A rigorous Mathematical Justification Of the 3D Finite Larmor Radius Approximation

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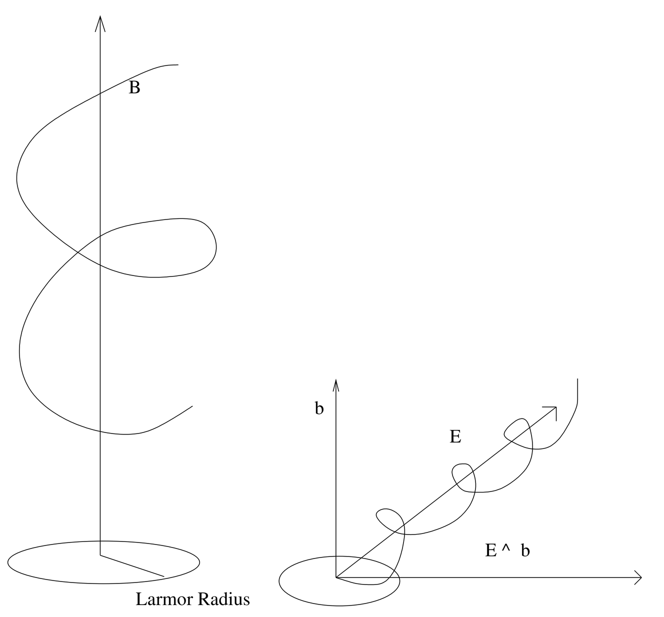
Context



The Larmor gyration VS The electric drift

We are interested in the behaviour of a plasma (that is to say a gaz made of ions and electrons) submitted to a large magnetic field. The effects we want to describe are non linear interaction between the particles and the electric field. In order to do this, we assume that the plasma is sufficiently rarefied so that one can use a kinetic description (Vlasov-Poisson system) :

$$\begin{cases} \partial_t f + v.\nabla_x f + (E + v \wedge B).\nabla_v f = 0 \\ f_{|t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \end{cases} \tag{0.1}$$

where $t \in \mathbb{R}^+, x \in \mathbb{R}^d, v \in \mathbb{R}^d$ (usually $d = 2$ or 3) are the usual physical variables. $f(t, x, v)$ is the density of ions : $f(t, x, v)dxdv$ gives the number of ions in the infinitesimal volume centered in (x, v) with length dx and dv. In our present study $B = |B|e_z$ is taken as a constant vector.

The guiding center approximation:

It is obtained as $|B|$ tends to infinity : take $|B| \sim \frac{1}{\epsilon}$ (then make it tend to 0). This leads to the following system :

$$\begin{cases} \partial_t f + v_{\|}.\nabla_x f + E_{\|}.\nabla_v f = 0 \\ f_{|t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \end{cases} \tag{0.2}$$

For some mathematical study see for example [3] or [5].

Unfortunately this approximation is not sufficient for simulation purposes since one can notice with real plasma experimentations that there is a drift appearing in long time scales. Some straightforward formal calculations with a one particle model show that the drift is actually created by the electric field. Note nevertheless that with this scaling but with a longer time scale it is still possible to make this electric drift appear (see [5] for a $2D$ study).

The finite Larmor Radius approximation:

A good way to make this drift appear is to consider the so called Finite Larmor Radius Scaling ([4]). Physically speaking it consists in not neglecting anymore the gradients in the scale of the Larmor radius in the plane perpendicular to the constant magnetic field. More precisely, the spacial observation lenght in the perpendicular plane is taken with the same order in ϵ as the Larmor Radius ($\sim \epsilon$) :

$$\begin{cases} \partial_t f_\epsilon + v_{\|}.\nabla_x f_\epsilon + \frac{v_\perp}{\epsilon}.\nabla_x f_\epsilon + \left(E_\epsilon + \frac{v \wedge e_z}{\epsilon}\right).\nabla_v f_\epsilon = 0 \\ f_{\epsilon,|t=0} = f_0 \\ E_\epsilon = \left(-\epsilon \nabla_{x_\|} V_\epsilon, -\nabla_{x_\perp} V_\epsilon\right) \\ \underline{V_\epsilon} - \epsilon^2 \Delta_{x_\|} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \rho_\epsilon \end{cases}$$

In [4], Frénod and Sonnendrücker assumed that nothing depends on $x_\|$ and $v_\|$ and proved the convergence in some weak sense to a solution of some pseudo $2D$ system.

Note here that unlike Frénod and Sonnendrücker, we consider, for technical reasons, a Poisson equation with the term V_ϵ in the left hand side. (it can be physically seen as the contribution of the electrons to the Poisson equation)

The proved result:

We assume that $f_0 \geq 0, f_0 \in L^1_{x,v} \cap L^p_{x,v}$ (for some $p > 7/2$) and the initial kinetic energy bounded $0 < \int f_0(1 + |v|^2)dv < \infty$

For each ϵ, let (f_ϵ, E_ϵ) in $L^\infty_t(L^1_{x,v} \cap L^p_{x,v}) \times L^\infty_t(L^{7/5}_{x_\|}(W^{1,\frac{7}{5}}_{x_\perp}))$ be a solution to the previous system.

Then up to a subsequence we have the following convergence in some weak sense (which will be stated more precisely after) to $(G, \mathcal{E})$ which is solution to :

$$\partial_t G + v_\|.\nabla_x G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(-\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(-\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(\tau)v)d\tau\right).\nabla_v G = 0$$

$$G_{|t=0} = \frac{1}{2\pi} f_0$$

$$\mathcal{E} = (-\nabla_\perp \Phi, 0), \ \Phi - \Delta_\perp \Phi = \int_v G(t, x + \mathcal{R}(-\tau)v, R(-\tau)v)dv$$

denoting by R and $\mathcal{R}$ the linear operators defined by :

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = -\left(R(\pi/2 + \tau) - R(\pi/2)\right)$$

The difficulty comes from the fact that there is no full elliptic regularity for the electric field because of the factor ϵ^2 in front of $\Delta_{x_\|}$. In particular there is no a priori regularity on $x_\|$ and therefore no strong compactness. Nevertheless, we actually prove that due to the particular form of the asymptotic equation, the moments with respect to $v_\|$ of the solution are more regular in $x_\|$ than the solution itself. We can then easily pass to the weak limit.

The mathematical tools

2 scale convergence from homogenization theory ([1]):

Definition 1 *Let X be a separable Banach space, X' be its topological dual space and $(.,.)$ the duality bracket between X' and X. For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L^{q'}_\alpha(\mathbb{R}; X')$) the space of α periodic continuous (respectively $L^{q'}$) functions on $\mathbb{R}$ with values in X. Let $q \in [1; \infty[$.*

Given a sequence (u_ϵ) of functions of $L^{q'}(0, t; X')$ and a function $U^0(t, \theta) \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X')$ we say that

$$u_\epsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0, T; \mathcal{C}_\alpha(\mathbb{R}, X))$ we have :

$$\lim_{\epsilon \to 0} \int_0^T \left(u_\epsilon(t), \Psi\left(t, \frac{t}{\epsilon}\right)dt\right) = \frac{1}{\alpha}\int_0^T \int_0^\alpha \left(U^0(t, \theta), \Psi(t, \theta)\right) d\theta dt$$

Theorem 1 *Given a sequence (u_ϵ) bounded in $L^{q'}(0, t; X')$, there exists for all $\alpha > 0$ a subsequence $U^0_\alpha \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X')$ such that up to a subsequence,*

$$u_\epsilon \text{ 2-scale converges to } U^0_\alpha$$

The profile U^0_α is called the α periodic two scale limit of u_ϵ and the link between U^0_α and the weak- limit u of u_ϵ is given by :*

$$\frac{1}{\alpha}\int_0^\alpha U^0 d\theta = u$$

Averaging lemma ([2]):

Theorem 2 *Let $1 < p \leq 2$. Let $f, g \in L^p(dt \otimes dx \otimes dv)$ solutions of the following transport equation*

$$\partial_t f + v.\nabla_x f = (\Delta_{t,x})^{\tau/2}(\Delta_v)^{m/2} g$$

with $m \in \mathbb{R}^+, \tau \in [0, 1[$. Then $\forall \Psi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $\rho(t, x) = \int f(t, x, v)\Psi(v)dv \in \dot{W}^{s,p}(\mathbb{R} \times \mathbb{R}^d)$ where

$$s = \frac{1 - \tau}{(1 + m)p'}$$

Moreover,

$$\|\rho\|_{W^{s,p}(\mathbb{R} \times \mathbb{R}^d)} \leq C(\|f\|_{L^p(dt \otimes dx \otimes dv)} + \|(\Delta_{t,x})^{\tau/2}(\Delta_v)^{m/2} g\|_{\dot{W}^{-\tau,p}_{t,x}(\dot{W}^{-m,p}_v)})$$

A priori estimates

Conservation of L^p norms:

For all $1 \leq p \leq \infty$,

$$\forall t \geq 0, \|f(t)\|_{L^p_{x,v}} = \|f(0)\|_{L^p_{x,v}} \tag{0.3}$$

Conservation of the energy:

$$\frac{d}{dt}\left(\int f_\epsilon |v|^2 dv dx + \epsilon \int |\nabla_{x_\perp} V_\epsilon|^2 dx + \epsilon^3 \int |\nabla_{x_\|} V_\epsilon|^2 dx\right) = 0 \tag{0.4}$$

Regularity of the ions density (interpolation):

$$\rho_\epsilon \in L^\infty_t(L^{7/5}_x)$$

Regularity of the electric field (elliptic estimates):

$$E_\epsilon \in L^\infty_t(L^{7/5}_{x_\|}(W^{1,7/5}_{x_\perp}))$$

Note that in the usual guiding center approximation, we do have $E_\epsilon \in L^\infty_t(W^{1,7/5}_{x_\|}(W^{1,7/5}_{x_\perp}))$

Ideas of the proof

The first two steps are the same as in [4] :

Getting the constraint equation:

Thanks to Theorem 1, f_ϵ 2-scale converges to $F_{2\pi} \in L^\infty(0, T; L^\infty_{2\pi}(\mathbb{R}; L^p_{x,v}))$ Considering oscillating test functions and passing to the 2-scale limit we get the usual constraint equation:

$$\partial_\tau F_{2\pi} + v_\perp.\nabla_x F_{2\pi} + v \wedge e_z.\nabla_v F_{2\pi} = 0$$

Writing that F is constant along the characteristics we get:

$$F_{2\pi}(t, \tau, x, v) = G(t, x + \mathcal{R}(-\tau)v, R(-\tau)v) \tag{0.5}$$

Filtering the essential oscillation:

Take $g_\epsilon(t, x, v) = f_\epsilon(t, x + \mathcal{R}(t/\epsilon)v, R(t/\epsilon)v)$ We easily compute the equation satisfied by w_ϵ :

$$\partial_t g_\epsilon + v_\|.\nabla_x g_\epsilon + \mathcal{R}(-t/\epsilon)E_\epsilon(t, x + \mathcal{R}(t/\epsilon)v).\nabla_x g_\epsilon$$
$$+R(-t/\epsilon)E_\epsilon(t, x + \mathcal{R}(t/\epsilon)v).\nabla_v g_\epsilon = 0$$

Getting some regularity for the moments in $v_\|$:

Define $\eta_\epsilon(t, x, v_\perp) = \int g_\epsilon(t, x, v)\Psi(v_\|)dv_\|$ with $\Psi \in \mathcal{D}(\mathbb{R})$. Then, thanks to the special structure of the previous equation, we can prove, using the averaging lemma:

$$\eta_\epsilon \text{ is uniformly bounded in } \epsilon \text{ in } W^{s,\gamma}_{t,\text{loc}}(W^{s,\gamma}_{x_\|,\text{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\text{loc}})) \tag{0.6}$$

with $\frac{1}{\gamma} = \frac{5}{7} + \frac{1}{p}$ and some $s \in]0; 1[$ (depending on γ)

Using interpolation arguments we then prove that there exists $\theta \in]0, 1[$ such that $\eta_\epsilon \in W^{s\theta,7/2}_{t,\text{loc}}(W^{s\theta,7/2}_{x_\|,\text{loc}}(W^{-\theta,7/2}_{x_\perp,v_\perp,\text{loc}}))$ Finally a "Aubin-Lions" type of lemma shows that there exists η such that up to a subsequence :

$$\|\eta_\epsilon - \eta\|_{L^{7/2}_{t,\text{loc}}(L^{7/2}_{x_\|,\text{loc}}(W^{-\theta-\xi,7/2}_{x_\perp,v_\perp,\text{loc}}))} \to 0$$

(with $\theta + \xi \leq 1$)

Passing to the limit:

Finally it is an easy game to study the convergence of the non linear term $R(-t/\epsilon)E_\epsilon(t, x + \mathcal{R}(t/\epsilon)v).\nabla_v g_\epsilon$. The idea is that we manage to compensate for the lack of compactness of the electric field with respect to the $x_\|$ variable by getting some on the moment with respect to the $v_\|$ variable which is the relevant quantity when one passes to the weak limit in the sense of distributions.

Prospects

A lot of other issues remain to be seen : what asymptotic equation do we get if we consider a "slowly" varying magnetic field ? Which model should we consider for the electrons ? How should we choose the scaling for the Debye Lenght and the Larmor Radius ?

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