Gyrokinetic Turbulence: Generalised Energy Cascade, Fluid Limits and Nonlinear Phase Mixing.

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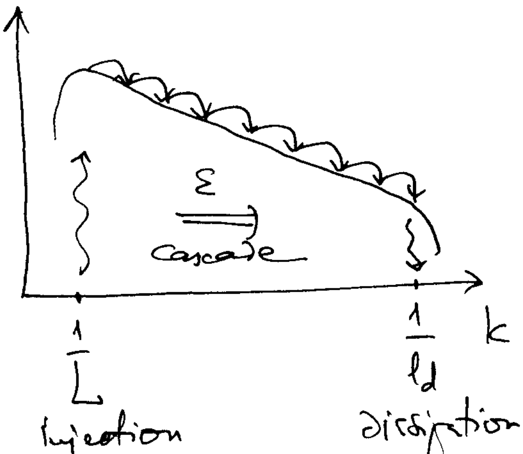
§1. Generalised Energy Cascade.

[arXiv: 0806.1069]

Turbulence = multiscale disorder
(defined by its symptoms, like a syndrome)
Energy injected at large scales (outer)
dissipated at small scale (inner)
One way of bridging this gap is to fill up intermediate scales with fluctuations = turbulent cascade

Big whorls have little whorls
That feed on their velocity
And little whorls have lesser whorls
And so on to viscosity
(L.F. Richardson 1922)



What is this scale?

In neutral fluid,

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \vec{f}$$

$$\frac{d}{dt}\int\frac{d^3\vec{r}}{V}\,\frac{u^2}{2} = \underbrace{\int\frac{d^3\vec{r}}{V}\,\vec{u}\cdot\vec{f}}_{\varepsilon\text{ - finite energy injection}} - \underset{\text{small}}{\nu}\int\frac{d^3\vec{r}}{V}\,\underset{\text{large gradients}}{|\nabla\vec{u}|^2} \quad (\emptyset)$$

← dissipation

$$\boxed{l_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim \frac{L}{Re^{3/4}} \ll L}$$

Kolmogorov scale

and
$$\left.\begin{array}{l}\dfrac{\delta u_\lambda^2}{\tau_\lambda} \sim \varepsilon = \text{const} \\ \tau_\lambda \sim \dfrac{\lambda}{\delta u_\lambda}\end{array}\right\} \Rightarrow \boxed{\delta u_\lambda \sim (\varepsilon\lambda)^{1/3}}$$

K41

That was the reference example — K41 still defines our philosophical attitude to turbulence.

So what happens in a plasma?

Plasmas of interest are kinetic (turbulence at collisionless scales). So

— What is cascading?

— What is dissipation? — conversion of energy into heat

Start from the beginning: Vlasov-Maxwell Boltzmann! (or Landau)

$$\begin{cases} \dfrac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s + \dfrac{q_s}{m_s}\left(\vec{E} + \dfrac{\vec{v}\times\vec{B}}{c}\right)\cdot\dfrac{\partial f_s}{\partial \vec{v}} = \left(\dfrac{\partial f_s}{\partial t}\right)_c \\ \nabla\cdot\vec{E} = 4\pi\sum_s q_s n_s, \quad n_s = \int d^3\vec{v}\, f_s \\ \nabla\times\vec{B} - \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} = \dfrac{4\pi}{c}(\vec{j} + \vec{j}_{ext}), \quad \vec{j} = \sum_s q_s \int d^3\vec{v}\,\vec{v} f_s \\ \dfrac{\partial\vec{B}}{\partial t} = -c\nabla\times\vec{E}, \quad \nabla\cdot\vec{B} = 0 \end{cases}$$

Total particle energy:

field energy

$$\underbrace{\frac{d}{dt}\int\frac{d^3\vec{r}}{V}\sum_s\int d^3\vec{v}\,\frac{m_s v^2}{2} f_s}_{\text{particle energy}} = \underbrace{\int\frac{d^3\vec{r}}{V}\,\vec{E}\cdot\vec{j}}_{\text{work done}} = \varepsilon - \frac{d}{dt}\int\frac{d^3\vec{r}}{V}\frac{E^2+B^2}{8\pi} \quad (1)$$

$$\varepsilon = -\int\frac{d^3\vec{r}}{V}\,\vec{E}\cdot\vec{j}_{ext}$$

This is not yet heating! (because reversible)

Irreversibility is associated with collisions:

equality for Maxwellian

$$\frac{dS_s}{dt} \equiv \frac{d}{dt}\left[-\int\frac{d^3\vec{r}}{V}\int d^3\vec{v}\, f_s \ln f_s\right] = -\int\frac{d^3\vec{r}}{V}\int d^3\vec{v}\,\ln f_s\left(\frac{\partial f_s}{\partial t}\right)_c \geq 0$$

Boltzmann 1872 (2)

Let $f_s = F_{0s} + \delta f_s$ — In GK: this is an expansion in $\epsilon \sim \frac{\omega}{\Omega_i}$

Maxwellian

$$F_{0s} = \frac{n_{0s}}{(\pi v_{ths}^2)^{3/2}} e^{-v^2/v_{ths}^2}$$

$$v_{ths}^2 = 2T_{0s}/m_s$$

Equilib. quantities vary on $\sim \epsilon^2 \omega$

Collisions formally ordered $\underbrace{\nu_i \sim \omega \epsilon^{1/2}}$, $\nu_e \sim \omega$

[see Steve Cowley's lectures]

$\nu_i v_{thi}^2 \frac{\partial^2}{\partial v^2} \sim \omega$

$\delta v / v \sim \epsilon^{1/4}$

Then the lhs of (2) is

$$T_{0s} \frac{dS_s}{dt} = T_{0s} \frac{d}{dt} \left[- \iint f_s \ln (F_{0s} + \delta f_s) \right] =$$

$$f_s \left[\ln F_{0s} + \ln \left(1 + \frac{\delta f_s}{F_{0s}}\right) \right] = -f_s \frac{v^2}{v_{ths}^2} + f_s \ln \frac{n_{0s}}{(\pi v_{ths}^2)^{3/2}} + (F_{0s} + \delta f_s)\left(\frac{\delta f_s}{F_{0s}} - \frac{\delta f_s^2}{2F_{0s}^2} + \dots\right)$$

$$\ln\left(1+\frac{\delta f_s}{F_{0s}}\right) = \frac{\delta f_s}{F_{0s}} - \frac{\delta f_s^2}{2F_{0s}^2} + \dots$$

$$= -\frac{mv^2}{2T_{0s}} f_s + F_{0s} \left[\cancel{\ln \frac{n_{0s}}{(2\pi/m_s)^{3/2}}} - \frac{3}{2} \ln T_{0s} \right]$$

$$+ \delta f_s \left[1 + \ln \frac{n_{0s}}{(\pi v_{ths}^2)^{3/2}} \right] + \frac{\delta f_s^2}{2F_{0s}} + \dots$$

$\int F_{0s} = n_{0s}$

$\iint \delta f_s = \int \delta n_s = 0$

$\frac{d}{dt} \int n_{0s} \ln \frac{n_{0s}}{(2\pi/m_s)^{3/2}} = 0$

$$= T_{0s} \frac{d}{dt} \left[+ \iint \frac{mv^2}{2T_{0s}} f_s + \frac{3}{2} n_{0s} \ln T_{0s} - \iint \frac{\delta f_s^2}{2F_{0s}} \right]$$

$$= \frac{d}{dt} \iint \left[\frac{mv^2}{2} f_s - \frac{\delta f_s^2 T_{0s}}{2F_{0s}} \right] + \cancel{\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt}} - \cancel{\frac{dT_{0s}}{dt} \underbrace{\iint \frac{mv^2}{2T_{0s}} f_s}_{\approx \frac{3}{2} n_{0s}}} + \dots$$

Thus,

$$\boxed{T_{0s} \frac{dS_s}{dt} = \frac{d}{dt} \iint \left[\underbrace{\frac{mv^2}{2} f_s}_{\substack{\text{total particle energy}}} - \underbrace{\frac{T_{0s} \delta f_s^2}{2F_{0s}}}_{T_{0s}\delta S_s \text{ perturbed entropy}} \right] = \underbrace{- \iint \frac{T_{0s} \delta f_s}{F_{0s}} \left(\frac{\partial \delta f_s}{\partial t} \right)_c}_{\text{positive definite collisions}} + \text{interspecies collisions}} \quad (3)$$

(use (2))

see (1)

Now use (1) in (3): get

$$\boxed{\frac{d}{dt}\int\frac{d^3\vec{r}}{V}\underbrace{\left[\sum_s\int d^3\vec{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}}+\frac{E^2+B^2}{8\pi}\right]}_{\text{positive definite quantity}}=\underbrace{\varepsilon}_{\text{energy injection}}+\underbrace{\sum_s\int\frac{d^3\vec{r}}{V}\int d^3\vec{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_c}_{\text{neg. definite collisional dissipation}}} \quad (4)$$

positive definite quantity
generalised energy (free energy of the particle + fields system $-T\delta S+U$)

Note that time averaging of (3) (over fluct. timescales) gives

$$\underbrace{\frac{3}{2}n_{0s}\frac{dT_{0s}}{dt}}_{\epsilon^2\omega} = -\underbrace{\overline{\iint\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_c}}_{\epsilon^2\nu_i\frac{v_{thi}^2}{\delta v^2}\sim\epsilon^2\omega} + \begin{array}{c}\text{interspecies}\\ \text{collisions}\\ -n_{0s}\nu_E^{ss'}(T_{0s}-T_{0s'})\end{array} \quad (5)$$

as originally assumed

- Eq. (4) is the generalisation of Eq. (0) to the case of plasma turbulence. We can think of the turbulence as a generalised energy cascade from some large injection scales to dissipation (collisional) scales in velocity space:

} Thus, coll. dissipation of generalised energy exactly corresponds to heating of equilibrium

$$\nu_{ii}\, v_{thi}^2\frac{\partial^2\delta f_i}{\partial v^2}\sim\omega \implies \frac{\delta v}{v_{thi}}\sim\left(\frac{\nu_{ii}}{\omega}\right)^{1/2}\ll 1 \quad \left[\sim\epsilon^{\frac{1}{4}}\ \text{in S. Cowley's ordering}\right]$$

"Kolmogorov scale" in velocity space.

The cascade is, generally, in phase space — 6D (or 5D in GK)

- Formation of small scales in velocity space is known as phase mixing. The simplest type of phase mixing is linear: "ballistic response" / Van-Kampen modes

$$\partial_t \delta f_s + \vec{v}\cdot\nabla\delta f_s + \ldots = 0$$

Hom. solution: $\delta f_s \propto e^{i\vec{k}\cdot\vec{v}t}$ $\quad \frac{\partial}{\partial v} \sim kt$ grows with time [Landau 1946]

There are also other phase mixing mechanisms

- on nonlinear perp. phase mixing, later in this lecture (+ see talk by T. Tatsuno, poster by G. Plunk)
- on parallel phase mixing in electrostatic plasma turbulence see talk by H. Sugama.

• Landau damping is not dissipation — it is a redistribution of generalised (free) energy from U to $-T\delta S$
fields — entropy

Ballistic part of the response contains ever smaller scales in velocity space, so eventually, collisions are activated and heating is effected.

• There are many previous talks on this subject:

Fowler 1968 Adv. Plasma Phys. 1, 201

Krommes & Hu 1994 PoP 1, 3211

Krommes 1999 PoP 6, 1477

Sugama et al. 1996 PoP 3, 2379

Hallatschek 2004 PRL 93, 125001

Howes et al. 2006 ApJ 651, 590

Schekochihin et al. 2007 arXiv:0704.0044

Scott 2007 arXiv:0710.4899

Schekochihin et al. 2008 PPCF, in press / arXiv:0806.1069

§2. Gyrokinetic Cascade and Fluid/Hybrid Limits.

[arXiv: 0704.0044]

So now the question is how this generalized cascade going to get energy from large to small scales.

There we need some* assumptions/conjectures/insights

- Low frequency turbulence $\frac{\omega}{\Omega_i} \sim \epsilon \ll 1$
- Critical balance

linear frequency $\omega \sim k_\parallel v_{thi} \sim k_\perp u_\perp \sim$ nonlinear decorr. rate

(or v_A, we order $\beta_i \sim 1$)

- This is a physically reasonable conjecture:

$\omega \gg k_\perp u_\perp \Rightarrow$ weak turbulence $\Rightarrow$ goes strong at suff. small scales

e.g. [Galtier et. al. 2000]

$\omega \ll k_\perp u_\perp \Rightarrow$ "quasi-2D" $\Rightarrow$ initially correlated $\perp$ planes cannot stay correlated if they are separated by $l_\parallel > \frac{v_{thi}}{k_\perp u_\perp}$, so $k_\parallel v_{thi} \sim k_\perp u_\perp$

- GK can be constructed if critical balance is used as an ordering assumption:

$$k_\parallel v_{thi} \sim k_\perp u_\perp \Rightarrow \frac{k_\parallel}{k_\perp} \sim \frac{u_\perp}{v_{thi}} \sim \frac{\delta f}{F_0} \sim \frac{\omega}{\Omega_i} \sim \epsilon \ll 1$$

$\to k_\perp \rho_i \sim 1$

(so ion-scale turbulence is included)

In GK, $f_s = F_{0s} + \delta f_s$

$$\delta f_s = -\frac{q_s \varphi}{T_{0s}} F_{0s} + h_s(t, \vec{R}_s, \varepsilon_s, \mu_s)$$

Boltzmann response

gyrocentre distribution

$$\vec{R}_s = \vec{r} + \frac{\vec{v}_\perp \times \hat{b}}{\Omega_s}$$

$$\varepsilon_s = \frac{m_s v^2}{2}$$

$$\mu_s = \frac{m_s v_\perp^2}{2B_0}$$

Then the kinetic eqn is

GK collision operator [see posters by Abel, Barnes] arXiv:0808.1300

$$\frac{\partial h_s}{\partial t} + v_\parallel \hat{b}\cdot\nabla h_s + \frac{c}{B_0}\left\{\langle\chi\rangle_{R_s}, h_s\right\} = \frac{q_s F_{0s}}{T_{0s}}\frac{\partial\langle\chi\rangle_{R_s}}{\partial t} + \left(\frac{\partial h_s}{\partial t}\right)_c$$

$$\chi = \varphi - \frac{\vec{v}\cdot\vec{A}}{c}, \quad \langle\chi\rangle_{R_s} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta\, \chi\left(\vec{R}_s - \frac{\vec{v}_\perp\times\hat{b}}{\Omega_s}\right) \qquad (6)$$

gyroaverage.

Eq. (4) in GK becomes

$$\frac{dW}{dt} \equiv \frac{d}{dt}\int\frac{d^3\vec{r}}{V}\left[\sum_s \underbrace{\left(\int d^3\vec{v}\,\frac{T_{0s}\langle h_s^2\rangle_r}{2F_{0s}} - \frac{q_s^2\varphi^2 n_{0s}}{2T_{0s}}\right)}_{\int d^3\vec{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}} \,=\, -T\delta S \text{ entropy}} + \frac{|\delta\vec{B}|^2}{8\pi}\right] =$$

$$= \varepsilon + \sum_s \int d^3\vec{v}\int\frac{d^3\vec{R}_s}{V}\,\frac{T_{0s}h_s}{F_{0s}}\left(\frac{\partial h_s}{\partial t}\right)_c \qquad (7)$$

heating → goes into the temperature eqn derived in S. Cowley's lecture on GK transport.

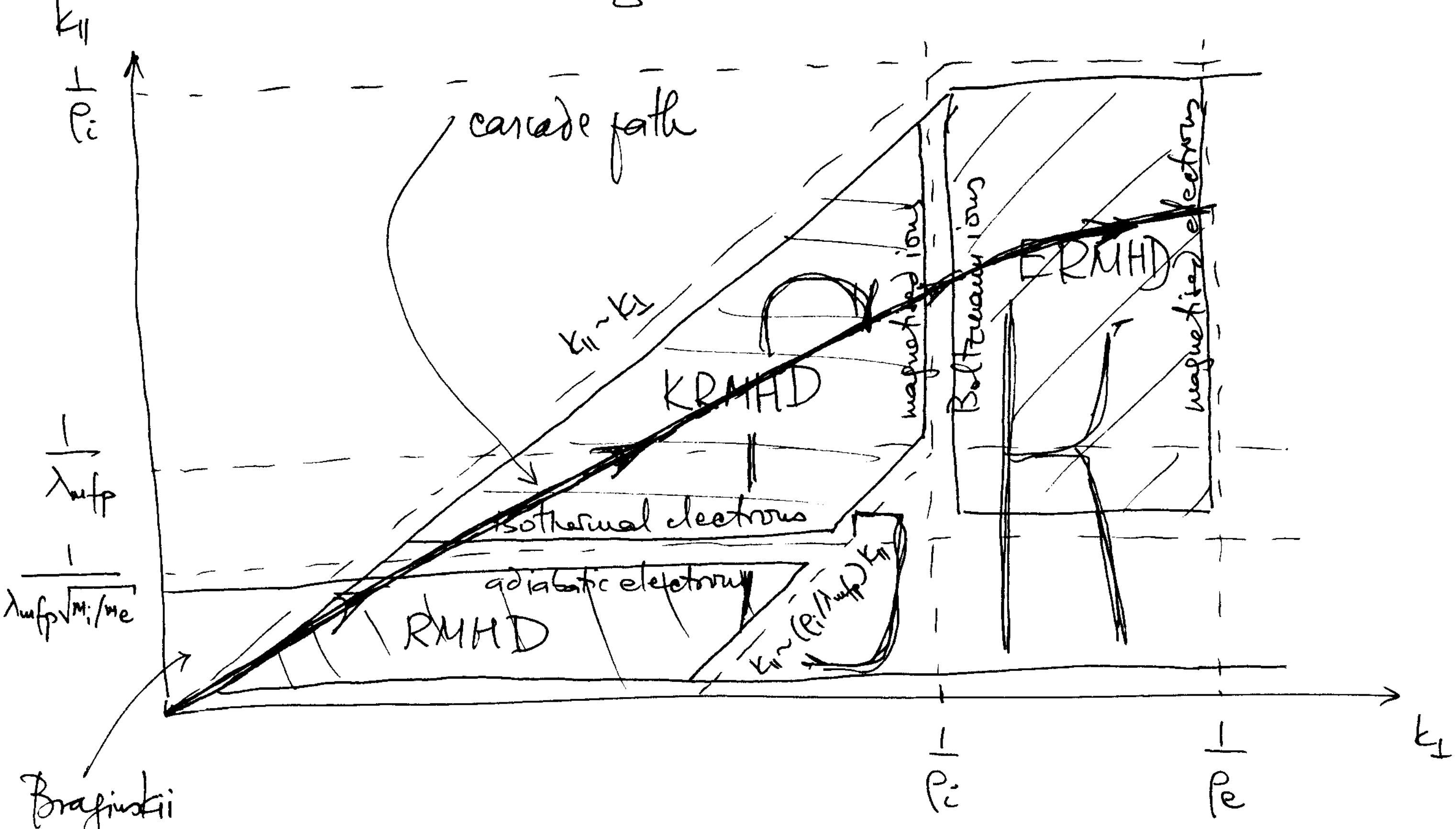
[see talk by H. Sugama about further forms of this in the electrostatic limit]

So W will cascade from large to small scales along some anisotropic ($k_\parallel \ll k_\perp$) path in k space.

This cascade passes between various physically distinct regimes as it crosses special scales:

$k_\parallel \lambda_{mfp}\sqrt{\frac{m_i}{m_e}} \sim 1$ el. diffusion scale $\qquad k_\perp \rho_i \sim 1$ ← same as d_i and ρ_s because $\beta_i \sim 1$, $\frac{T_i}{T_e} \sim 1$

$k_\parallel \lambda_{mfp} \sim 1$ coll. scale (mfp) $\qquad k_\perp \rho_e \sim 1$



This is all worked out in painful detail in arXiv:0704.0044, but here is a summary:

- GK covers all except possibly the largest scales
- For $k_\parallel \lambda_{mfp}\sqrt{\frac{m_i}{m_e}} \ll 1$, $k_\perp \rho_i \ll 1$ we have standard MHD
- For $k_\parallel \lambda_{mfp}\sqrt{\frac{m_i}{m_e}} \gg 1$, $k_\perp \rho_e \ll 1$ electrons are isothermal and magnetised
  ↳ they become fluid while ions remain kinetic
- For $\boxed{k_\parallel \lambda_{mfp} \ll 1,\ k_\perp \rho_i \ll 1}$, ions also become fluid and one gets the Reduced MHD equations, which, by virtue of the anisotropy decouple (nonlinearly!) the MHD modes: Alfvén waves from the slow modes and entropy fluctuations.

Alfvén waves: $\vec{u}_\perp = \hat{b}_0 \times \nabla_\perp \Phi$, $\dfrac{\delta \vec{B}_\perp}{\sqrt{4\pi m_i n_{0i}}} = \hat{b}_0 \times \nabla_\perp \Psi$

Strauss - Kadomtsev - Pogutse equations

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \dfrac{\partial \Phi}{\partial z} \\[2ex] \dfrac{\partial \nabla_\perp^2 \Phi}{\partial t} + \{\Phi, \nabla_\perp^2 \Phi\} = v_A \dfrac{\partial}{\partial z} \nabla_\perp^2 \Psi + \{\Psi, \nabla_\perp^2 \Psi\} \end{cases} \quad (8)$$

- Two interacting but non-energy-exchanging cascades:

$$W_\perp^\pm = \int \frac{d^3\vec{r}}{V} \frac{m_i n_{0i}}{2} |\nabla \zeta^\pm|^2 , \quad \zeta^\pm = \Phi \pm \Psi$$

Elsasser potentials

[Exercise: Write equs for $\zeta^\pm$]

- Decoupled from other MHD modes
- Indifferent to collisions! — so valid for all $k_\parallel$ and $k_\perp \rho_i \ll 1$

Slow waves and entropy mode:

$$\begin{cases} \dfrac{d u_\parallel}{dt} = v_A^2 \hat{b} \cdot \nabla \dfrac{\delta B_\parallel}{B_0} \\[2ex] \dfrac{d}{dt} \dfrac{\delta B_\parallel}{B_0} = \dfrac{1}{1 + v_A^2/c_s^2} \hat{b} \cdot \nabla u_\parallel \\[2ex] \dfrac{d}{dt} \dfrac{\delta n}{n} = -\dfrac{1}{1 + c_s^2/v_A^2} \hat{b} \cdot \nabla u_\parallel \end{cases} \quad (9)$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \{\Phi, \ldots\}$$

$$\hat{b} \cdot \nabla = \frac{\partial}{\partial z} + \frac{1}{v_A} \{\Psi, \ldots\}$$

$$\frac{d}{dt} \frac{\delta s}{s_0} = 0$$

- Passive! — advected by Alfvén waves
- Three decoupled cascades:

Slow waves $\longrightarrow$ $W_\parallel^\pm = \displaystyle\int \frac{d^3\vec{r}}{V} \frac{m_i n_{0i}}{2} |z_\parallel^\pm|^2 , \quad z_\parallel^\pm = u_\parallel \pm \frac{\delta B_\parallel}{\sqrt{4\pi m_i n_{0i}}} \left(1 + \frac{v_A^2}{c_s^2}\right)^{1/2}$

Entropy mode $\longrightarrow$ $W_s = \left(\dfrac{3}{4}\right)^2 n_{0i} T_{0i} \displaystyle\int \frac{d^3\vec{r}}{V} \frac{\delta s^2}{s_0^2} , \quad \frac{\delta s}{s_0} = -\frac{8}{3} \left(\frac{\delta n}{n_0} + \frac{v_A^2}{c_s^2} \frac{\delta B_\parallel}{B_0}\right)$

entropy mode

One can show that

$$W = W_\perp^+ + W_\perp^- + W_\parallel^+ + W_\parallel^- + W_s \qquad (10)$$

This is a ~~general~~ typical situation: the generalised energy cascade, when considered in various asymptotic limits, splits into several non-energy-exchanging (and in some cases also non-interacting) channels. When these "subcascades" reach some physical transition scale, they get mixed together and then ~~re~~emerge on the "other side" of the transition in a different configuration.

• Thus, for $\boxed{k_\parallel \lambda_{mfp} \gtrsim 1}$ the slow waves and entropy mode cease to be decoupled and are now described by a kinetic system (ion drift kinetics, $k_\parallel \ll k_\perp$) — Alfvén waves

"Kinetic RMHD" : $\delta f_i = -\frac{q_i}{T_{0i}}(\varphi - \langle\varphi\rangle_{R_i})F_{0i} + \delta\tilde{f}_i$

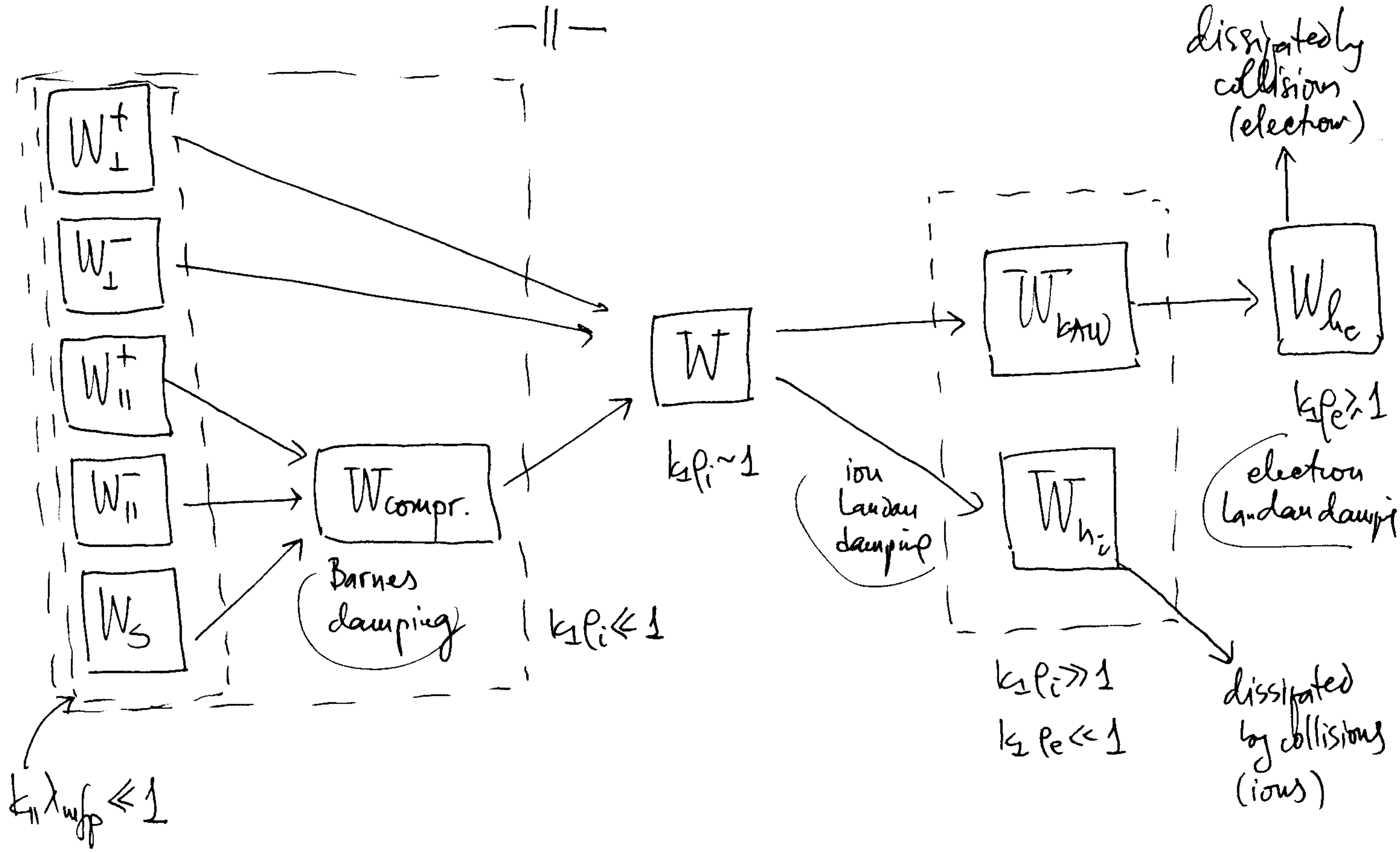
$$\left\{\begin{aligned} &\frac{d}{dt}\left(\delta\tilde{f}_i - \frac{v_\perp^2}{v_{thi}^2}\frac{\delta B_\parallel}{B_0}F_{0i}\right) + v_\parallel \hat{b}\cdot\nabla\left(\delta\tilde{f}_i + \frac{\delta n}{n}F_{0i}\right) = \left(\frac{\partial \delta\tilde{f}_i}{\partial t}\right)_c \\ &\frac{\delta n}{n} = \frac{1}{n_{0i}}\int d^3\vec{v}\,\delta\tilde{f}_i \\ &\frac{\delta B_\parallel}{B_0} = -\frac{\beta_i}{2}\frac{1}{n_{0i}}\int d^3\vec{v}\left(1+\frac{v_\perp^2}{v_{thi}^2}\right)\delta\tilde{f}_i \end{aligned}\right. \qquad (11)$$

AW nonlinearities

perturbed Maxwellian (hence indifference to collisions)

These are aperiodic, passive (to AW), Landau-damped (a Barnes) fluctuations.

$$W = \underbrace{W_\perp^+ + W_\perp^-}_{\text{Alfvén waves}} + \underbrace{\frac{n_{0i}T_{0i}}{2}\left(\frac{\delta n_e^2}{n_{0e}^2} + \frac{2}{\beta_i}\frac{\delta B_\parallel^2}{B_0^2} + \frac{1}{n_{0i}}\int d^3\vec{v}\,\frac{\delta\tilde{f}_i^2}{F_{0i}}\right)}_{\text{generalised energy of compressive fluctuations}} \qquad (12)$$



- For $\boxed{k_\perp\rho_i \sim 1}$ everything is mixed together and subject to ion Landau damping.
- At $\boxed{k_\perp\rho_i \gg 1,\ k_\perp\rho_e \ll 1}$ we again have two cascades

Kinetic Alfvén Waves (KAW)

$$\omega = \pm\frac{k_\perp\rho_i k_\parallel v_A}{\sqrt{1+\beta_i}}$$

$$\begin{cases} \dfrac{\partial\Psi}{\partial t} = 2v_A\,\hat{b}\cdot\nabla\Phi, \quad \Phi = \dfrac{c\varphi}{B_0} \\[2ex] \dfrac{\partial\Phi}{\partial t} = -\dfrac{v_A}{2(1+\beta_i)}\,\hat{b}\cdot\nabla\left(\rho_i^2\nabla_\perp^2\Psi\right) \end{cases} \qquad (13)$$

[This is related to EMHD and Hall MHD in ways explained in arXiv:0704.0044]

Now

$$W = \underbrace{\int\frac{d^3\vec{r}}{V}\left[|\nabla_\perp\Psi|^2 + \frac{n_{0i}T_{0i}}{2}\,2(1+\beta_i)\frac{\Phi^2}{\rho_i^2}\right]}_{W_{KAW}} + \underbrace{\int\frac{d^3\vec{r}}{V}\int d^3\vec{v}\,\frac{T_{0i}\langle h_i^2\rangle_r}{2F_{0i}}}_{\text{ion entropy fluctuation}} \qquad (14)$$

$\Rightarrow$ This is the part of W that got Landau-damped at $k_\perp\rho_i \sim 1$

So the second cascade channel is the ion entropy cascade

$$\frac{\partial h_i}{\partial t} + v_\parallel \frac{\partial h_i}{\partial z} + \frac{c}{B_0}\left\{\langle\varphi\rangle_{R_i}, h_i\right\} - \left(\frac{\partial h_i}{\partial t}\right)_c = \frac{\partial}{\partial t}\frac{q_i\langle\varphi\rangle_{R_i}}{T_{0i}} F_{0i} \quad (15)$$

NB: these fluctuations are invisible in (electron) fluid models

passive advection by gyro-averaged E×B flows associated with KAW or, in the absence of KAW, advection by φ associated with h_i:

$$\underbrace{2}_{\left(1+\underbrace{\frac{T_i}{T_e}\frac{q_e}{q_i}}_{=1 \text{ for simplicity}}\right)} \frac{q_i\varphi}{T_{0i}} = \frac{1}{n_{0i}}\int d^3\vec{v}\, \underbrace{\langle h_i\rangle_r}_{=\sum_k e^{i\vec{k}\cdot\vec{r}} J_0\left(\frac{k_\perp v_\perp}{\Omega_{0i}}\right) h_i(\vec{k})} \quad \text{quasineutrality} \quad (16)$$

This brings us to the last topic:

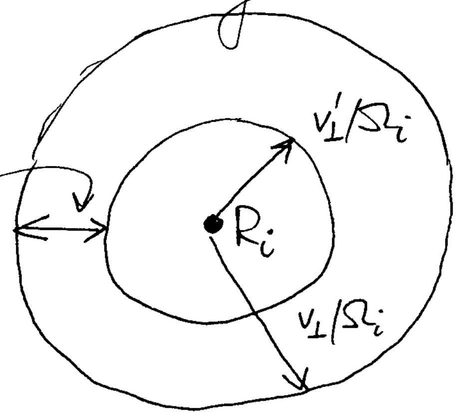
§3. Nonlinear Perpendicular Phase Mixing.

Eq. (15) describes the ion entropy cascade: it is in fact obvious that the nonlinear terms conserve

$$W_{h_i} = \int \frac{d^3\vec{R}}{V} \frac{T_{0i} h_{0i}^2}{2F_{0i}}$$

How do we get to coll. scales — and what are the coll. scales? For simplicity, consider the case with no KAW so eqns (15)-(16) are a closed system describing low-freq. electrostatic fluctuations.

- Gyroaveraged $\vec{E}\times\vec{B}$ flows $\langle\varphi\rangle_{R_i}$ mix h_i in $\vec{R}_i$ space via the nonlinear term, so h_i develops small scales in $\vec{R}_i$.
- h_i is also decorrelated in velocity ($v_\perp$) space: $h_i(\vec{R}_i, v_\perp)$ and $h_i(\vec{R}_i, v_\perp')$ are mixed by $\langle\varphi\rangle_{R_i}(v_\perp)$ and $\langle\varphi\rangle_{R_i}(v_\perp')$



If the difference between Larmor radii is bigger than the corr. scale of $\varphi(\vec{r})$ fluctuations, then $\langle\varphi\rangle_{R_i}(v_\perp)$ and $\langle\varphi\rangle_{R_i}(v_\perp')$ are completely decorrelated, so they mix h_i in a decorrelated way:

$$\left|\frac{v_\perp}{\Omega_i} - \frac{v_\perp'}{\Omega_i}\right| \sim \frac{1}{k_\perp} \Rightarrow \boxed{\frac{\delta v_\perp}{v_{thi}} \sim \frac{1}{k_\perp \rho_i}}$$

This is the perp. phase mixing mechanism with small scale structure forming simultaneously in gyrocentre and velocity space
[anticipated in paper by Dorland & Hammett 1993 PFB 5, 812]

- Structure of h_i in $\vec{R}_i$ gives rise to structure of φ in $\vec{r}$:

$$\frac{q_i\varphi_k}{T_{0i}} \sim \frac{1}{n_{0i}}\int d^3\vec{v}\; \underbrace{J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)}_{\left(\frac{\Omega_i}{k_\perp v_\perp}\right)^{1/2}\cos\left(\frac{k_\perp v_\perp}{\Omega_i}-\frac{\pi}{4}\right)} h_k \sim \frac{v_{thi}^3}{n_{0i}}\frac{1}{\sqrt{k_\perp\rho_i}}\left(\frac{\delta v_\perp}{v_{thi}}\right)^{1/2} h_k \sim \frac{v_{thi}^3}{n_{0i}}\frac{h_k}{k_\perp\rho_i}$$

This integral accumulates like random walk $\propto\left(\frac{\delta v_\perp}{v_{thi}}\right)^{1/2}$

period $\frac{\delta v_\perp}{v_{thi}} \sim \frac{2\pi}{k_\perp\rho_i}$

– but this is also the corr. scale of $h_k(v_\perp)$ in $v_\perp$ space

Now $$\frac{q_i\langle\varphi\rangle_k}{T_{0i}} \sim J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)\frac{q_i\varphi_k}{T_{0i}} \sim \frac{v_{thi}^3}{n_{0i}}\frac{h_k}{(k_\perp\rho_i)^{3/2}}$$

Thus, a fully nonlinear system:

small-scales in $\varphi(\vec{r})$ $\longrightarrow$ small scales in $\langle\varphi\rangle_{R_i}$ $\longrightarrow$ small scales in $h(\vec{R}_i, v_\perp)$ $\longrightarrow$ small-scales in $\varphi(\vec{r})$

$$\frac{c\varphi_\lambda}{B_0} \sim v_{thi}^4 h_\lambda \lambda / n_{0i}$$

- Cascade argument to get scalings:

– Entropy cascades: $\frac{m_i v_{thi}^8}{n_{0i}}\frac{h_\lambda^2}{\tau_\lambda} \sim \varepsilon$ = const flux of entropy

– Cascade time $\tau_\lambda \sim \sqrt{\frac{\rho_i}{\lambda}}\frac{\lambda^2}{c\varphi_\lambda/B_0} \sim \frac{\rho_i^{1/2}\lambda^{1/2}n_{0i}}{v_{thi}^4 h_\lambda}$

— Combining these two relations, we get

$$l_\lambda \sim \frac{n_{0i}}{v_{thi}^3}\,\frac{\rho_i^{1/6}\lambda^{1/6}}{l_0^{1/3}} \quad \Rightarrow \quad \text{spectrum } E_h(k_\perp) \sim k_\perp^{-4/3}$$

$$l_0 \equiv m_i n_{0i} v_{thi}^3/\varepsilon$$

$$\frac{q_i\varphi_\lambda}{T_{0i}} \sim \frac{\lambda^{7/6}}{\rho_i^{5/6}\, l_0^{1/3}} \quad \Rightarrow \quad \text{spectrum } E_\varphi(k_\perp) \sim k_\perp^{-10/3}$$

Measurable predictions

⇓

- See T. Tatsuno's talk for numerical confirmation
- See also G. Plunk's poster for a detailed spectral theory both in position space (Fourier) and $v_\perp$ space (Hankel)
- See T. Görler's poster for $k_\perp^{-3.2} \dots k_\perp^{-4}$ spectra for ITG and ETG — perhaps entropy cascade?

$$\tau_\lambda \sim \frac{l_0^{1/3}\rho_i^{1/3}\lambda^{1/3}}{v_{thi}}$$

⇓

From here we can estimate the collisional cutoff:

$$\frac{\delta v_\perp}{v_{thi}} \sim \left(\frac{\nu_{ii}}{\omega}\right)^{1/2} \sim (\nu_{ii}\tau_\lambda)^{1/2}$$

⇓

$$\boxed{\frac{\delta v_{\perp c}}{v_{thi}} \sim \frac{1}{k_{\perp c}\rho_i} \sim (\nu_{ii}\tau_{\rho_i})^{3/5}}$$

$$\tau_{\rho_i} \sim \left(\frac{m_i n_{0i}\rho_i^2}{\varepsilon}\right)^{1/3}$$ char. time at ion gyroscale

$\mathrm{Do} \equiv \nu_{ii}\tau_{\rho_i}$ Dorland Number

cf. $\frac{\delta v_{\perp c}}{v_{thi}} \sim \frac{1}{k_{\perp c}\rho_i} \sim \mathrm{Do}^{3/5}$ vs. $k_c L \sim Re^{3/4}$ in Kolmogorov turbulence

What is the Dorland Number of your GK simulation?

~~Note: Parallel phase mixing appears to be~~

NB: The coll. cutoff is reached in one turnover time (τ_{ρ_i}) – independent of ν_{ii}, like always in turbulence

Note: Parallel phase mixing appears to be less efficient:

$$h_i \sim e^{i k_\parallel v_\parallel t} \Rightarrow \frac{\delta v_\parallel}{v_{thi}} \sim \frac{1}{k_\parallel v_{thi} t} \sim \frac{1}{k_\parallel v_{thi} \tau_\lambda} \sim 1$$

t_λ

if the critical balance principle is obeyed (p.6)

This suggests that in this simple consideration perp. phase mixing dominates [see, however, H. Sugama's talk!]

see arXiv:0704.0044

Note: In the presence of KAW, a similar theory can be developed except now the ion entropy cascade is passive to the gyroaveraged flows associated with KAW

Note: For $k_\perp \rho_e \gg 1$, a similar theory can be developed for the electron entropy fluctuations.