Max-Planck-Institut
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Rotation in tokamaks and stellarators

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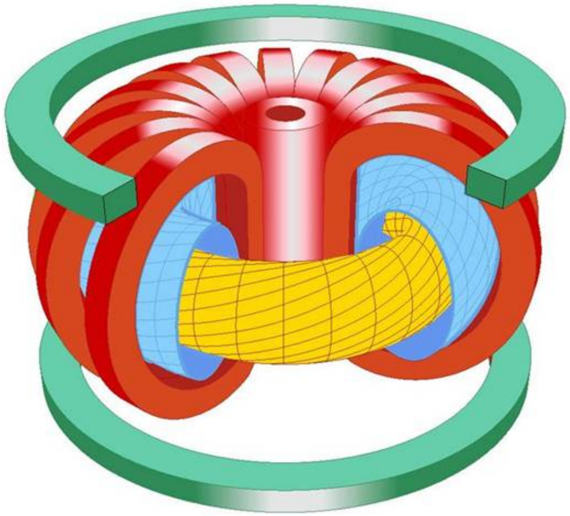
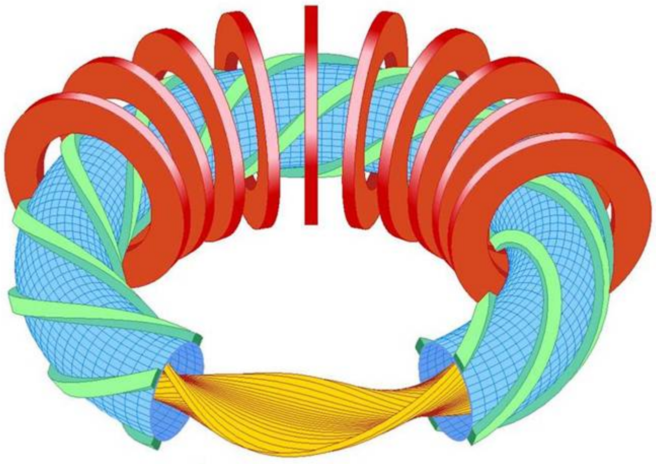
What is a stellarator?

- In a tokamak (Tamm and Sakharov, 1951), the magnetic field lines twist around the torus because of the toroidal plasma current.

- In the stellarator (Spitzer, 1951), this twist is imposed by external coils.
  - Magnetic field is necessarily 3D.
  - No toroidal current is necessary.
  - Less „free energy“ in the plasma, no dangerous instabilities.



Tokamak



Stellarator

Advantages with stellarators

- No current drive – inherently steady state
- No dangerous instabilities:
  - no disruptions, sawteeth or large ELMs
- No Greenwald density limit
- High-density operation possible
  - Much lower alpha-particle pressure $(p_\alpha \sim n_e^{-5/2})$
  - No fast-ion-driven instabilities?
  - Easier divertor operation?
- Greater deal of control over the plasma
- For a theoretician, more interesting!
  - theory plays a much stronger role.

Questions answered in this talk

- Can a tokamak or stellarator plasma rotate?
- If so, how rapidly and in what direction?
- What determines the rotation?
  - What is the relative role of collisional (neoclassical) processes and gyrokinetic turbulence?

Tokamaks

$$\mathbf{B} = I(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi$$

$$\psi = \text{poloidal flux}$$

$$\varphi = \text{toroidal angle}$$

Intrinsic ambipolarity in tokamaks

- In a tokamak, the collisional transport is independent of the radial electric field,

$$\mathbf{E} = -\nabla\Phi = -\Phi'(\psi)\nabla\psi$$

- Why? A Galilean transformation

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}$$

to a toroidally rigidly rotating frame

$$\mathbf{V} = \omega(\psi) R\hat{\varphi}$$

gives

$$\mathbf{E}' = -\Phi'\nabla\psi + \omega R\hat{\varphi} \times (\nabla\varphi \times \nabla\psi) = 0$$

if

$$\omega(\psi) = -\frac{d\Phi}{d\psi}$$

But the physics is the same in the rotating frame as in the lab frame, except for

- the centrifugal force, which is small (quadratic in E/Bv_{Ti})
- the Coriolis force, which gives rise to a new drift that is odd in $v_{||}$

- Radial electric field = toroidal rotation is set by the radial transport of toroidal angular momentum.

- Fundamentally, there are three conserved quantities undergoing transport:
  - particles, energy and angular momentum

$$\begin{pmatrix} \mathbf{\Gamma}_i \cdot \nabla\psi \\ \mathbf{q}_i \cdot \nabla\psi \\ R\hat{\varphi} \cdot \pi \cdot \nabla\psi \end{pmatrix} \propto - \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} p_i'/p_i \\ T_i'/T_i \\ \omega'/\omega \end{pmatrix}$$

- We expect the turbulent momentum transport to be important for determining E_r.

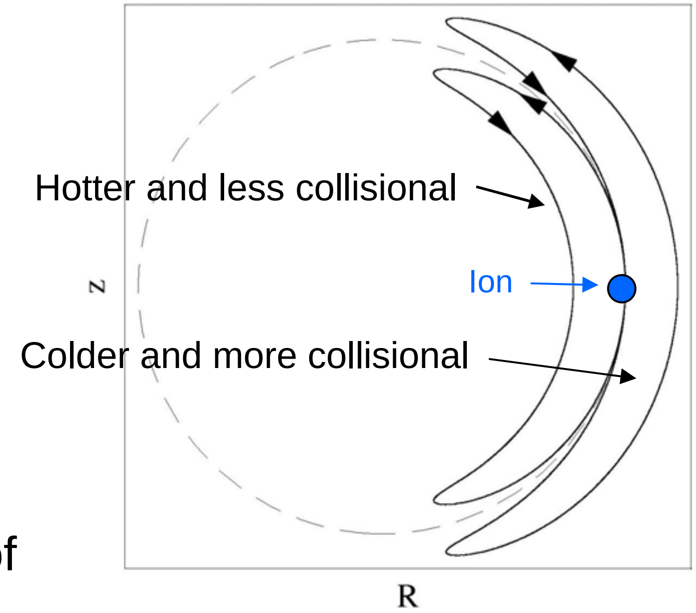
Neoclassical poloidal rotation

- Poloidal rotation is predicted to be slow due to friction between circulating and trapped particles.
  - Residual due to themal force:

$$V_p = \frac{k}{eB}\frac{dT_i}{dr}$$

$$k = 1.17 \quad \text{(banana regime)}$$

- Note that the poloidal rotation is independent of the radial electric field!



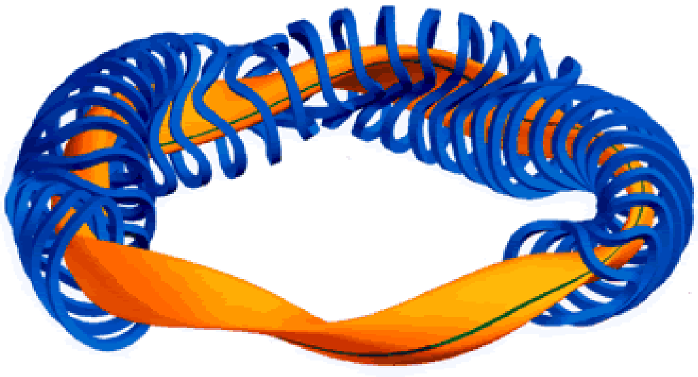
Summary so far

- A tokamak plasma may rotate toroidally but not poloidally (in lowest order)
  - Because of friction between trapped and circulating ions, poloidal rotation is damped to the level

$$V_\theta \sim O(\rho_i v_{Ti}/a)$$

- More precisely,
  - poloidal rotation is damped on the ion collision time scale
  - toroidal rotation is damped on the confinement time scale

Stellarators



Rapid rotation

$$(V \sim v_{Ti})$$

Drift kinetics

- The most robust predictions of plasma theory are made in the limit of vanishing gyro-radius. What do we learn about equilibrium?

- Start with the kinetic equation in a frame moving with the local velocity $\mathbf{V} \sim v_T$

$$\frac{\partial f}{\partial t}+(\mathbf{V}+\mathbf{v})\cdot\nabla f+\frac{e}{m}\left(\mathbf{E}+(\mathbf{V}+\mathbf{v})\times\mathbf{B}-\frac{\partial \mathbf{V}}{\partial t}-(\mathbf{V}+\mathbf{v})\cdot\nabla\mathbf{V}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}=C(f)+S,$$

- Look for magnetised equilibrium

$$\delta = v_T/\Omega L \ll 1 \qquad \partial f_0/\partial t \ll (v_T/L) f_0$$

- In lowest order

$$\frac{e}{m}(\mathbf{E}+\mathbf{V}\times\mathbf{B}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_0}{\partial \mathbf{v}}=0$$

implies that f_0 is independent of gyroangle and

$$\mathbf{E}+\mathbf{V}\times\mathbf{B}=0, \qquad \mathbf{B}\cdot\nabla\Phi_0=0 \qquad \Rightarrow \qquad \Phi_0=\Phi_0(\psi)$$

0‘th order distribution function

- In next order, one obtains the drift kinetic equation

$$(v_{\|}\mathbf{b}+\mathbf{V})\cdot\nabla f_0 + \dot{w}\frac{\partial f_0}{\partial w} + \dot{\mu}\frac{\partial f_0}{\partial \mu} = C(f_0),$$

with $w = mv^2/2, \mu = mv_\perp^2/2B,$

$$\dot{w} = eE_{\|}v_{\|} - mv_{\|}\mathbf{V}\cdot\nabla\mathbf{V}\cdot\mathbf{b} - mv_{\|}^2\mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} + \mu B\mathbf{V}\cdot\nabla\ln B$$

$$\dot{\mu} = 0$$

- Entropy balance: multiply by ln f_o and integrate over velocity space

$$\nabla\cdot\mathbf{G} = -\int \ln f_0 C(f_0)\, 2\pi v_\perp dv_\perp dv_{\|}, \qquad \text{where}$$

$$\mathbf{G} = -\int (\mathbf{V}+v_{\|}\mathbf{b})f_0(\ln f_0 - 1)\, 2\pi v_\perp dv_\perp dv_{\|}$$

- But the flux-surface average of $\nabla\cdot\mathbf{G}$ vanishes, so f_0 is Maxwellian.

Constraints on f_0

Substituting f_0 back into the drift kinetic equation gives the constraints

$$\mathbf{b}\cdot\nabla T = 0$$
$$\mathbf{b}\cdot\nabla n = 0$$
$$\nabla\cdot\mathbf{V} = 0$$
$$\mathbf{b}\cdot\nabla\mathbf{V}\cdot\mathbf{b} = 0$$

Substituting $\mathbf{V}_\perp = \mathbf{b}\times\nabla\Phi(\psi)/B$ gives

$$(\nabla\psi\times\nabla B)\cdot\nabla(\mathbf{b}\cdot\nabla B) = 0 \qquad \text{or} \qquad \mathbf{V} = 0$$

Thus, $\nabla_{||}B$ should not change when moving along a line of constant B, or

$$B = B(\psi, l),\ \text{where l = arc length} \qquad \text{or} \qquad \mathbf{V} = 0$$

Conclusion about fast rotation

- Rapid plasma rotation (comparable to ion thermal speed) is only possible is certain magnetic fields.
  - The parallel variation of |B| must be the same for all field lines on each flux surface.
- This follows from the kinetic equation <u>in zeroth order</u> (in gyroradius).
  - is therefore independent of collisionality and of turbulence!
- Physical reason:
  - In this order, the parallel transport is infinitely faster than cross-field transport.
  - Flux surfaces must therefore be isothermal and isobaric.
  - Also, there must be no parallel transport of momentum.
  - Rotation is only possible if a gyrating particle „cannot tell“ the difference between different field lines.
    - Magnetic field strength and mirror force must be the same on all field lines on a flux surface.

Slow rotation

$$V \sim (\rho_i/L) v_{Ti}$$

Intrinsic ambipolarity in stellarators

- Consider the drift kinetic equation in the absence of density or temperature gradients

$$v_\parallel \nabla f_{a1} - C_a(f_{a1}) = -\mathbf{v}_d \cdot \nabla f_{a0} = (\mathbf{v}_d \cdot \nabla \psi) \frac{e_a}{T_a} \frac{d\Phi_0}{d\psi} f_{a0}$$

  – Multiply by $T_a f_{a1}$, sum over all species, integrate over velocity space and take the flux-surface average.

- This gives an entropy theorem

$$\frac{d\Phi_0}{d\psi} \langle \mathbf{j} \cdot \nabla \psi \rangle = \sum_a T_a \langle \int f_{a1} C_a(f_{a1}) d^3 v \rangle \leq 0$$

with equality if, and only if,

$$\bar{f}_{a1} = (\alpha_a + \beta_a v_\parallel + \gamma_a v^2) f_{a0} \qquad (T_{a0}\beta_a / m_a = T_{b0}\beta_b / m_b)$$

  – But then f_{a1} must be odd in parallel velocity, so $\alpha_a = \gamma_a = 0.$

But then

$$v_\| \nabla_\| \bar{f}_{a1} = v^2 \left[(1-\lambda B) \nabla_\| \beta_a - \beta_a \lambda \nabla_\| B/2 \right] f_{a0}$$

must equal

$$-\mathbf{v}_d \cdot \nabla f_{a0} = \frac{m_a v^2}{e_a B^3} \frac{\partial f_{a0}}{\partial \psi} \left(1 - \frac{\lambda B}{2} \right) (\mathbf{B} \times \nabla B) \cdot \nabla \psi$$

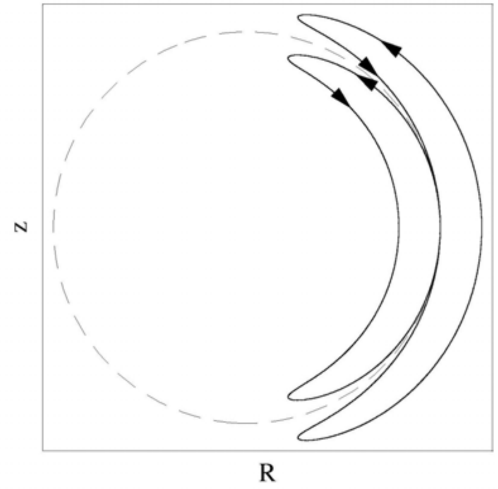
for all values of $\lambda = 2\mu / m_a v^2$ which requires

$$\frac{(\mathbf{B} \times \nabla \psi) \cdot \nabla \ln B}{\nabla_\| B} = \text{flux function} = F(\psi)$$

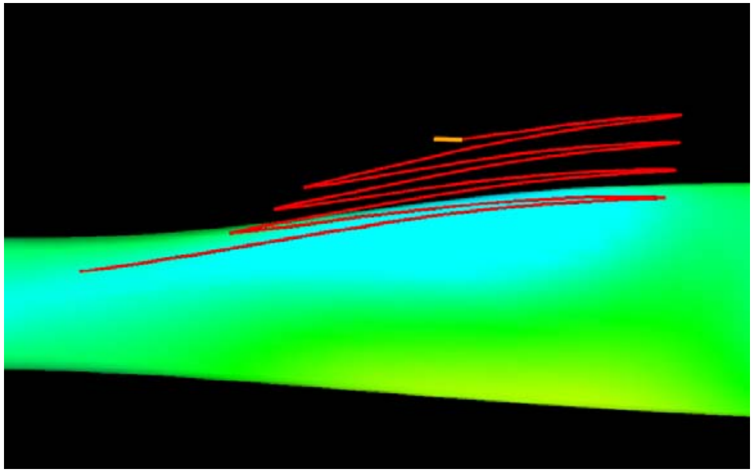
What does this mean?

Particle orbits

- In tokamaks, particles always return to the flux surface they started on.
  - mathematically, because of toroidal symmetry (angular momentum conservation)



- In stellarators, particles generally drift out of the plasma.

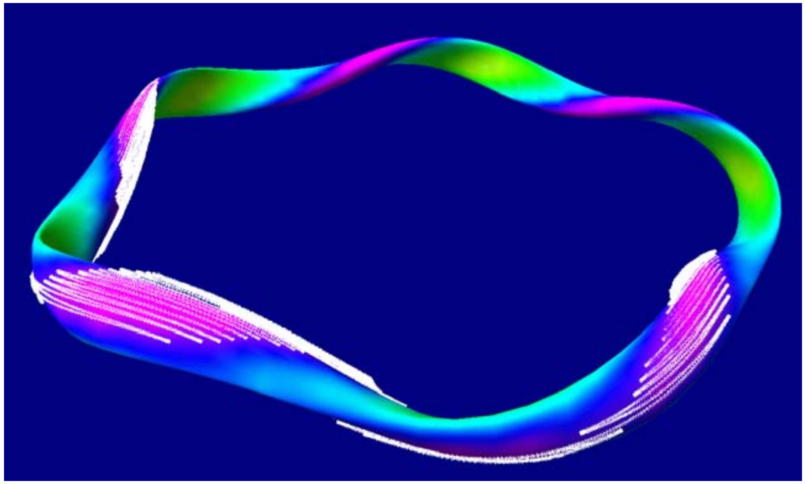


- The mean-free path of a fusion-produced alpha particle is

$$l_\alpha \sim 10^4 \text{ km}$$

Orbit confinement and symmetry

- In the tokamak, trapped particles precess toroidally around the torus
- Fundamental reason for confinement: |B| is toroidally symmetric.
- Topologically, precession can only occur in three ways:
  - Toroidally
  - Poloidally
  - Helically
- Could try to make B symmetric in these directions.



When expressed in so-called Boozer coordinates,

$$\mathbf{B} = \nabla\varphi \times \nabla\psi_p + \nabla\psi_t \times \nabla\theta = \beta(\psi_t, \theta, \varphi)\nabla\psi_t + I(\psi_t)\nabla\theta + J(\psi_t)\nabla\varphi$$

the guiding centre Lagrangian

$$L = \frac{mv_\parallel^2}{2} + e\mathbf{A}\cdot\mathbf{v} - \mu B$$

depends only on the modulus B

$$L = \frac{m}{2B^2}\left(I\dot{\theta} + J\dot{\varphi}\right)^2 + Ze\left(\psi_t\dot{\theta} - \psi_p\dot{\varphi}\right) - \mu B$$

For example, if B is „axisymmetric“ in Boozer coordinates

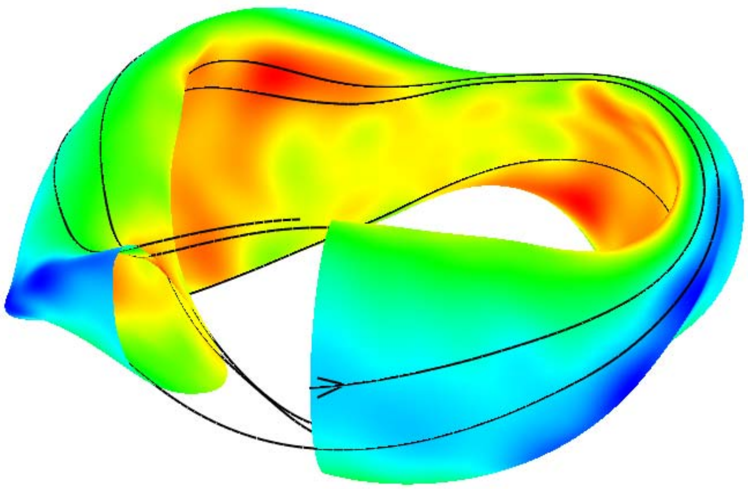
$$B = B(\psi_t, \theta)$$

there is a correponding constant of motion

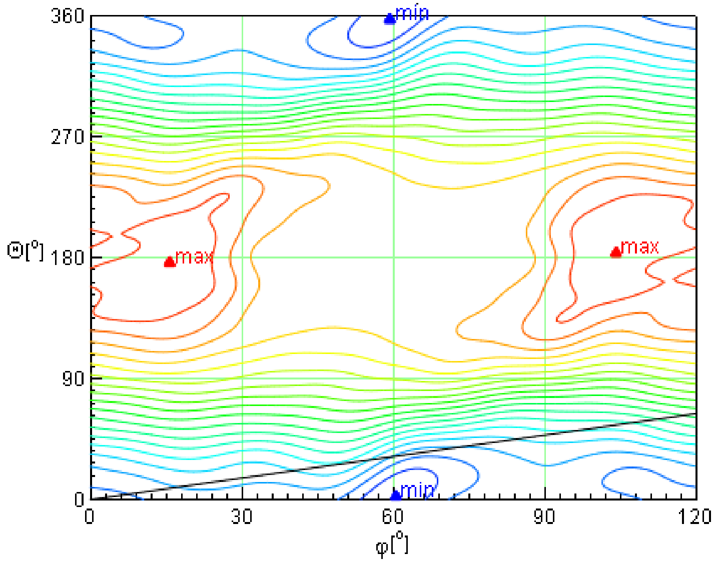
$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \frac{mIv_\parallel}{B} - e\psi_p = \text{constant}$$

|B| can be symmetric in Boozer coordinates but not in real space

NCSX (Princeton):

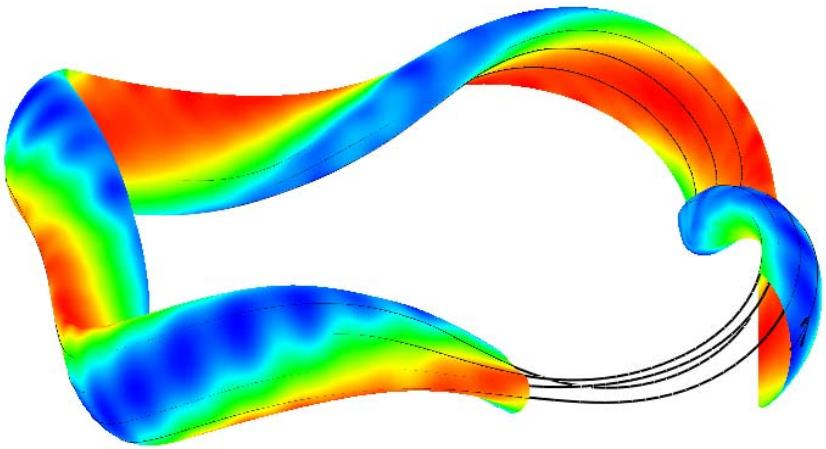


B on last closed flux surface

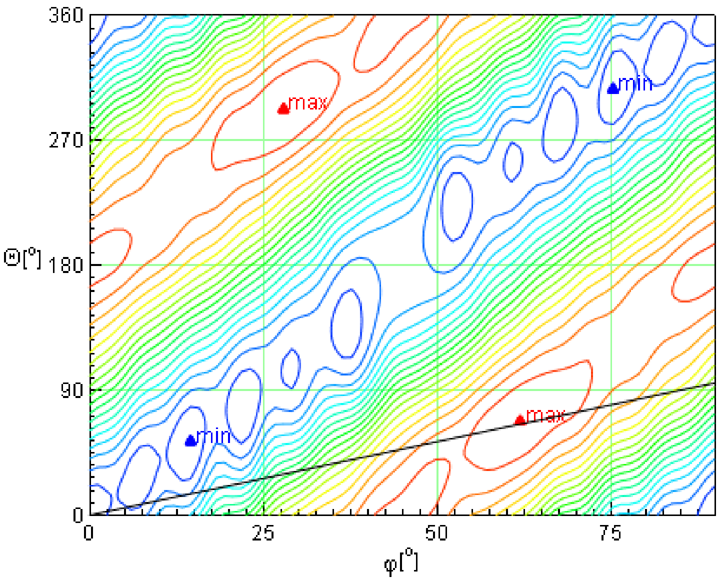


$B(\theta, \varphi)$ at $\psi/\psi_{\text{tot}} = 0.4$

- HSX (Madison)



B on last closed flux surface



$B(\theta, \varphi)$ at $\psi/\psi_{\rm tot} = 0.4$

Requirement for intrinsic ambipolarity

Recall that intrinsic ambipolarity requires

$$\frac{(\mathbf{B} \times \nabla\psi) \cdot \nabla \ln B}{\nabla_{\|} B} = \text{flux function} = F(\psi)$$

In Boozer coordinates

$$J \frac{\partial B}{\partial \theta} - I \frac{\partial B}{\partial \varphi} = F(\psi) \left(\iota \frac{\partial B}{\partial \theta} + \frac{\partial B}{\partial \varphi} \right)$$

After taking a Fourier transform

$$B(\psi, \theta, \varphi) = \sum_{m,n} b_{m,n}(\psi) e^{i(m\theta - n\varphi)}$$

we find

$$[mJ + nI - F(m\iota - n)]\, b_{m,n} = 0$$

i.e.

$$b_{m,n} = 0 \qquad \text{or} \qquad F(\psi) = \frac{(m/n)J(\psi) + I(\psi)}{(m/n)\iota(\psi) - 1}$$

which can only be satisfied if all non-zero Fourier components have the same helicity,

$$B = \sum_k \alpha_{kM,kN}(\psi) e^{ik(M\theta - N\varphi)}$$

Thus, for collisional transport

Intrinsic ambipolarity = quasisymmetry

Effect of turbulence

- Consider electrostatic, gyrokinetic turbulence:

$$\frac{e\tilde{\phi}}{T} \sim \delta = \frac{\rho_i}{L}, \qquad k_\perp \rho_i \sim 1$$

- The turbulent transport of particles, energy, and momentum is then at the gyro-Bohm level.
  - Same order in δ as neoclassical transport.
  - In the tokamak, the radial electric field is set by the balance of momentum sources and neoclassical + turbulent transport
  - What about stellarators?

General considerations

Consider the momentum equation

$$\frac{\partial(\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V}\mathbf{V} + p\mathbf{I} + \pi) = \mathbf{j} \times \mathbf{B} \qquad (1)$$

Write

$$\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_1, \qquad p = p_0(\psi) + p_1, \qquad \text{where } \mathbf{j_0} \times \mathbf{B} = \nabla p_0$$

Multiply (1) by $\mathbf{G} = \mathbf{j}_0 / p_0'(\psi)$, and take an average over time and over the flux surface:

$$\begin{aligned}\langle \mathbf{j} \cdot \nabla \psi \rangle &= - \langle \nabla \cdot (\rho \mathbf{V}\mathbf{V} + p\mathbf{I} + \pi) \cdot \mathbf{G} \rangle \\ &= \langle (\rho \mathbf{V}\mathbf{V} + \pi) : \nabla \mathbf{G} \rangle - \frac{1}{V'(\psi)} \frac{\partial}{\partial \psi} \langle V'(\psi) \, (\rho \mathbf{V}\mathbf{V} + \pi) : \mathbf{G} \nabla \psi \rangle\end{aligned}$$

$$V(\psi) = \text{ volume inside flux surface } \psi$$

Consequences of the gyrokinetic ordering

Normally in gyrokinetics

$$\pi = (p_{||} - p_{\perp})\,(\mathbf{bb} - \mathbf{I}/3) + O(\delta^2 p)$$

$$p_{||} - p_{\perp} \sim \delta p$$

$$\mathbf{V} \sim \delta v_{Ti}$$

$$\nabla \sim 1/(\delta L)$$

As a result, $\rho \mathbf{VV} \ll \pi$ and

$$\langle \mathbf{j} \cdot \nabla \psi \rangle = \langle (\rho \mathbf{VV} + \pi) : \nabla \mathbf{G} \rangle - \frac{1}{V'(\psi)} \frac{\partial}{\partial \psi} \langle V'(\psi)\,(\rho \mathbf{VV} + \pi) : \mathbf{G} \nabla \psi \rangle$$

small

comparable

Radial current

Again

$$\langle \mathbf{j}\cdot\nabla\psi\rangle = \langle \pi : \nabla\mathbf{G}\rangle - \frac{1}{V'(\psi)}\frac{\partial}{\partial\psi}\langle V'(\psi)\left(\rho\mathbf{V}\mathbf{V}+\pi\right):\mathbf{G}\nabla\psi\rangle$$

Neoclassical transport vanishes in quasisymmetric B

Reynolds stress

Fluctuating parts of the viscosity

Integrate over the volume, ΔV, between two flux surfaces several gyroradii apart.

$$\int_{\psi_1}^{\psi_2}\langle \mathbf{j}\cdot\nabla\psi\rangle\, V' d\psi = \int_{\psi_1}^{\psi_2}\langle \pi : \nabla\mathbf{G}\rangle\, V' d\psi - \left[\langle \rho\mathbf{V}\mathbf{V}+\pi) : \mathbf{G}\nabla\psi V'\rangle\right]_{\psi_1}^{\psi_2}$$

$$\sim \delta p \Delta V$$

$$\sim \delta p \Delta V (\rho_i/\Delta r) = \mathrm{small}$$

Effect of turbulent Reynolds stress

- Locally, the turbulent Reynolds stress is as important as the non-ambipolar neoclassical current, but

- On a radial average, taken over several gyroradii, the current created by parallel viscosity dominates:

$$\int_{\psi_1}^{\psi_2} \langle \mathbf{j} \cdot \nabla \psi \rangle \, V' d\psi \simeq \int_{\psi_1}^{\psi_2} \langle \pi_{\|} : \nabla \mathbf{G} \rangle \, V' d\psi$$

- This current is the neoclassical current.

- Only if the contribution from parallel viscosity vanishes for some reason does Reynolds stress become important.
  - This is the case in quasisymmetric B.

Conclusions

- In tokamaks, the plasma can rotate almost freely in the toroidal direction
  - Intrinsically ambipolar transport
  - Toroidal rotation damped on the confinement time scale, can be comparable to the ion thermal speed.
  - Poloidal rotation is small and damped on the collision time scale
- In stellarators, radial ion and electron particle fluxes are not automatically equal.
  - Radial electric field determined by ambipolarity
  - Exception: quasisymmetric stellarators have exactly the same neoclassical properties as tokamaks
- The plasma is free to rotate only if B is quasisymmetric. Otherwise
  - Electrostatic turbulence is unlikely to affect the rotation.
  - Small-scale zonal flows are possible, however.
  - It is easer to calculate the radial electric field than in a tokamak!

References: P. Helander, Phys. Plasmas **14**, 104501 (2007);
P. Helander and A.N. Simakov, to appear in Phys. Rev. Lett. (2008).