What place for mathematicians in plasma physics

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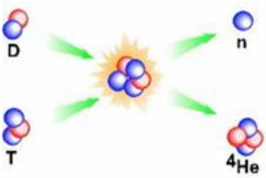
Introduction

- What motivates a mathematician to work on plasma physics problems
  - Challenging programs with important applications
  - Interesting new problems: difficult, novel in mathematics
- What kind of mathematics are needed in plasma physics
  - Modeling: approximate models, links between different scales
  - Analysis: existence, uniqueness, long time behavior, regularity of solutions,...
  - Numerical analysis: Analyze numerical methods, develop new methods.
  - High performance scientific computing: very large problems need to be solved. At the edge of available computing power.
  - Hamiltonian systems: Tokamak, long time numerical integration.

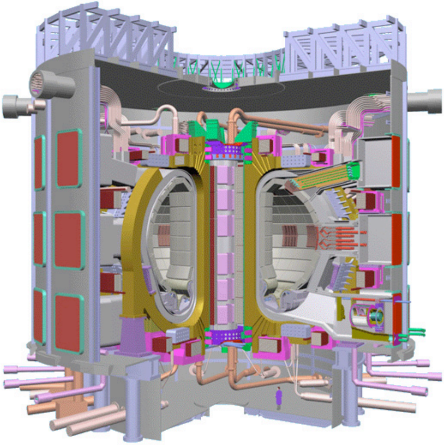
Outline

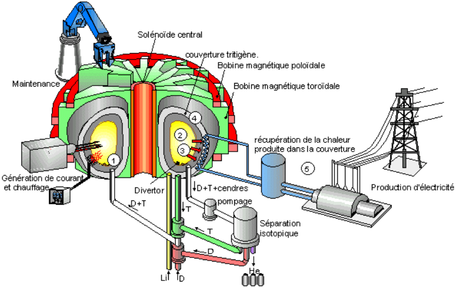
Controlled thermonuclear fusion



- Confinement magnétique (ITER)
- Confinement inertiel
  - par laser (LMJ)
  - par ions lourds

The ITER project

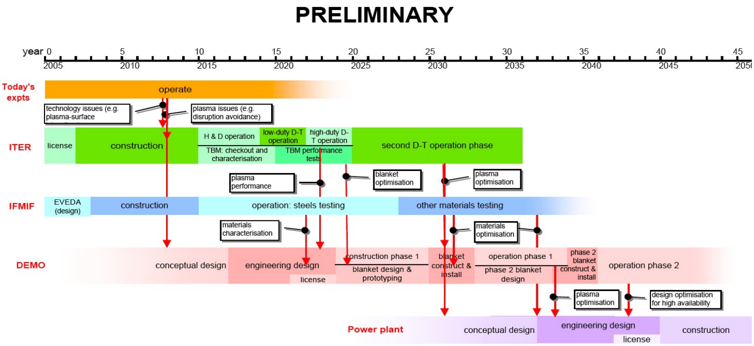




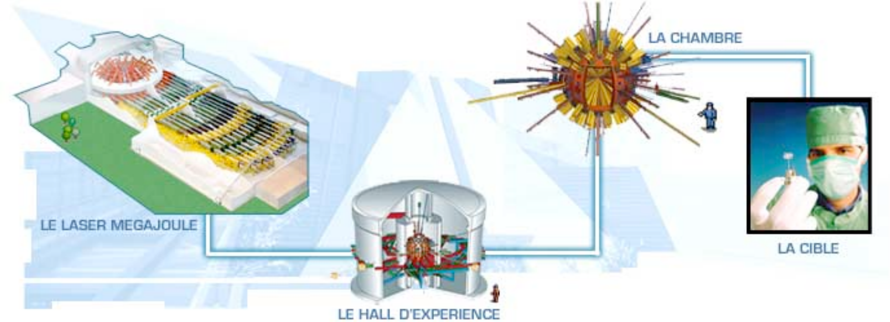
Political context

- ITER project launched, partnership between European Union, Japan, China, South Korea, Russia, USA and India. International agreement signed 21st november 2006.
- Enlarged approach. Convention between EU and Japan on companion program. ITER in Cadarache, Japon becomes
  - Small experimental Tokamak
  - Particle accelerator generating 14 MeV neutrons for material testing.
  - Computing center dedicated to magnetic fusion.
- Ambitious scientific and technological development program in the 7th european Framework Program (FP7) (EURATOM, association EURATOM-CEA, EFDA).
- Magnetic fusion federation founded in France grouping CEA, le CNRS, l'INRIA and several universitys to pilot ITER related research

Roadmap towards a fusion powerplant



Laser MegaJoule



Outline

Mathematical tools of fundamental plasma physics

- Complex analysis: Landau Damping, and in general derivation of kinetic dispersion relations. Residue theorem, integration along contours in complex plane.
- Differential geometry: Toroidal form of equilibria in magnetic field (Poincaré). Coordinate systems used in Tokamaks and stellerators.

- Old mathematical tools.
- No real interaction between physicists and mathematicians on these subjects today.

Outline

Huge span of time and space scales

- Typical space scales in Tokamak

$$\lambda_D = \rho_e \sim 5 \times 10^5, \quad \rho_i \sim 10^{-3}, \quad a \sim 1, \quad mfp \sim 10^3.$$

- Simulation time scales in codes
  - Gyrokinetic 10^{-6} to 10^{-3}
  - MHD 10^{-4} to 1
  - Fluid transport 10^{-3} to 10.

Analysis of models from plasma physics

- Vast literature on analysis of kinetic models of plasma physics.
  - Existence of strong solutions of Vlasov-Poisson, Vlasov-Maxwell and related models.
  - Existence of weak solutions of Vlasov-Poisson, Vlasov-Maxwell and related models.
- Analysis of singular limits in plasma physics
  - The quasi-neutral limit of Vlasov-Poisson.
  - Plasmas in a large magnetic field. Guiding-center, drift-kinetic, gyrokinetic.
- Analysis of numerical methods
  - Convergence proofs for Particle-In-Cell methods.
  - Convergence proofs for semi-Lagrangian methods in different frameworks.

Outline

Derivation of gyrokinetic model

- Gyrokinetic equations used in codes have been most of the time obtained using a Lie transform method in order to find a coordinate system allowing to decouple fast and slow motions in a Hilbert expansion of the characteristics. Littlejohn, Hahm, Brizard, Qin...
- In codes mostly 5D model describing the evolution of guiding centers $f(r, \theta, \phi, v_{\|}, \mu)$ obtained and used by Hahm (Phys. Fluids 1988), Hatzky et al. (Phys. Plasmas 2002), and Brizard-Hahm in a review article (Reviews of Modern Physics 2007).
- The limit model is Hamiltonian and in particular verifies exact conservation of particles and energy which are important for long time simulations.
- An other approach (Sosenko) is based on Boboliugov-Mitropolsky averaging techniques.
- The rigorous mathematical justification of these models is still far from complete. Partial results have been obtained by. Brenier-Grenier, Golse-Saint-Raymond, Frénod-Sonnendrücker, Bostan.

Outline

Model coupling

- Enhanced fluid models where most particles are treated as fluids, but some population is extracted and modeled with kinetic equations.
- Integrated tokamak modeling: model full tokamak discharge using a suite of several codes.

Asymptotics preserving schemes

- Based on idea of Shi Jin.
- Very rich developments for plasma physics applications in Degond's team.
- See Pierre Degond's lectures on gyro-fluid and gyro-kinetic limit.
- Principle is to find a numerical scheme that is robust for two models the one being the limit of the other.
- Useful when both models are needed in the problem. Coupling automatically performed by the numerical scheme.

Outline

Operator splitting

- Consider e.g. the non relativistic Vlasov-Poisson equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m} \mathbf{E} \cdot \nabla_v f = 0.$$

- We decompose the equation into the two following steps.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = 0, \tag{1}$$

with $\mathbf{v}$ fixed and

$$\frac{\partial f}{\partial t} + \frac{q}{m} \mathbf{E}(\mathbf{x}, t) \cdot \nabla_v f = 0, \tag{2}$$

with $\mathbf{x}$ fixed.

- We solve the two equations successively on one time step. At least dimension reduction and in our example constant coefficient advections for reduced equations.

Conservativity

- Consider abstract Vlasov equation where $\mathbf{z}$ are all the phase space variables
$$\frac{\partial f}{\partial t} + \mathbf{a}(\mathbf{z}, t) \cdot \nabla_z f = 0 \quad \text{with } \nabla \cdot \mathbf{a} = 0.$$
- The equation is conservative: $\frac{d}{dt} \int f \, d\mathbf{z} = 0.$
- Consider splitting the equations by decomposing the variables into $\mathbf{z}_1$ and $\mathbf{z}_2$. Then the split equations read
$$\frac{\partial f}{\partial t} + \mathbf{a}_1(\mathbf{z}, t) \cdot \nabla_{z_1} f = 0, \quad \text{and} \quad \frac{\partial f}{\partial t} + \mathbf{a}_2(\mathbf{z}, t) \cdot \nabla_{z_2} f = 0.$$
- We have $\nabla \cdot \mathbf{a} = \nabla_{z_1} \cdot \mathbf{a}_1 + \nabla_{z_2} \cdot \mathbf{a}_2 = 0$, but in general $\nabla_{z_1} \cdot \mathbf{a}_1$ and $\nabla_{z_2} \cdot \mathbf{a}_2$ do not vanish separately.
- One or more of the split equations may not be conservative.

Example 1: Vlasov-Poisson

- In this case the Vlasov equation reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0.$$

- So $\mathbf{a} = (\mathbf{v}, \mathbf{E}(\mathbf{x}, t))$
- Standard splitting yields:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f = 0 \quad \text{and} \quad \frac{\partial f}{\partial t} + \mathbf{E} \cdot \nabla_v f = 0.$$

- So that $\mathbf{a}_1 = \mathbf{v}$ and $\mathbf{a}_1 = \mathbf{E}(\mathbf{x}, t)$.
- In this case $\nabla_x \cdot \mathbf{a}_1 = 0$ and $\nabla_v \cdot \mathbf{a}_2 = 0$.
- Splitting yields two conservative equations.

Example 2: guiding-center model

- Classical model for magnetized plasmas. Describes motion in plane perpendicular to magnetic field.

$$\frac{\partial \rho}{\partial t} + \mathbf{v}_D \cdot \nabla \rho = 0, \qquad -\Delta \phi = \rho,$$

$$\mathbf{v}_D = \frac{-\nabla \phi \times \mathbf{B}}{B^2} = \begin{pmatrix} -\frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial x} \end{pmatrix} \text{ if } \mathbf{B} = \mathbf{e}_z \text{ unit vector in direction } z.$$

- The model is conservative: $\nabla \cdot \mathbf{v}_D = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0.$
- Split equations become

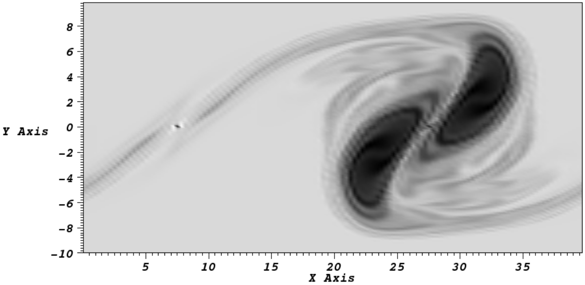
$$\frac{\partial \rho}{\partial t} - \frac{\partial \phi}{\partial y}\frac{\partial \rho}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x} = 0.$$

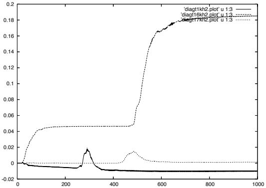
- In general $\frac{\partial^2 \phi}{\partial x \partial y} \neq 0$.
- The split equations are not conservative.

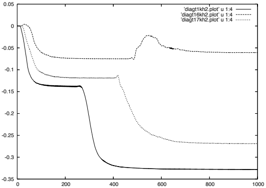
Problem with non conservative Vlasov solver

- When non conservative splitting is used for the numerical solver, the solver is not exactly conservative.
- Does generally not matter when solution is smooth and well resolved by the grid. The solver is still second order and yields good results.
- However: Fine structures develop in non linear simulations and are at some point locally not well resolved by the phase space grid.
- In this case a non conservative solvers can exhibit a large numerical gain or loss of particles which is totally unphysical.
- Lack of robustness.

Vortex in Kelvin-Helmholtz instability







Outline

Supercomputing

- Two supercomputers dedicated for magnetic fusion in near future (Jülich 2009, Japan 2012).
- More than 10000 processors usable for gyrokinetic codes.
- New programming constraint (with respect to < 100 processors) for efficient use:
  - No transfer from one processor to all.
  - No global data redistribution.
  - Sophisticated adaptive methods probably less competitive due to overhead.
  - Charge balance problems for particle methods.
  - Advantage to local methods with static charge balance.

Conclusions

- Whole range of interesting mathematical problems arising from plasma physics.
- Most research in mathematics and plasma physics without much real time interaction: mathematical results not used directly by physicists.
- Direct collaboration and interaction is more on codes: numerical algorithms and parallel computing.
- Physicists and mathematicians have a very different approach in problem solving.
- Different ways of thinking complementary and beneficiary in the long term.