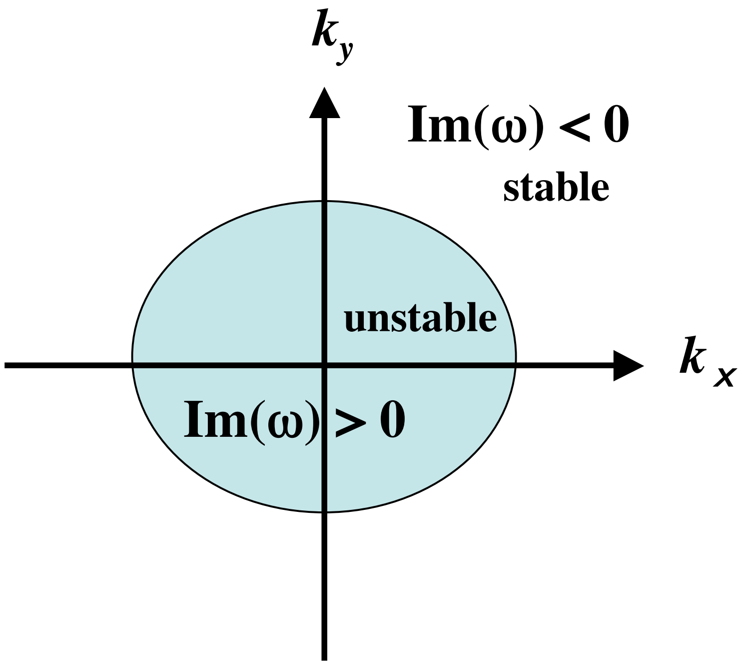
Slab Configuration

homogeneity in the z-direction

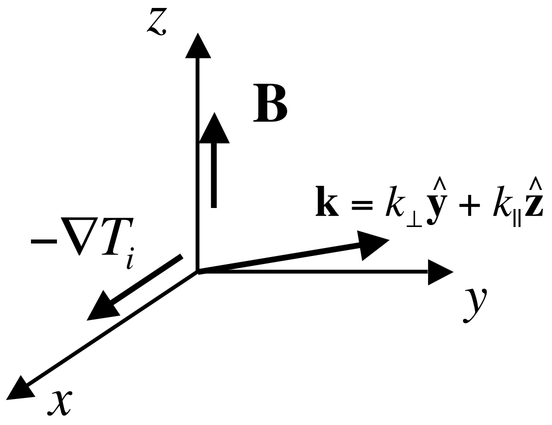
no magnetic shear



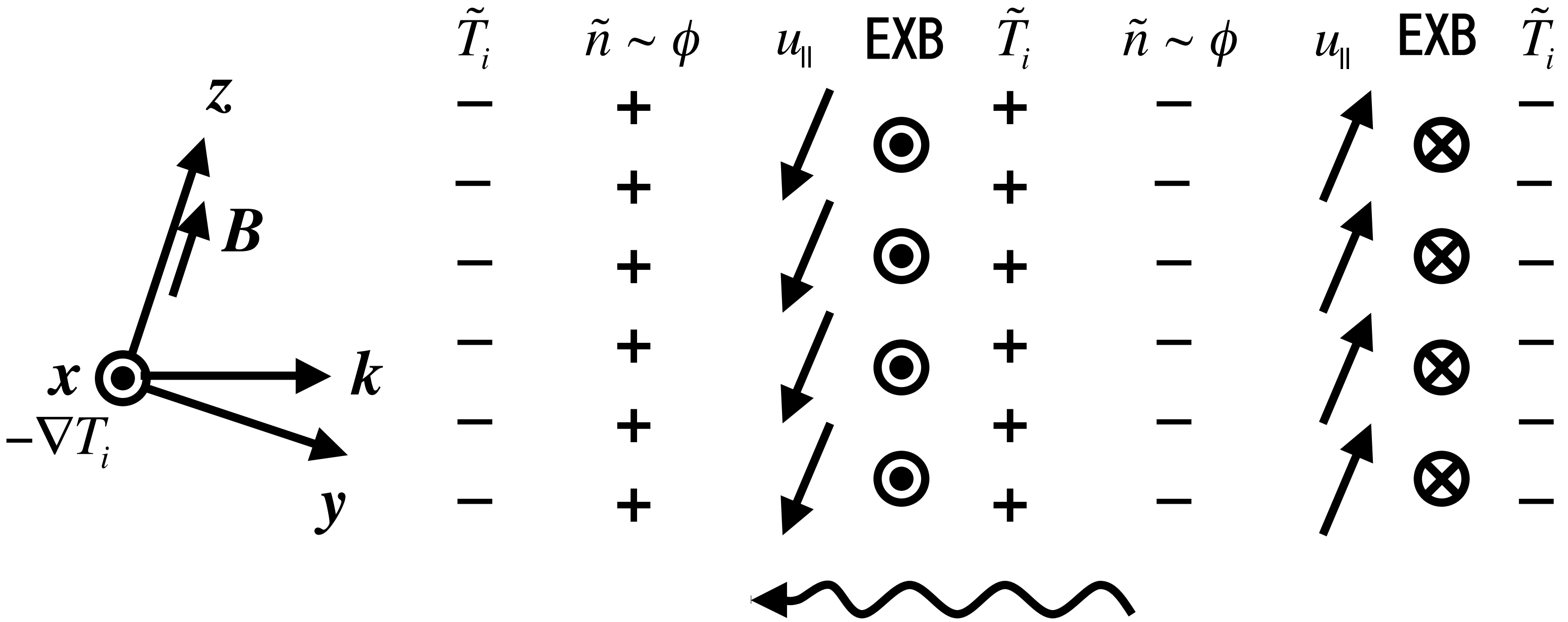
(k_x, k_y) -plane



Physical mechanism of the slab ITG mode



$u_\parallel$	**ion parallel flow**
$\frac{\tilde{n}_i}{n_i} = \frac{\tilde{n}_e}{n_e} = \frac{e\phi}{T_e}$	**quasineutrality and adiabatic electrons**
$-c\frac{\nabla\phi \times \mathbf{b}}{B}$	**EXB drift**



propagation in the ion diamagnetic direction

An entropy balance equation is obtained by taking the phase-space integral of the basic equation multiplied by $\tilde{f}_{\mathbf{k}}(v_{\parallel})$.

$$\boxed{\frac{d}{dt}\left\{\delta S + W\right\} = \eta_i Q_i + D}$$

$$\left\{\begin{array}{ll}
\delta S = \displaystyle\sum_{\mathbf{k}} \int dv_{\parallel} \left|\tilde{f}_{\mathbf{k}}\right|^2 \Big/ 2F_M & \textbf{(entropy variable)} \\
Q_i = \displaystyle\sum_{\mathbf{k}} \int dv_{\parallel} \left(-ik_y e^{-k^2/2} \Phi_{\mathbf{k}}\right) v_{\parallel}^2 \tilde{f}_{-\mathbf{k}} / 2 & \textbf{(turbulent energy flux)} \\
W = \displaystyle\sum_{\mathbf{k}} \left(1 - \Gamma_0(k^2) + \left(T_i / T_e\right)\left[1 - \delta(k_y)\right]\right) \left|\Phi_{\mathbf{k}}\right|^2 \Big/ 2 & \textbf{(potential energy)} \\
D = \displaystyle\sum_{\mathbf{k}} \int dv_{\parallel} \left(\tilde{f}_{-\mathbf{k}} / F_M\right) C\left[\tilde{f}_{\mathbf{k}}\right] \quad \mathbf{< 0} & \textbf{(collisional dissipation)}
\end{array}\right.$$

Entropy paradox [Krommes & Fu, PoP(1994)]

For collisionless case,
we have no transport $\boldsymbol{Q_{\mathrm{i}} = 0}$ in steady state $\boldsymbol{d(\delta S + W)/dt = 0}$
or constant transport $\boldsymbol{Q_{\mathrm{i}}}$ = const with monotonic increase in $\boldsymbol{\delta S}$

Constant thermal flux

$$Q_i = \text{const.} \qquad \Longrightarrow \qquad \frac{d}{dt}\delta S = \text{const.}$$

generation of the fine-scale structures in $\tilde{f}$

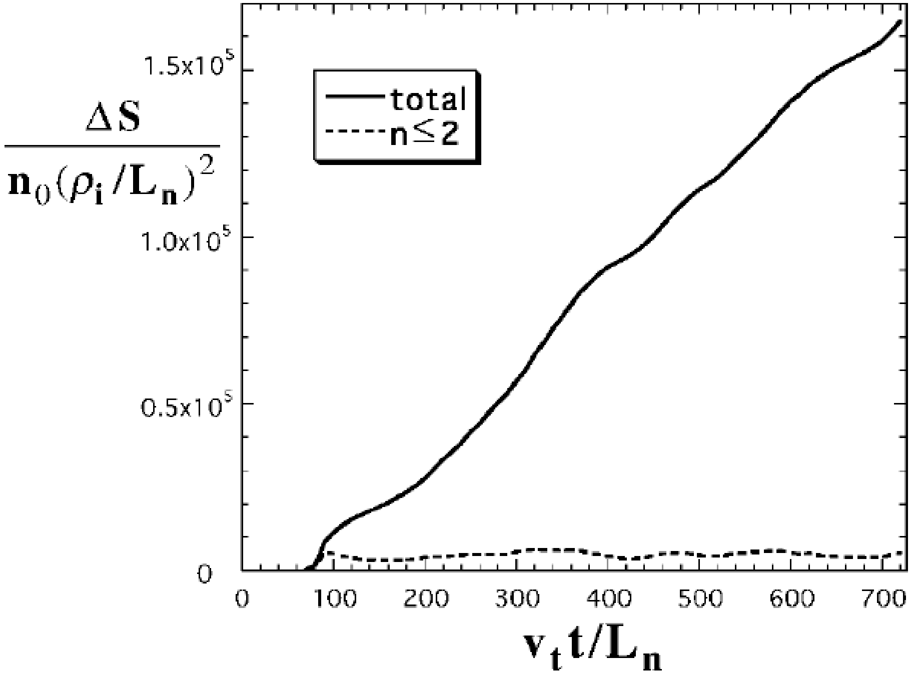
Entropy variable cosists of all-order fluid variables.

Entropy variable

$$\delta S = \sum_{\mathbf{k}} \left(\frac{|n_{\mathbf{k}}|^2}{2} + \frac{|u_{\mathbf{k}}|^2}{2} + \frac{|T_{\mathbf{k}}|^2}{4} + \frac{|q_{\mathbf{k}}|^2}{12} + \sum_{n \geq 4} \frac{n!}{2} |\varphi_{n\mathbf{k}}|^2 \right)$$

Fluid variables

$$\begin{cases} n_{\mathbf{k}} = \int dv_{\|} \tilde{f}_{\mathbf{k}} \\ u_{\mathbf{k}} = \int dv_{\|} \tilde{f}_{\mathbf{k}} v_{\|} \\ T_{\mathbf{k}} = \int dv_{\|} \tilde{f}_{\mathbf{k}} \left(v_{\|}^2 - 1 \right) \\ q_{\mathbf{k}} = \int dv_{\|} \tilde{f}_{\mathbf{k}} \left(v_{\|}^3 - 3v_{\|} \right) \end{cases}$$



[Sugama, Watanabe & Horton, PoP (2003)]

Colisionless slab ITG simulation shows a quasisteady state with

constant thermal flux $Q_i = \text{const.}$

monotonic increase in δS $\frac{d}{dt}\delta S = \text{const.}$

saturation in amplitudes of low-order fluid variables: n_k, u_k, T_k, …

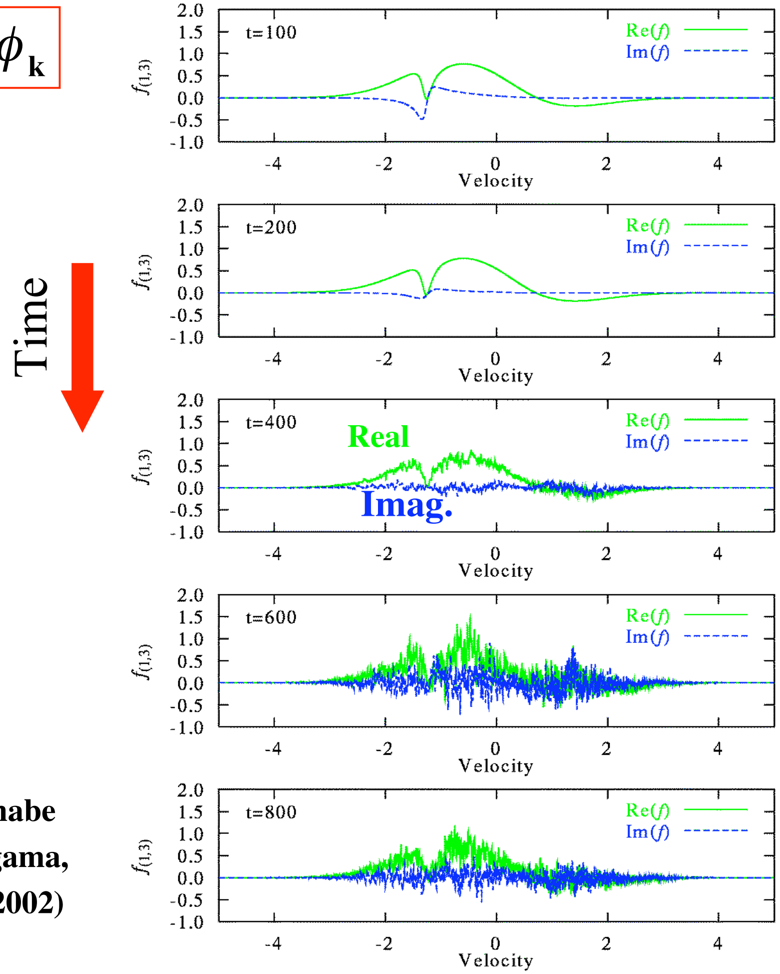
⇩

Generation of high-n moments $|\varphi_{n\mathbf{k}}|^2$ leads to increase of δS.

Ion distribution function in the velocity space

the linearly most unstable mode normalized by potential
in the **collisionless** case with **no zonal flow**

$f_{\mathbf{k}} / \phi_{\mathbf{k}}$



Watanabe & Sugama, PoP (2002)

The monotonical increase of δS results from continuous generation of fine-scale fluctuations of $\widetilde{f}_{\mathbf{k}}(v_{\|})$ due to the phase mixing.

Macroscopit structure of $\mathrm{Im}(f_{\mathbf{k}} / \phi_{\mathbf{k}})$ in the nonlinear stages is different from that in the linear stage.

The phase relation between $T_{\mathbf{k}}$ and $q_{\mathbf{k}}$ in the nonlinear stages can change from that in a linear stage.

Number of grid points in the parallel velocity space = 8193

When fine-scale structures of ballistic modes reach the grid scale in the $v_{\|}$-space, stop the Vlasov simulation !

Collisionless Kinetic-Fluid Model Equations

Equations for the ion gyrocenter density, parallel velocity, and temperature are obtained by taking velocity-space moments of the ion gyrokinetic equation.

$$\partial_t n_{\mathbf{k}} + i k_{\|} n_0 u_{\mathbf{k}} - i\omega_{*i} n_0 \left(1 - \frac{b_{\mathbf{k}}}{2}\eta_i\right)\frac{e\Psi_{\mathbf{k}}}{T_i} - \frac{c}{B}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [\mathbf{b}\cdot(\mathbf{k}'\times\mathbf{k}'')]\Psi_{\mathbf{k}'} n_{\mathbf{k}''} = 0,$$

$$n_0 m_i \partial_t u_{\mathbf{k}} + i k_{\|}\left(T_i n_{\mathbf{k}} + n_0 T_{\mathbf{k}} + n_0 e\Psi_{\mathbf{k}}\right) - \frac{n_0 m_i c}{B}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [\mathbf{b}\cdot(\mathbf{k}'\times\mathbf{k}'')]\Psi_{\mathbf{k}'} u_{\mathbf{k}''} = 0,$$

$$n_0 \partial_t T_{\mathbf{k}} + i k_{\|}\left(2 n_0 T_i u_{\mathbf{k}} + q_{\mathbf{k}}\right) - i\omega_{*i}\eta_i n_0 e\Psi_{\mathbf{k}} - \frac{n_0 c}{B}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [\mathbf{b}\cdot(\mathbf{k}'\times\mathbf{k}'')]\Psi_{\mathbf{k}'} T_{\mathbf{k}''} = 0,$$

$$\Psi_{\mathbf{k}} \equiv \phi_{\mathbf{k}} \exp(-b_{\mathbf{k}}/2) \qquad b_{\mathbf{k}} \equiv k_{\perp}^2 T_i/(m_i \Omega_i^2)$$

Closure models for $q_{\mathbf{k}}$

Nondissipative closure model (NCM) **[Sugama, Watanabe & Horton, PoP (2001)]**

$$q_{\mathbf{k}} = C_{T\mathbf{k}} n_0 v_t T_{\mathbf{k}} + C_{u\mathbf{k}} n_0 T_i u_{\mathbf{k}}$$ **for unstable modes**

Hammett-Perkins model
[PRL(1990)]

$$q_{\mathbf{k}} = -n_0 \chi_{\|}^{hp} i k_{\|} T_{\mathbf{k}}$$ **for stable modes**

[FLR closure by Dorland & Hammett PoF B (1993), Toroidal closure by Beer & Hammett PoP (1996)]

From equations of fluid moments, $n_{\mathbf{k}}$, $u_{\mathbf{k}}$, and $T_{\mathbf{k}}$, we obtain

$$\frac{d}{dt}\sum_{\mathbf{k}}\left(\frac{|n_{\mathbf{k}}|^2}{2}+\frac{|u_{\mathbf{k}}|^2}{2}+\frac{|T_{\mathbf{k}}|^2}{4}\right)+\frac{dW}{dt}=\eta_i Q_i+\sum_{\mathbf{k}}\mathrm{Re}\left(\frac{ik_\parallel}{2}T_{\mathbf{k}}q_{\mathbf{k}}^*\right)$$

$$\boxed{\begin{cases} n_{\mathbf{k}} = \int dv_\parallel \tilde{f}_{\mathbf{k}} \\ u_{\mathbf{k}} = \int dv_\parallel \tilde{f}_{\mathbf{k}} v_\parallel \\ T_{\mathbf{k}} = \int dv_\parallel \tilde{f}_{\mathbf{k}} \left(v_\parallel^2 - 1\right) \\ q_{\mathbf{k}} = \int dv_\parallel \tilde{f}_{\mathbf{k}} \left(v_\parallel^3 - 3v_\parallel\right) \end{cases}}$$

Using the Hermite polynomial expansion of $\tilde{f}_{\mathbf{k}}(v_\parallel)$, entropy balance equation in the collisionless case is written as

$$\frac{d}{dt}\sum_{\mathbf{k}}\left(\frac{|n_{\mathbf{k}}|^2}{2}+\frac{|u_{\mathbf{k}}|^2}{2}+\frac{|T_{\mathbf{k}}|^2}{4}+\frac{|q_{\mathbf{k}}|^2}{12}+\sum_{n\geq 4}\frac{n!}{2}|\varphi_{n\mathbf{k}}|^2\right)+\frac{dW}{dt}=\eta_i Q_i$$

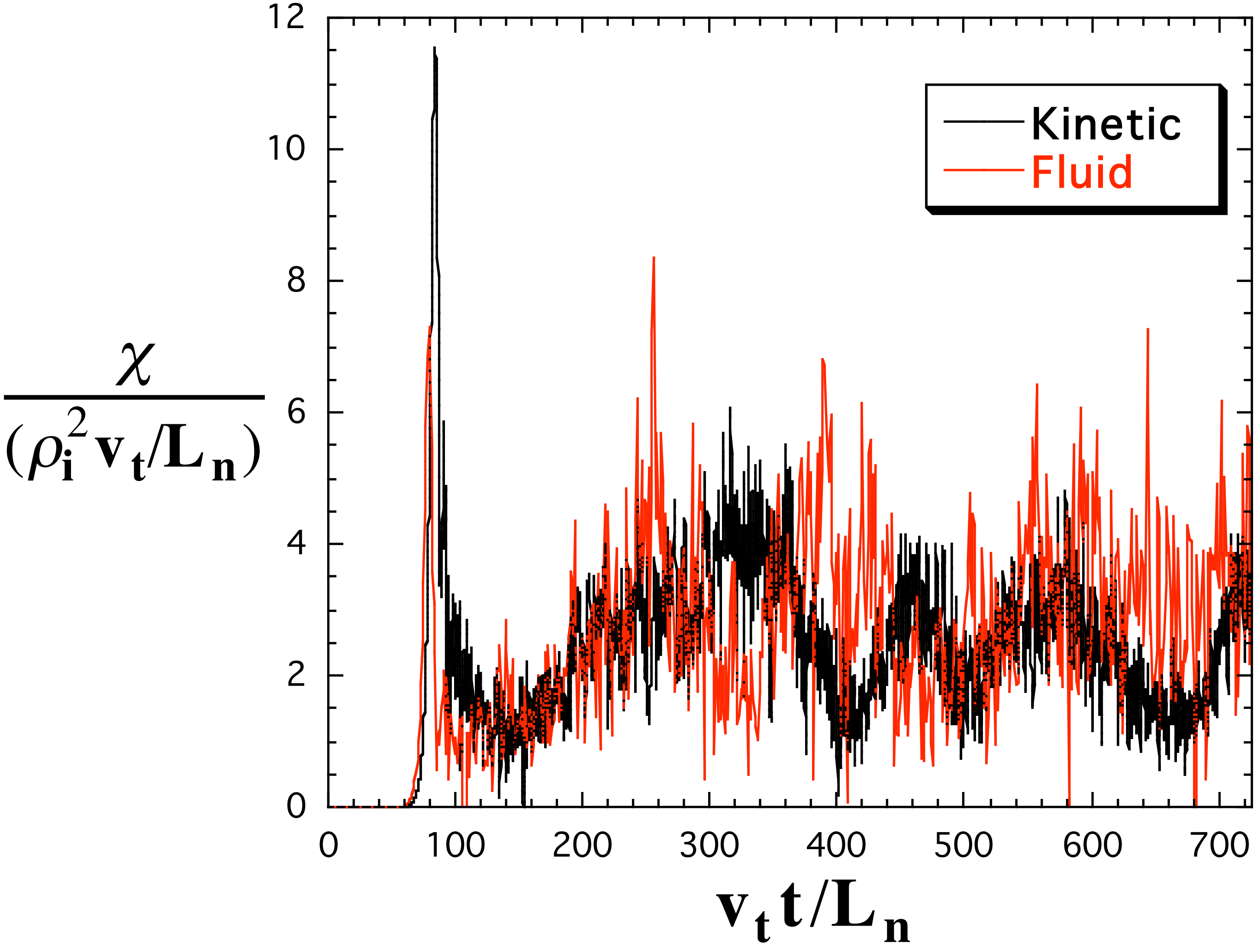
In the case that the lower-order (n = 0,1,2,3) moments are constant, comparison of the above two equations gives

$$\boxed{\eta_i Q_i = -\sum_{\mathbf{k}}\mathrm{Re}\left(\frac{ik_\parallel}{2}T_{\mathbf{k}}q_{\mathbf{k}}^*\right)=\frac{d}{dt}\sum_{\mathbf{k}}\sum_{n\geq 4}\frac{n!}{2}|\varphi_{n\mathbf{k}}|^2}$$

The above relation represents that growth of the high-n moments is driven by the transport through the correlation between $T_{\mathbf{k}}$ and $q_{\mathbf{k}}$. When one considers a steady transport in a collisionless fluid model, thus, it implicitly assumes existence of the quasi-steady state where $n_{\mathbf{k}}$, $u_{\mathbf{k}}$, $T_{\mathbf{k}}$ and $q_{\mathbf{k}}$ are saturated but the high-n moments continue to grow.

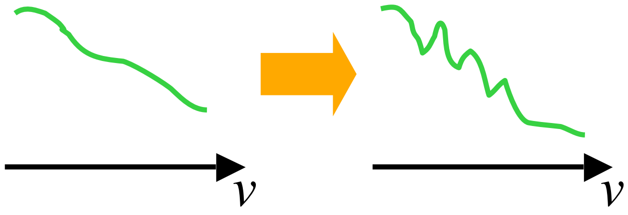
Time Evolution of Turbulent Thermal Diffusivity $\chi = q_\perp / (-n\nabla_\perp T)$

[Sugama, Watanabe & Horton, PoP (2003)]



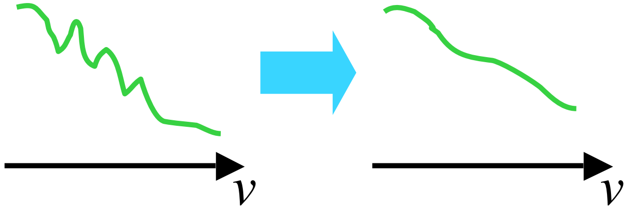
Phase mixing & collisional dissipation

Phase mixing



$v_\parallel \nabla_\parallel f$ term generates fine-scale fluctuations of f in the velocity space, *e.g.*, ballistic mode.

Collisional dissipation



Fokker-Planck-type collision operator smoothes out fine-scale fluctuations of f in the velocity space.

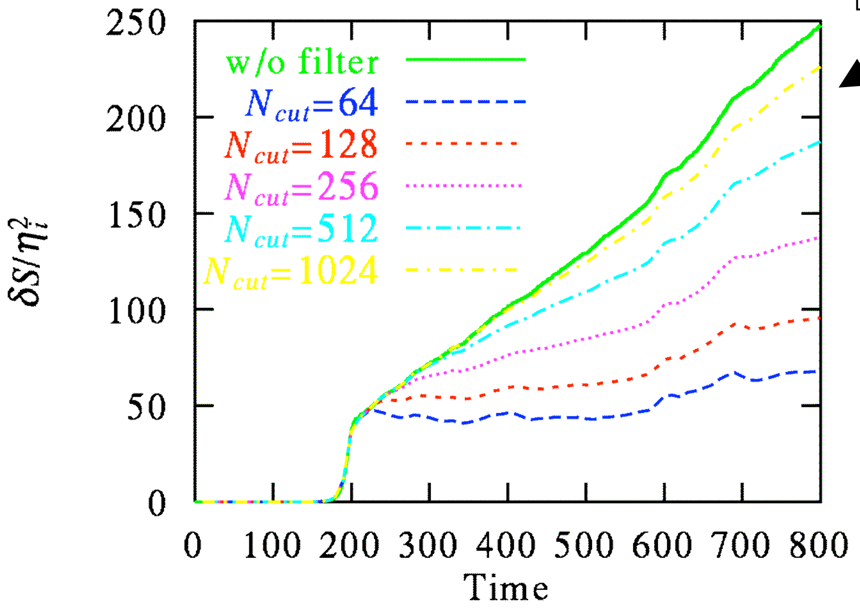
- A balance of the two effects gives a statistically steady state of weakly-collisional turbulence with constant drive of instability.
- In collisionless turbulence, low-order moments of f are constant in average, while high-order ones continue to grow (a quasi-steady state).

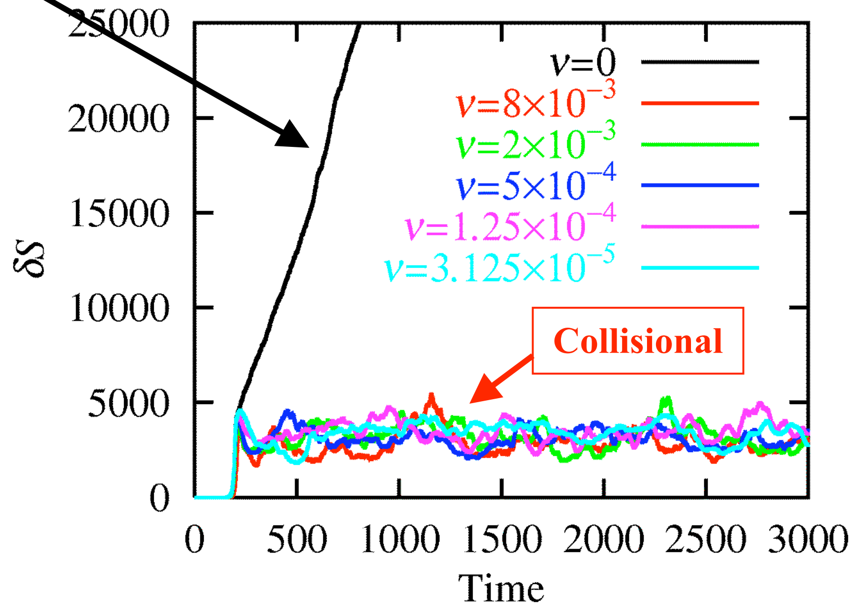
Time evolution of $\delta S = \left\langle \int dv_{\parallel} \tilde{f}^2 / 2F_M \right\rangle$

Time-histories of δS and their low-pass filtered values in the collisionless case

Statistically steady states are realized for the case with finite collisionality.

Collisionless





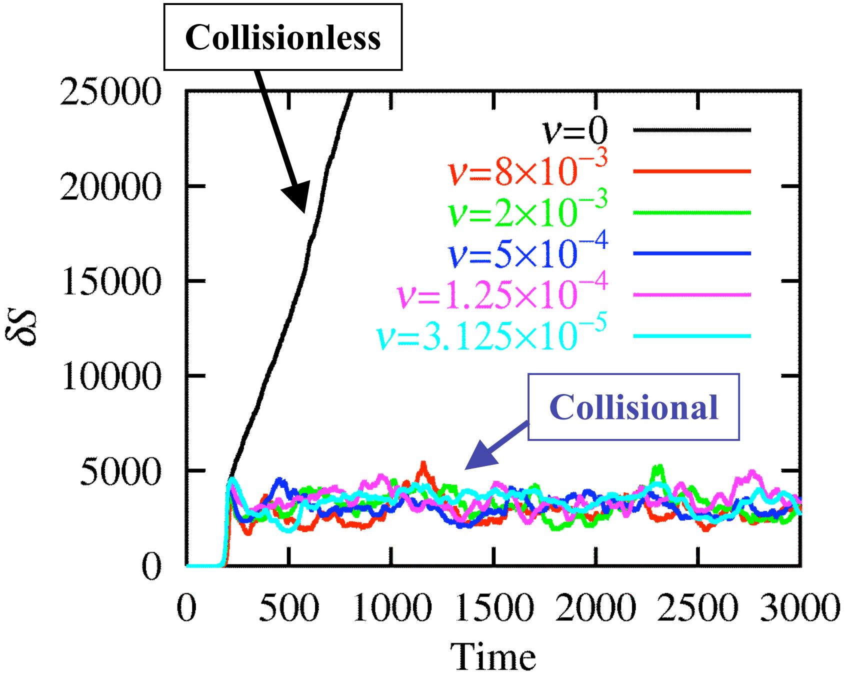
$$\boxed{\frac{d}{dt}\delta S = \eta_i Q_i \approx \text{const}}$$

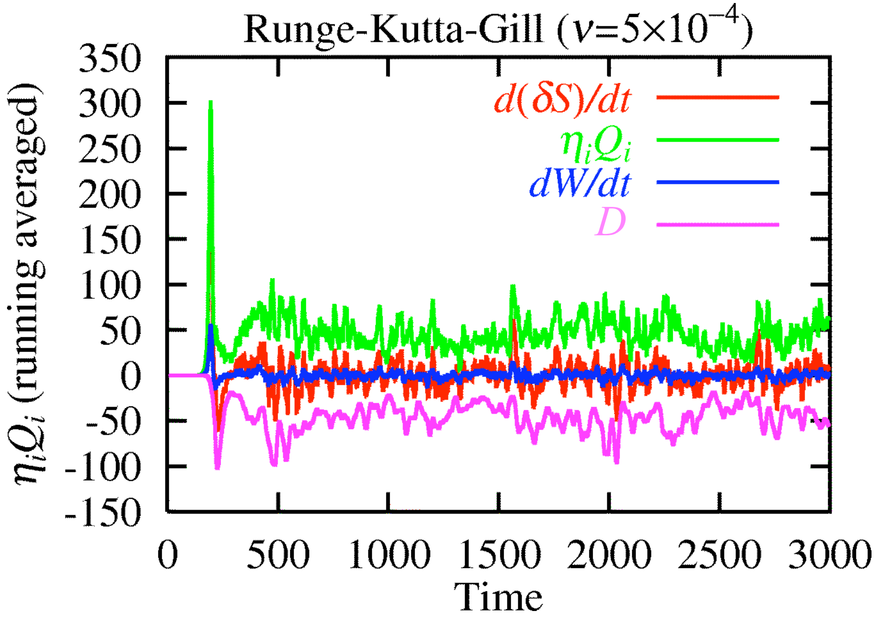
Fine-scale fluctuations develop in the velocity space.

$$\boxed{\frac{d}{dt}\delta S \approx 0}$$

Entropy Balance in Slab ITG Turbulence

$$\boxed{\frac{d}{dt}\left\{\delta S+W\right\}=\eta_i Q_i+D}$$





Statistically steady state for the case with finite collisionality.

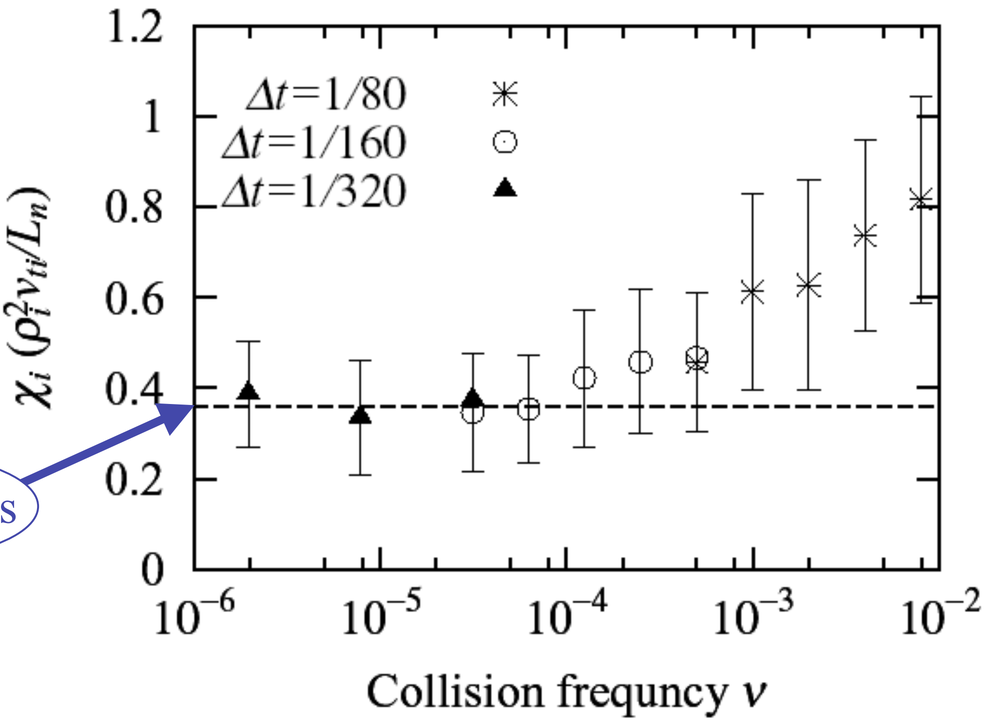
With finite collisionality, the transport flux balances with the collisional dissipation.

$$\frac{d}{dt}\left\{\delta S+W\right\}=\eta_i Q_i+D\approx 0 \qquad \Longrightarrow \qquad \eta_i Q_i\approx -D$$

[Watanabe & Sugama, PoP (2004)]

Collision frequency dependence of transport coefficient

For small ν , ion heat transport coefficient χ_i approaches a value found in the collisionless simulation, while it has a logarithmic dependence on larger ν .



Collisionless

➡ The quasi-steady state in collisionless turbulence is the *ideal limit* of the real steady state in weakly-collisional turbulence.

[Watanabe & Sugama, PoP (2004)]

Spectral Analysis of the distribution function

Hermite-polynomial expansion

$$\tilde{f}_{\mathbf{k}}(v_{\parallel}) = \sum_{n=0}^{\infty} \hat{f}_{\mathbf{k},n} H_n(v_{\parallel}) F_M(v_{\parallel})$$

spectrum of entropy variable
in the *n*-space

$$\delta S_n \equiv \sum_{\mathbf{k}} \delta S_{\mathbf{k},n} \equiv \sum_{\mathbf{k}} \frac{1}{2} n! \left|\hat{f}_{\mathbf{k},n}\right|^2$$

Entropy balance equation in the *n*-space

collisional dissipation

$$\frac{d}{dt}\left[\delta S_n + \delta_{n,1}\frac{1}{2}\sum_{\mathbf{k}}\left|\phi_{\mathbf{k},n}\right|^2\{1-\Gamma_0(b_{\mathbf{k}})\}\right] = J_{n-1/2} - J_{n+1/2} + \delta_{n,2}\,\eta_i Q_i - 2\nu n \delta S_n$$

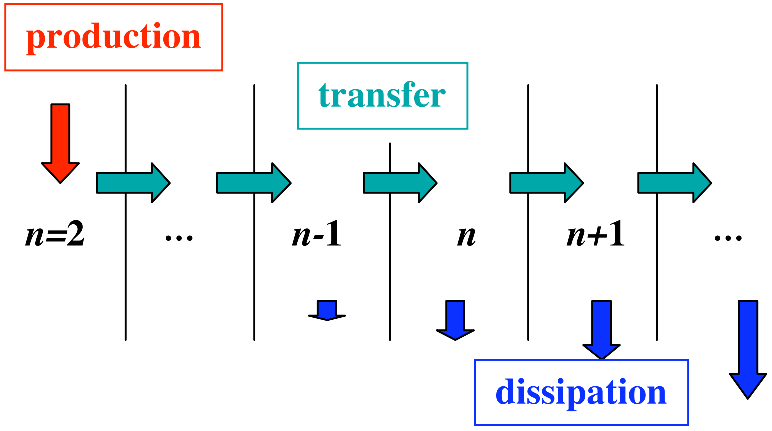
$$\delta_{n,m} = 1(n = m), 0(n \neq m)$$

production (*n*=2)
by turbulent tansport in the temperature gradient

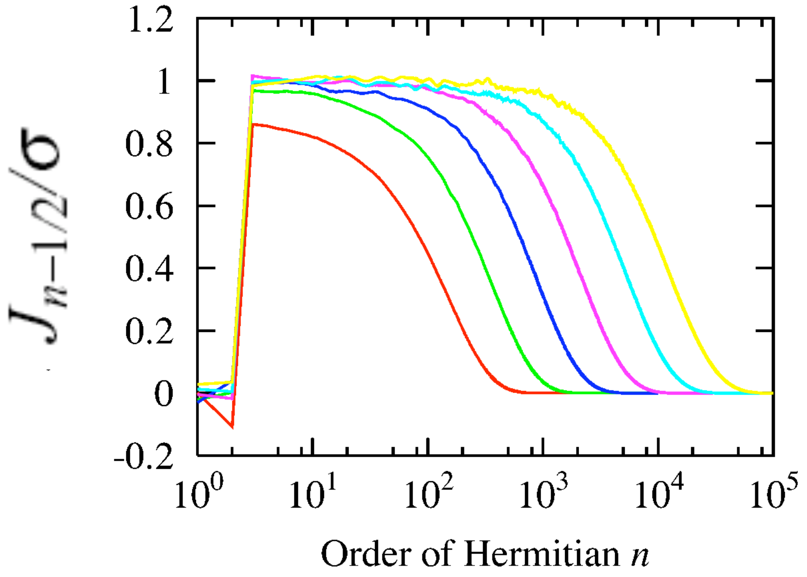
entropy transfer
by phase mixing

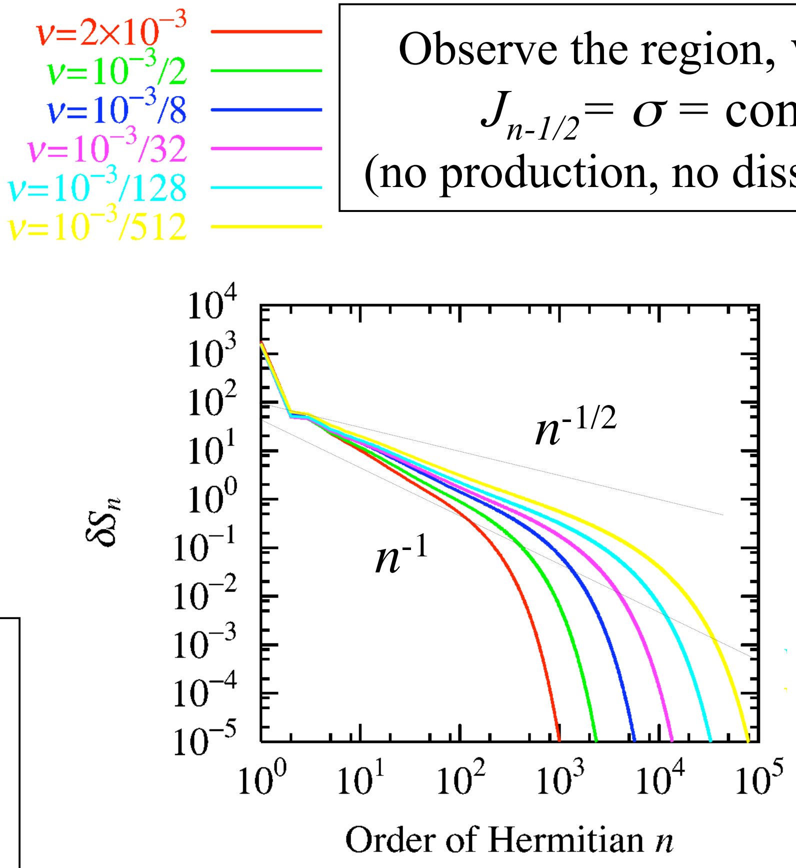
$$J_{n-1/2} \equiv \sum_{\mathbf{k}} \Theta k_y\, n!\, \mathrm{Im}(\hat{f}_{\mathbf{k},n-1}\, \hat{f}^*_{\mathbf{k},n})$$

$$J_{n+1/2} \equiv \sum_{\mathbf{k}} \Theta k_y\, (n+1)!\, \mathrm{Im}(\hat{f}_{\mathbf{k},n}\, \hat{f}^*_{\mathbf{k},n+1})$$



Profiles of $J_{n\text{-}1/2}$ and δS_n (simulation results)





Observe the region, where
$J_{n\text{-}1/2} = \sigma = \text{const}$
(no production, no dissipation)

For small collisionality, microscopic (high-n) structures, which are responsible for dissipation, adjust themselves to the steady state, while keeping macroscopic (low-n) ones and heat transport unchanged.

"*flux determines dissipation*"
(Krommes and Hu, 1994)

Analytical treatment of δS_n in the steady state

For $n > 2$ in the steady state, $-2\nu n \delta S_n = J_{n+1/2} - J_{n-1/2} \approx \dfrac{dJ_n}{dn}$

Here, we use

$$\frac{J_n}{\delta S_n} \equiv \frac{(n+1/2)! \sum_{\mathbf{k}} \Theta k_y \ \mathrm{Im}(\hat{f}_{\mathbf{k},n-1/2} \hat{f}^*_{\mathbf{k},n+1/2})}{(n!/2) \sum_{\mathbf{k}} \left| \hat{f}_{\mathbf{k},n} \right|^2} \approx 2\Theta\sqrt{n} \frac{\sum_{\mathbf{k}} |k_y| \left| \hat{f}_{\mathbf{k},n} \right|^2}{\sum_{\mathbf{k}} \left| \hat{f}_{\mathbf{k},n} \right|^2} \equiv 2\Theta\sqrt{n} \left\langle |k_y| \right\rangle_n$$

averaging $\langle \cdots \rangle \equiv \sum_{\mathbf{k}} \cdots \left| \hat{f}_{\mathbf{k},n} \right|^2 \Big/ \sum_{\mathbf{k}} \left| \hat{f}_{\mathbf{k},n} \right|^2$

approximations $(n+1/2)! \,/ n! = \Gamma(n+3/2)/\Gamma(n+1) \approx \sqrt{n}$ and

$\hat{f}_{\mathbf{k},n-1/2} \hat{f}^*_{\mathbf{k},n+1/2} \approx -i\left(k_y \big/ |k_y|\right) \left| \hat{f}_{\mathbf{k},n} \right|^2$ (from the phase mixing factor)

Then, we obtain

$$\boxed{d\,(\,2\,\Theta \langle |k_y| \rangle_n \, n^{1/2} \delta S_n) \,/\, d\,n = -\,2\,\nu\,n\,\delta S_n \quad (\text{ for } n >> 1\,)}$$

In analogy to the convection of a passive scalar in a fluid with large Prandtl number, the E x B convection of $f_{\boldsymbol{k},n}$ causes exponential growth of wave number :

$$k_y(t) \propto \exp\left(\gamma t\right)$$

Using $f \propto \exp\left[i\, l v_{\parallel}\right]$ $\quad \dfrac{d\,l}{d\,t} = -k_y\,\Theta$ (ballistic modes)

$$\Theta\left|k_y\right| \approx \gamma\left|l\right| \approx \gamma\sqrt{n} \qquad \Theta <|k_y|>_n = \gamma\, n^{1/2}$$

we obtain

the steady-state spectrum

$$\boxed{\delta S_n = \frac{\sigma}{2\gamma n}\exp\left(-\frac{\nu n}{\gamma}\right) \quad \text{for} \quad n >> 1}$$

Eq.(1)

where $\sigma = 2\nu\int_0^{\infty} n\,\delta S_n\,dn \approx 2\nu\sum_n n\,\delta S_n = \eta_i\,Q_i$

represents **the entropy production (or dissipation) rate**

Effects of finite k_{max} (the upper limit of $|\boldsymbol{k}|$)

In numerical simulation, there exists the maximum wave number k_{max}.

Therefore, saturation of $<|k_y|>_n$ with increasing n is anticipated.

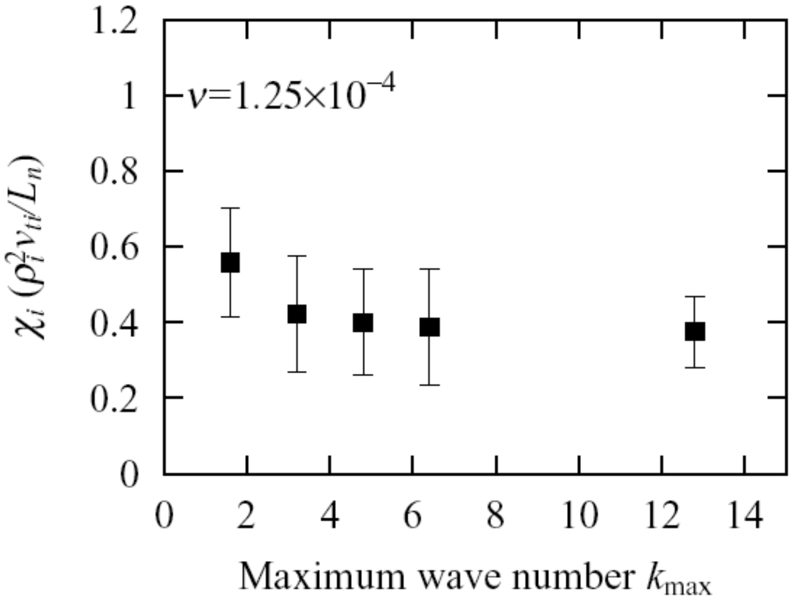
$$\Theta <|k_y|>_n = \gamma_M \qquad \text{(independent of } n \text{)}$$

we obtain

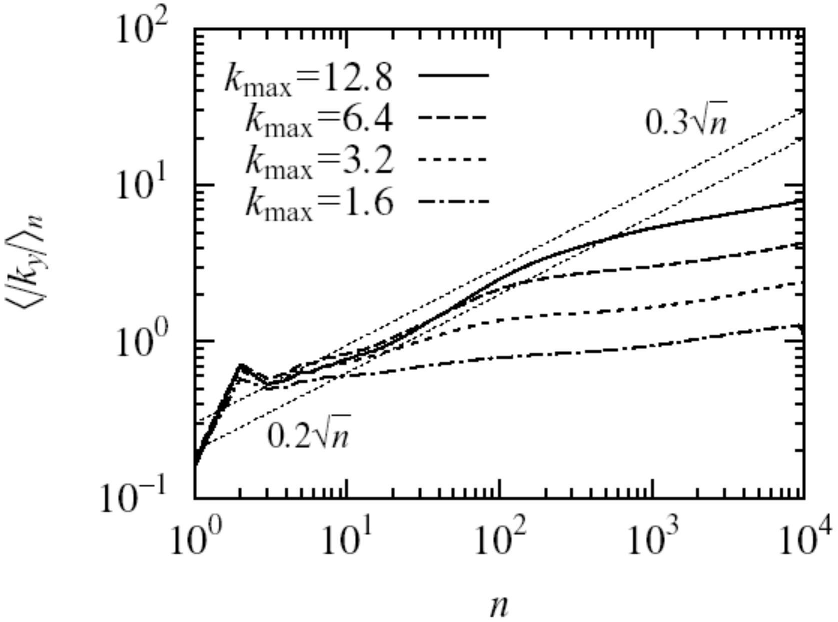
$$\boxed{\delta S_n = \frac{\sigma}{2\gamma_M \sqrt{n}} \exp\left(-\frac{2}{3} \frac{\nu n^{3/2}}{\gamma_M} \right) \quad \text{for} \quad n >> 1}$$

Eq.(2)

Effects of k_{max} (the upper limit of $|k|$) observed by simulation



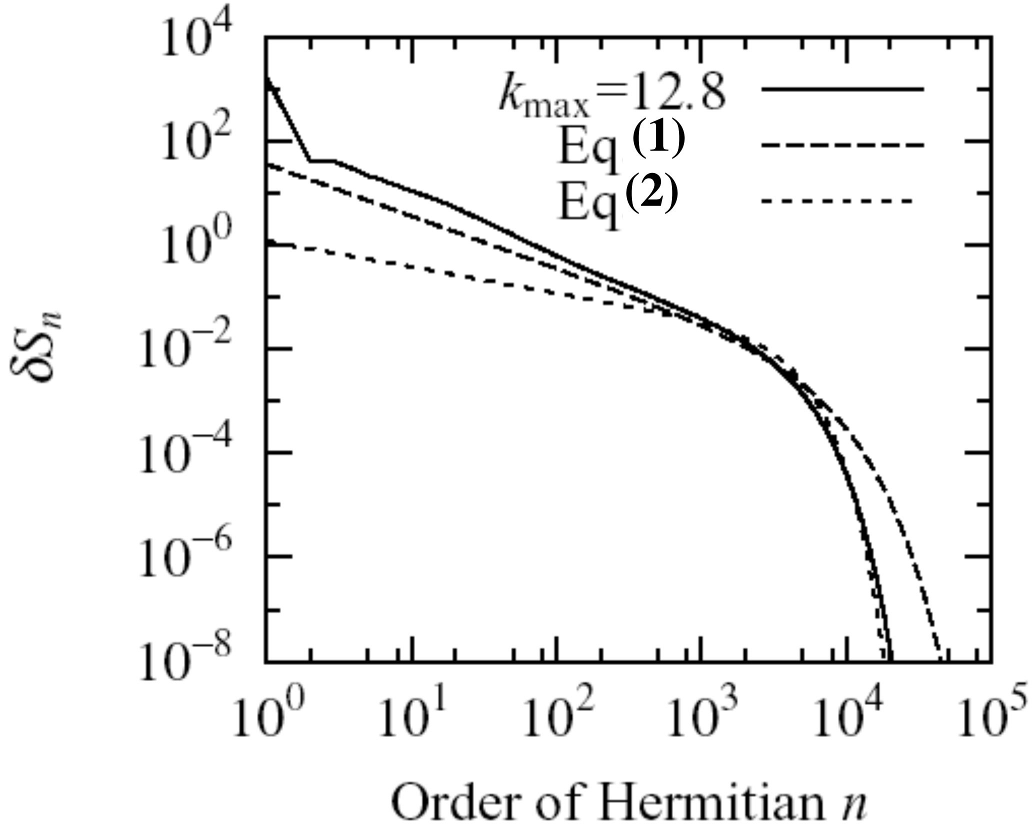
Turbulent ion heat diffusivity χ_i vs. k_{max}



Spectrum-averaged wave number $<|k_y|>_n$ as a function of n for different values of k_{max}

Spectrum δS_n for $k_{max} = 12.8$

Comparison of δS_n to formulas in Eqs.(1) and (2) for k_{max}=12.8



Spectrum (for *n* >> 1) obtained by simulation can be described by using the analytical formulas.

Gyrokinetic Equations (for Toroidal ITG Turbulence)

$$k_\perp \rho_i \approx 1, \quad k_\perp \rho_e << 1$$

Ion gyrokinetic equation for $\delta f(x, v_{||}, \mu, t)$

$$\boxed{\left[\frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}}\cdot\nabla + \mathbf{v}_d\cdot\nabla - \mu\left(\hat{\mathbf{b}}\cdot\nabla\Omega\right)\frac{\partial}{\partial v_\parallel}\right]\delta f + \frac{c}{B_0}\left\{\psi,\delta f\right\} = \left(\mathbf{v}_* - \mathbf{v}_d - v_\parallel\hat{\mathbf{b}}\right)\cdot\frac{e\nabla\psi}{T_i}F_M + C\left(\delta f\right)}$$

Gyrocenter drift
&
Diamagnetic drift

$$\mathbf{v}_d\cdot\nabla = -\frac{v_\parallel^2 + \Omega\mu}{\Omega R_0}\left[\left(\cos z + \hat{s}z\sin z\right)\frac{\partial}{\partial y} + \sin z\frac{\partial}{\partial x}\right],$$

$$\mathbf{v}_* = -\frac{cT_i}{eL_nB_0}\left[1+\eta_i\left(\frac{mv^2}{2T_i}-\frac{3}{2}\right)\right]\hat{\mathbf{y}}, \quad \mu = \frac{v_\perp^2}{2\Omega}$$

Quasineutrality condition & Adiabatic electron assumption

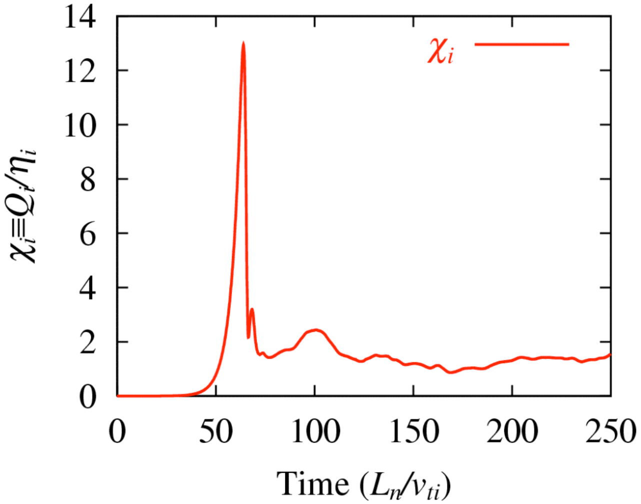
$$\boxed{\int J_0\left(k_\perp v_\perp/\Omega\right)\delta f\,\mathrm{d}^3 v - \left[1-\Gamma_0\left(k_\perp^2\right)\right]\frac{e\phi}{T_i} = \frac{e}{T_e}\left(\phi - \left\langle\phi\right\rangle\right), \quad k_\perp^2 = \left(k_x + \hat{s}zk_y\right)^2 + k_y^2}$$

Ion polarization

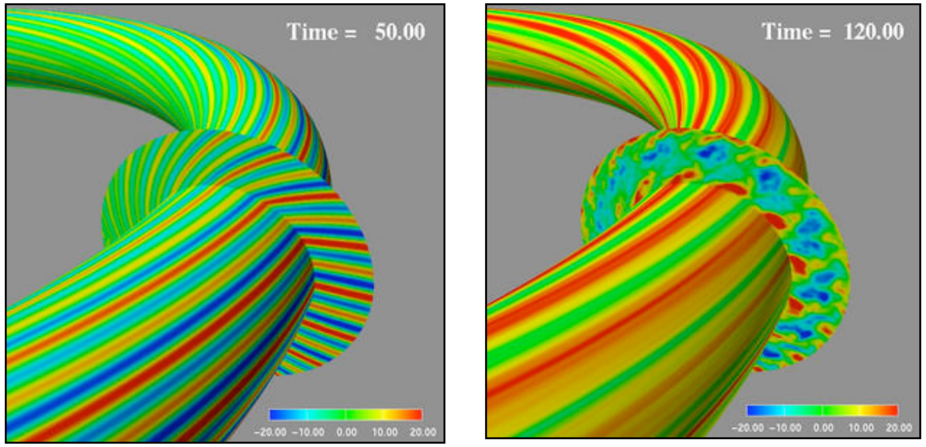
Gyrokinetic Simulation of Toroidal ITG Turbulence

[Watanabe & Sugama, NF (2006)]

Time evolution of anomalous ion heat diffusivity



Structures of electrostatic potential



Ion energy flux

$$\boxed{Q_i = \left\langle \frac{c}{B}\int d^3v \frac{1}{2} m_i v^2 \sum_{\mathbf{k}_\perp} J_0(k_\perp \rho_i)\,\mathrm{Im}\left[f_{i\mathbf{k}_\perp}^{(g)*}\phi_{\mathbf{k}_\perp}\right](\mathbf{k}_\perp \times \mathbf{b})\cdot \nabla r \right\rangle}$$

Cyclone DIII-D base case

$\varepsilon = r/R = 0.18$, $q = 1.4$, $s = (r/q)(dq/dr) = 0.78$

$T_e/T_i = 1$, $\eta_i = L_n/L_T = 3.114$, $R/L_T = 6.92$

Entropy Balance in the Toroidal ITG System

$$\boxed{\frac{d}{dt}\left(\delta S + W\right) = \eta_i Q_i + D_i}$$

$$\delta S = \frac{1}{2}\sum_{\mathbf{k}}\left\langle \int d^3v \left|\tilde{f}_{\mathbf{k}}\right|^2 \Big/ F_M \right\rangle$$

(Entropy Variable)

$$W = \frac{1}{2}\sum_{\mathbf{k}}\left[\left\langle\left(1 - \Gamma_0 + \frac{T_i}{T_e}\right)\left|\Phi_{\mathbf{k}}\right|^2\right\rangle - \frac{T_i}{T_e}\left|\left\langle\Phi_{\mathbf{k}}\right\rangle\right|^2 \delta_{k_y,0}\right]$$

(Potential Energy)

$$Q_i = \frac{1}{2}\sum_{\mathbf{k}}\left\langle -ik_y\Phi_{\mathbf{k}}\int d^3v v^2 J_0 \tilde{f}_{-\mathbf{k}}\right\rangle$$

(Heat Transport Flux)

$$D_i = \sum_{\mathbf{k}}\left\langle \int d^3v \left(\Phi_{-\mathbf{k}} + \frac{\tilde{f}_{-\mathbf{k}}}{F_M}\right) C\left(\tilde{f}_{\mathbf{k}}\right)\right\rangle$$

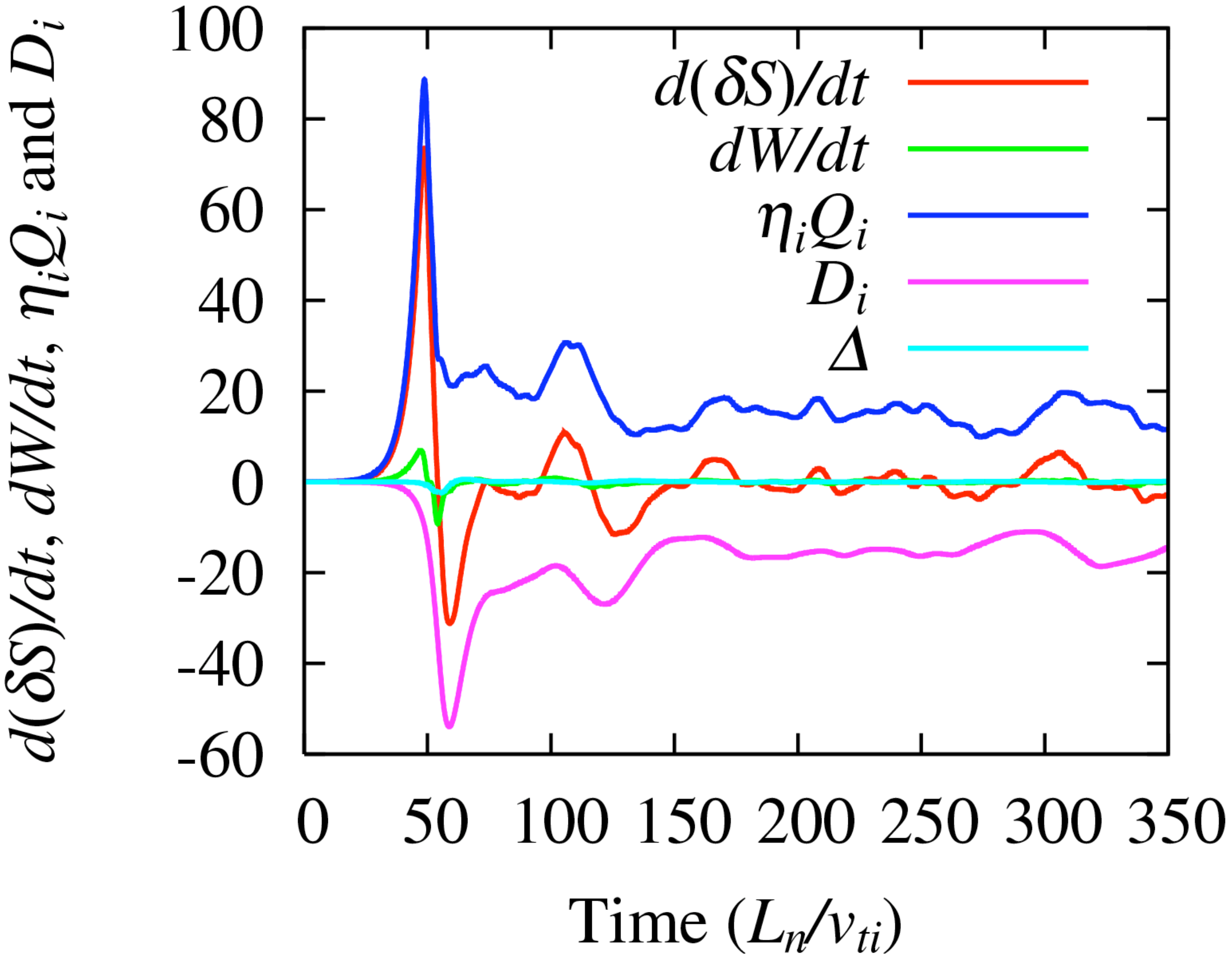
(Collisional Dissipation)

$$\boxed{\frac{d}{dt}\left(\delta S + W\right) = \eta_i Q_i + D_i}$$

Cyclone Base Case Parameters:

$r_0/R_0 = 0.18$, $r_0/\rho_i = 80$, $q_0 = 1.4$, $s = 0.8$,

$R_0/L_T = 6.92$, $\eta_i = 3.114$, $\tau_e = 1$



Summary of Part II

- **Macroscopic entropy $S^{(\mathrm{macro})}$ is transported and produced by classical, neoclassical, and anomalous (turbulent) transport processes.**

- **Entropy $\delta S = S^{(\mathrm{macro})} - S^{(\mathrm{micro})}$ associated with turbulent fluctuations is produced by turbulent transport fluxes and gradient forces while it is dissipated by collisions.**

- **δS consists of all-order moments of velocity-distribution function. Therefore, δS measures generation of fine-scale structures in velocity space and transfer of δS from macro- to microscopic velocity scale is an important process that should be correctly described by kinetic-fluid closure models.**

- **It is confirmed by velocity-space spectral analysis of gyrokinetic slab ITG turbulence gyrokinetic that δS is produced by transport fluxes in macroscopic velocity scale and transferred by phase mixing into microscopic velocity scale where collisional dissipation occurs.**

Summary of Part II (continued)

- **Analytical formulas for entropy spectral functions in slab ITG turbulence are derived and shown to agree with simulation results.**
- **Entropy balance in toroidal ITG turbulence is verified by gyrokinetic simulation.**

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