Gyro-fluid models: theory and numerics

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Summary

1. Introduction

Multiphysics problems

- Perturbed problem $P^\varepsilon \longrightarrow$ limit problem P^0
  - Standard scheme $P^{\varepsilon,h}$: stability $h \leq C\varepsilon$

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- Value of ε is local
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  - $\varepsilon = O(1)$ in other areas
- Multiphysics coupling
  - Use P^ε where $\varepsilon = O(1)$
  - Use P^0 where $\varepsilon \ll 1$

Problems with multiphysics coupling

- Location of the interface region
  - Find it (and move it)
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  - Choices often arbitrary (layer analysis provides only incomplete information)
  - Replacement of smooth by abrupt transition leads to wrong physics
- Results depend on these choices
  - Lack of reliability & robustness

Asymptotic-Preserving (AP) strategy

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  - Stability h independent of ε
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P^{\varepsilon,h} \arrow[r, "h\to 0"] \arrow[d, "\varepsilon\to 0"] & P^{\varepsilon} \arrow[d, "\varepsilon\to 0"] \\
P^{0,h} \arrow[r, "h\to 0"] & P^{0}
\end{tikzcd}$$

AP strategy

- AP scheme: use the perturbation problem P^ε
    - and discretize it with scheme $P^{\varepsilon,h}$ such that

$$h \sim \Delta x, \Delta t \quad \textbf{independent of} \quad \varepsilon \ll 1$$

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- Allows the simulation of limit regime P^0 with the perturbation problem P^ε
  - With **arbitrary large** time / space steps compared to ε
- No need to change the model from P^0 to P^ε
  - The transition is done by the scheme **automatically**

Examples

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- Non-QuasiNeutral to QN: $\varepsilon =$ Debye length
  - Euler-Poisson [Crispel, D. Vignal] & [D., Liu, Vignal]
  - Vlasov-Poisson (PIC) [D. Deluzet, Navoret, Sun, Vignal]
  - Vlasov-Poisson (Eulerian meth.) [Belaouar, Crouseilles, D., Sonnendrücker]
  - Euler-Maxwell [D. Deluzet, Savelief, Vignal]

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  - Euler-Maxwell [D. Deluzet, Savelief, Vignal]
- Drift-fluid limit: $\varepsilon =$ cyclotron freq. (inverse)
  - Euler-Lorentz [D. Deluzet, Sangam, Vignal]
  - This talk ...

AP schemes: general methodology

- P^ε singularly perturbed problem
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- Step 1: 'Discretization'
  - Dirscretize P^ε into $P^{\varepsilon,h}$ $\quad (h = \min(\Delta t, \Delta x))$ s.t.
  - $P^{0,h} := \lim_{\varepsilon\to 0} P^{\varepsilon,h}$ is a scheme for P^0
  - Find $R^{\varepsilon,h}$ a regular perturbation form. as $\varepsilon \to 0$

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    - But only there

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- $P^{0,h} := \lim_{\varepsilon \to 0} P^{\varepsilon,h}$ is a scheme for P^0 :
  - Difficult part: requires impliciteness where the problem is singularly perturbed
  - But only there

- To preserve properties such as conservation, positivity, . . . it is preferable to
  - 'Reformulate the discretization'
  - than to 'Discretize the reformulation'

2. Drift-Fluid limit of Euler-Lorentz model

The Euler-Lorentz model

- Isothermal pressure law for clarity

$$\begin{cases} \partial_t n + \nabla \cdot (n\,u) = 0, \\ m\left(\partial_t (n\,u) + \nabla\,(n\,u \otimes u)\right) + T\,\nabla n \\ \qquad\qquad\qquad\qquad\qquad = q\,n\,(E + u \times B), \end{cases}$$

- $n =$ ion density, $u =$ ion velocity,

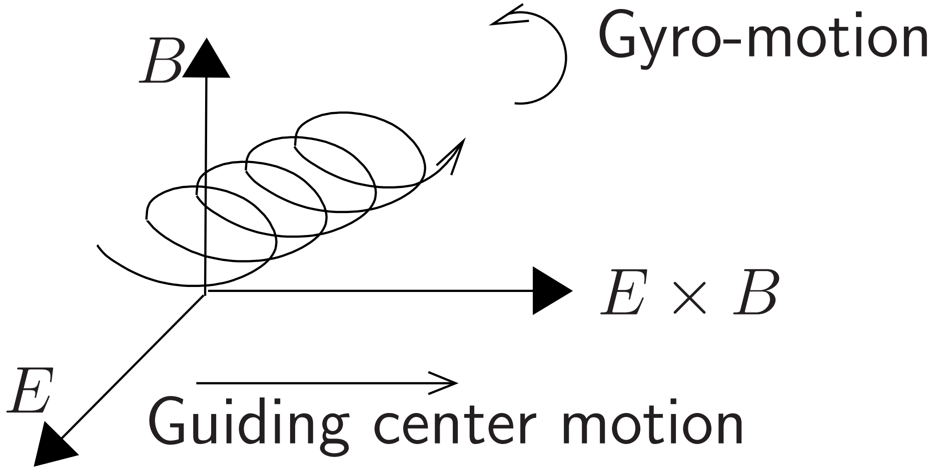
  $m =$ ion mass, $T =$ constant temperature,

  $q =$ ion charge, $E =$ electric field,

  $B =$ magnetic field.

The drift-fluid regime

- Motion of a particle in an electromagnetic field



- Regime such that:
  - Lorentz and pressure forces are very large
- Consequences:
  - gyro-period$\ll 1$.
  - Dynamics $\parallel B$ much quicker than $\perp B$.

The rescaled Euler-Lorentz model

- Lorentz and pressure forces very large

- Rescaling the problem

$$(EL_\varepsilon) \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0, \\ \varepsilon \Big(\partial_t (n\, u) + \nabla\, (n\, u \otimes u) \Big) + T\, \nabla n \\ \qquad = n\, (E + u \times B), \end{cases}$$

- $\varepsilon = \dfrac{\text{gyro-period}}{\text{carac. time}} = (\text{Mach number})^2 = \dfrac{m u_0^2}{T_0} \ll 1$

References

- Fluid models
  - Reference: [Hazeltine, Meiss]
  - [Ottaviani, Manfredi, PoP 1999], [Garbet et al, PoP 2001] [Falchetto, Ottaviani, PRL 2004]
  - Beer, Dorland, Snyder, . . .

References

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  - [Ottaviani, Manfredi, PoP 1999], [Garbet et al, PoP 2001] [Falchetto, Ottaviani, PRL 2004]
  - Beer, Dorland, Snyder, . . .
- Kinetic models (gyrokinetic)
  - Reference: [Hazeltine, Meiss] (again !)
  - Math analysis: [Frenod, Sonnendrücker, . . .]
  - Many codes: e.g. [Sonnendrücker & coworkers, Garbet, Grandgirard . . .]
  - Note, often in combination with δf method (e.g. [Chen & Parker])

Formal drift-fluid limit I

- $\varepsilon \to 0$ in $(EL_\varepsilon) \Rightarrow$ Drift-fluid model

$$(DF) \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 \\ T\, \nabla n = n\, (E + u \times B), \end{cases}$$

- Splitting the velocity according $b = \dfrac{B}{\|B\|}$

$$\Downarrow$$

$$u = \overbrace{(I - b \otimes b)\, u}^{u_\perp} + \overbrace{(u \cdot b)\, b}^{u_\| b}$$

Formal drift-fluid limit II

- Projection of the "momentum eq."
  - Perpendicular part

$$b\times(T\,\nabla n - nE = n\,u\times B) \Rightarrow nu_\perp = \frac{b}{\|B\|}\times(T\nabla n - n\,E)$$

$$\Rightarrow \text{Explicit eq. for } nu_\perp$$

  - Parallel part

$$b\cdot(T\,\nabla n - nE = n\,u\times B) \Rightarrow b\cdot(T\,\nabla n - nE) = 0$$

$$\Rightarrow \text{Implicit eq. for } nu_\parallel$$

- $u_\parallel$ = Lagrangian multiplier of $b\cdot(T\,\nabla n - nE) = 0$

Explicit eq. for the parallel velocity I

- For clarity B =constant (not necessary)

$$(DF) \Leftrightarrow \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 & (1) \\ n u_\perp = \dfrac{b}{\|B\|} \times (T \nabla n - n\, E) & (2) \\ b \cdot (T\, \nabla n - n E) = 0 & (3) \end{cases}$$

- $T\, b \cdot \nabla (1) \quad \Rightarrow \quad T\, b \cdot \nabla \partial_t n + T\, b \cdot \nabla (\nabla \cdot (n\, u)) = 0$

- $\partial_t (3) \quad \Rightarrow \quad T\, b \cdot \nabla \partial_t n - \partial_t (n\, b \cdot E) = 0$

Taking the difference $\Rightarrow$ Explicit elliptic eq. for $u_\|$

Explicit eq. for the parallel velocity II

- Explicit elliptic eq. for $u_\|$

$$-T \underbrace{(b \cdot \nabla)(\nabla \cdot (nu_\| b))}_{\nabla_\|(\nabla_\| \cdot (nu)_\|)} = T\, b \cdot \nabla(\nabla \cdot (nu_\perp)) + \partial_t(n\, b \cdot E)$$

$$\longrightarrow \text{dual operators}$$

- Reformulated Drift-Fluid model

$$(DF) \Leftrightarrow (RDF) \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0, \\ nu_\perp = \dfrac{b}{\|B\|} \times (T \nabla n - n\, E), \\ -T\, \nabla_\|(\nabla_\| \cdot (nu)_\|) = RHS. \end{cases}$$

Reformulated Euler-Lorentz system I

$$\begin{tikzcd}
\text{Euler-Lorentz} \arrow[r, "\varepsilon \to 0"] \arrow[d] & \text{Drift-Fluid} \arrow[d] \\
\text{Reformulated Euler-Lorentz} \arrow[r, "\varepsilon \to 0"'] & \text{Reformulated Drift-Fluid}
\end{tikzcd}$$

- In the Euler-Lorentz model

$$(T\, b \cdot \nabla) \text{ Mass eq.} - (b \cdot \partial_t) \text{ Momentum eq.}$$

$$\Downarrow$$

$$\varepsilon\, \partial^2_{tt}(n u_\parallel) - T(b \cdot \nabla)(\nabla \cdot (n u_\parallel\, b)) = RHS$$

Reformulated Euler-Lorentz system II

- Reformulated Euler-Lorentz model

$$
(REL_\varepsilon)\left\{\begin{array}{l}
\partial_t n + \nabla\cdot(n\,u) = 0, \\[2ex]
\Bigg(\varepsilon \Big(\partial_t(n\,u) + \nabla\,(n\,u\otimes u)\,\Big) + T\,\nabla n \\[2ex]
\qquad\qquad\qquad\qquad\qquad\qquad = n\,(E + u\times B)\Bigg)_\perp \\[2ex]
\varepsilon\,\partial^2_{tt}(nu_\parallel) - T(b\cdot\nabla)(\nabla\cdot(nu_\parallel\,b)) = RHS
\end{array}\right.
$$

- Equivalent to the Euler-Lorentz system

Reformulated Euler-Lorentz system III

- Reduces to (RDF) when $\varepsilon = 0 \Rightarrow$ consistency property.

- Wave Eq. on $nu_{\|}$
  - Explicit scheme $\Rightarrow$ conditional stability
  - Implicit scheme $\Rightarrow$ unconditional stability

3. AP-scheme for the Euler-Lorentz model in the DF limit

Classical scheme I

- If n^m and u^m known approx. at time t^m

$$\begin{cases} \dfrac{n^{m+1} - n^m}{\Delta t} + \nabla \cdot (n\,u)^m = 0, \\ \varepsilon \left(\dfrac{(n\,u)^{m+1} - (n\,u)^m}{\Delta t} + \nabla\,(n\,u \otimes u)^m \right) + T\,\nabla n^m \\ \qquad\qquad\qquad = n^{m+1}\,(E + u \times B)^{m+1}, \end{cases}$$

- Stable and consistant iff $\Delta t = O(\varepsilon)$

$\varepsilon = 0 \Rightarrow$ we lose $u_{\parallel}^{m+1} \Rightarrow$ consistency pb

Classical scheme II

- Discrete reformulation

$(T\, b \cdot \nabla)$ Mass eq. $-$ $(b \cdot \text{discrete}\, \partial_t)$ Momentum eq.

$$\Downarrow$$

$$\varepsilon\, \frac{(nu_\|)^{m+1} - 2(nu_\|)^{m} + (nu_\|)^{m-1}}{\Delta t^2}$$
$$-T(b \cdot \nabla)(\nabla \cdot (nu_\|\, b)^{m-1}) = RHS$$

- Explicit scheme ⇒ conditional stability

AP scheme I

$$\begin{cases} \dfrac{n^{m+1}-n^m}{\Delta t} + \nabla \cdot (n\,u)^{m+1} = 0, \\ \varepsilon\left(\dfrac{(n\,u)^{m+1}-(n\,u)^m}{\Delta t} + \nabla (n\,u \otimes u)\right)^m + T\,(\nabla n)^{m+1/2} \\ \qquad\qquad = n^m\,E^{m+1} + (n\,u \times B)^{m+1} \\ (\nabla n)^{m+1/2} = (\nabla n)^{m+1}_{//} + (\nabla n)^m_{\perp}, \end{cases}$$

- Asymptotically stable and consistant

$\varepsilon \to 0 \Rightarrow$ Discretization of (RDF)

AP scheme II

- Discrete reformulation

$$(T\,b\cdot\nabla)\ \text{Mass eq.} - (b\cdot\text{discrete}\,\partial_t)\ \text{Momentum eq.}$$

$$\Downarrow$$

$$\varepsilon\,\frac{(nu_\|)^{m+1} - 2(nu_\|)^{m} + (nu_\|)^{m-1}}{\Delta t^2} - T(b\cdot\nabla)(\nabla\cdot(nu_\|\,b)^{m+1}) = RHS$$

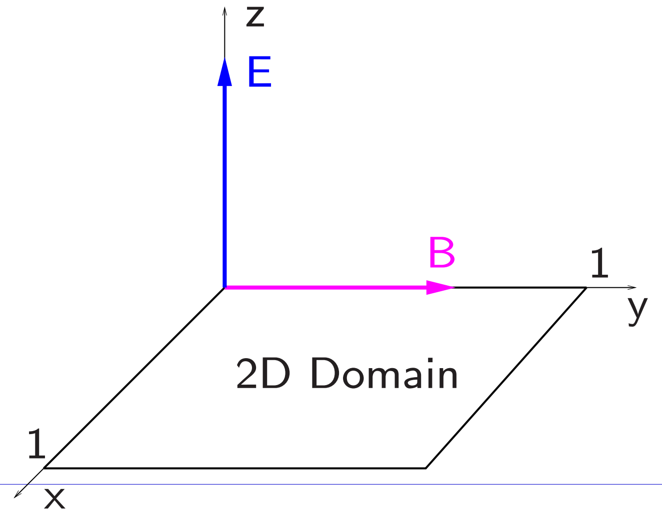
- Implicit scheme ⇒ unconditional stability

Model and parameters

$$(EL_\varepsilon)\left\{\begin{array}{l} \partial_t n + \nabla\cdot(n\,u) = 0, \\ \varepsilon\,\Big(\partial_t(n\,u) + \nabla\,(n\,u\otimes u)\Big) + T\,\nabla n \\ \qquad\qquad\qquad\qquad\qquad = n\,(E + u\times B), \end{array}\right.$$

➠ $T = 1,\qquad E = (0,0,1),\qquad B = (0,1,0),$

$\varepsilon = 10^{-6}$ or 1, $\qquad \Delta x = \Delta y = 1/100.$



Model and parameters

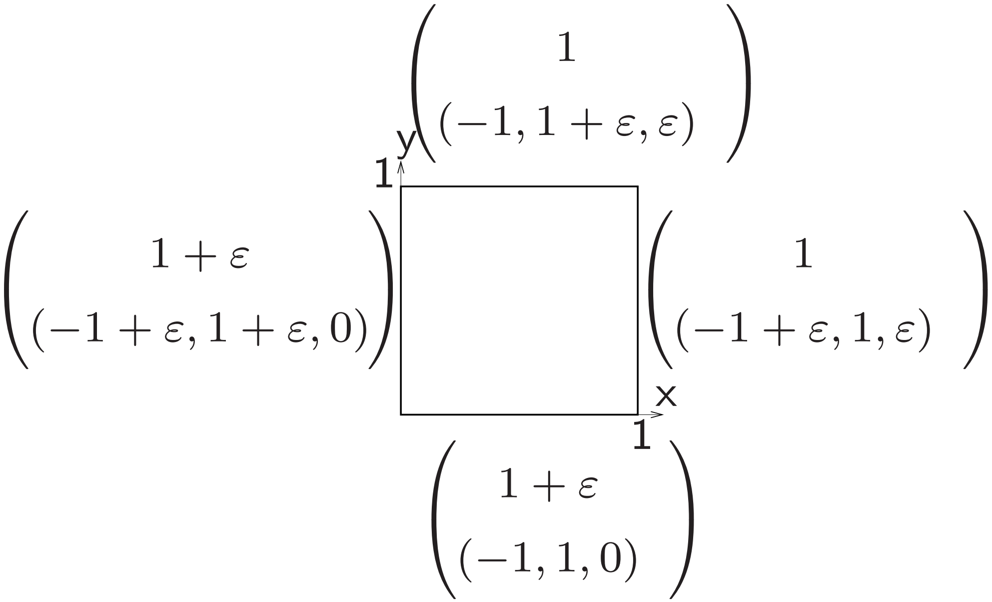
- Initial cond.

$$(n, nu) = (1, (0, 0, 0))$$

- Solution of the Drift limit model

$$(n, nu)(x, t) = (1, (-1, 1, 0))$$

- Boundary conditions



$$\begin{pmatrix} 1 \\ (-1, 1+\varepsilon, \varepsilon) \end{pmatrix}$$

$$\begin{pmatrix} 1+\varepsilon \\ (-1+\varepsilon, 1+\varepsilon, 0) \end{pmatrix} \qquad \begin{pmatrix} 1 \\ (-1+\varepsilon, 1, \varepsilon) \end{pmatrix}$$

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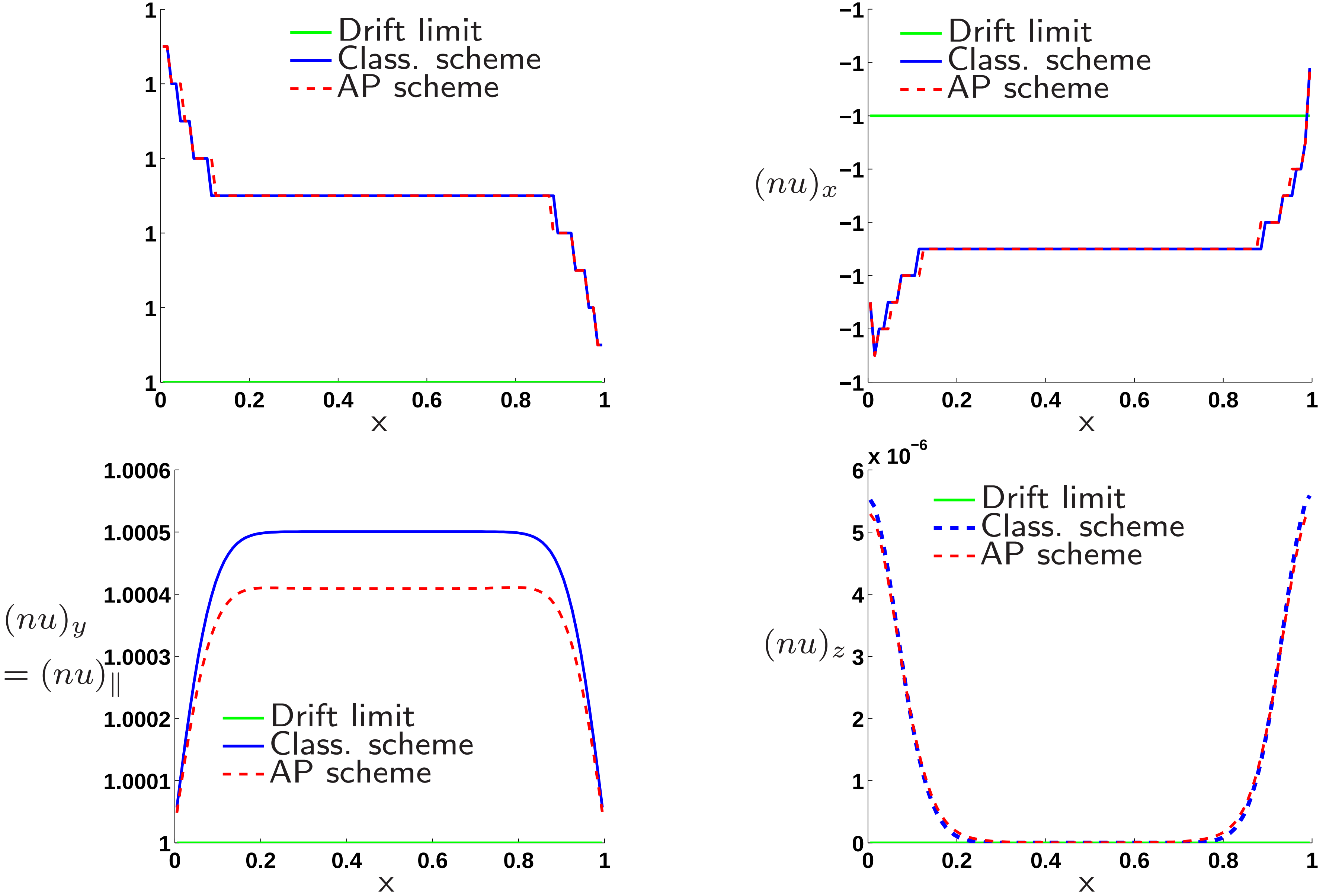
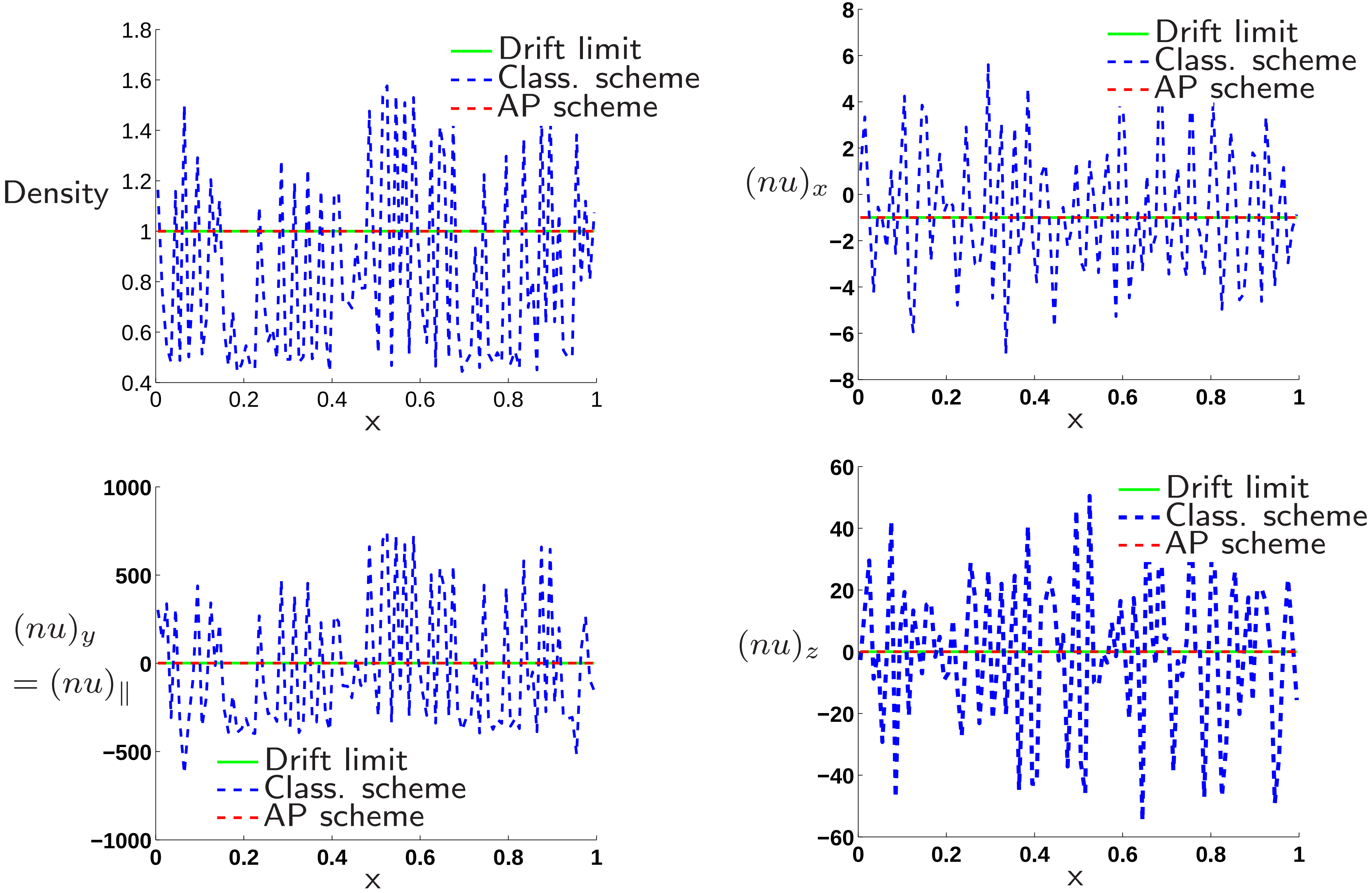
$\varepsilon = 10^{-6}$, Resolved case $\Delta t \leq \varepsilon$



$\varepsilon = 10^{-6}$, Non resolved case $\Delta t > \varepsilon$



Time-step comparison

ε	τ	AP	NAP
10^{-5}	-5	-5.09	-2.6
10^{-6}	-6	-5.6	-2.6
$1.5\,10^{-8}$	-7.83	-6.51	-2.6

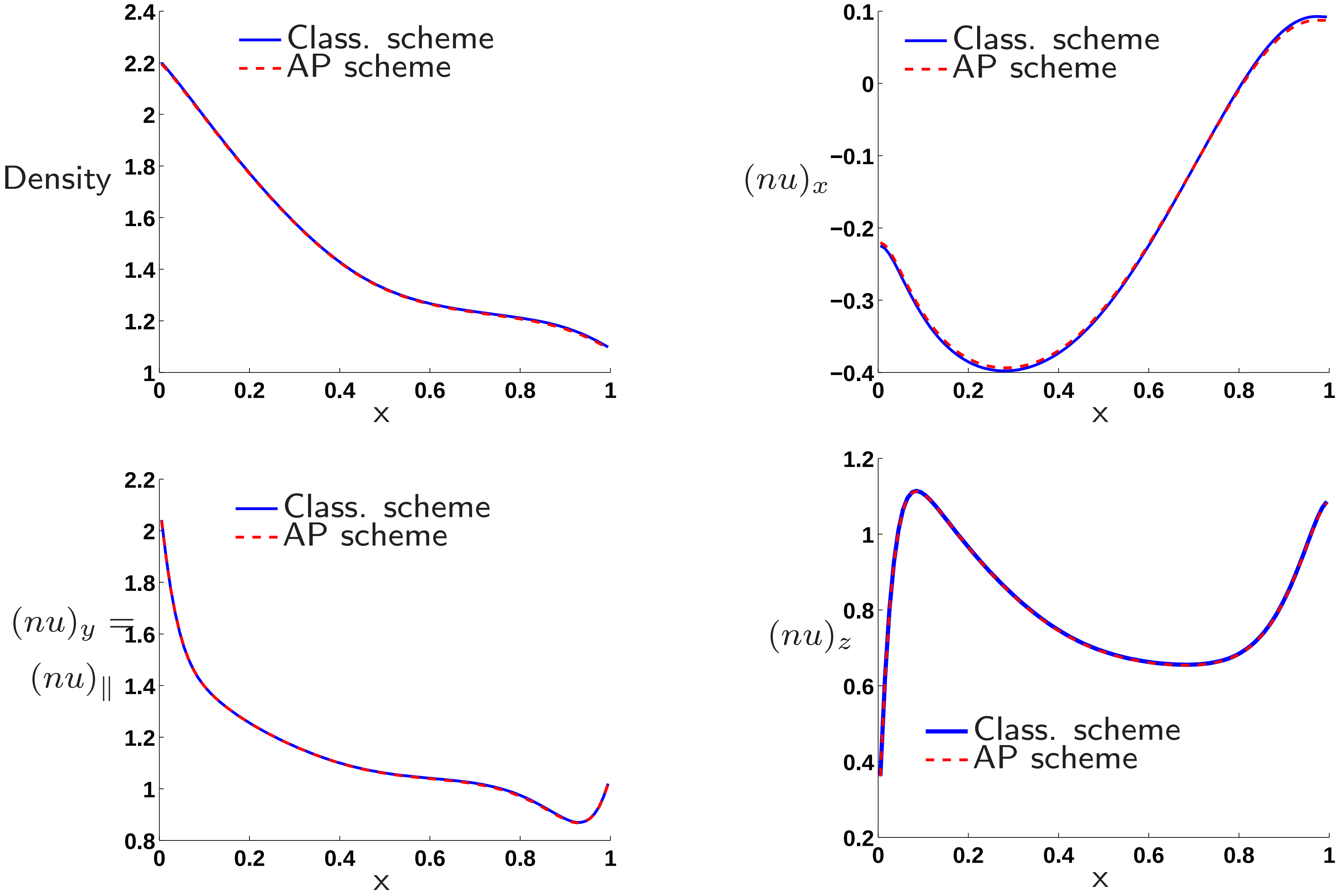
Logarithms of the gyro-period τ, maximum of time-steps used in the resolved AP scheme (AP) and non-resolved AP scheme (NAP)

CPU time comparison

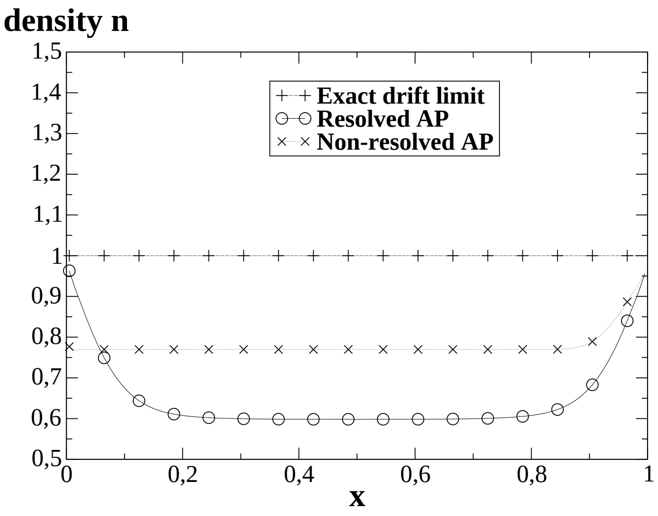
ε	t_{fin}	CONV	NAP	CONV/NAP
10^{-5}	1.00	4940.32	13.84	357
10^{-6}	0.1	1584.21	1.39	1140
$1.5\,10^{-8}$	0.01	1149.54	0.17	6762

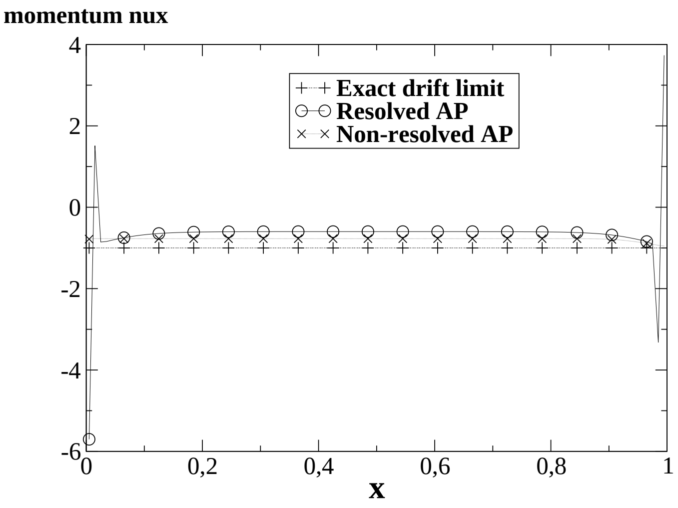
CPU time (in s): resolved conventional scheme (CONV) and non-resolved AP scheme (NAP). Final time t_{fin} (in s). Ratio of CPU times.

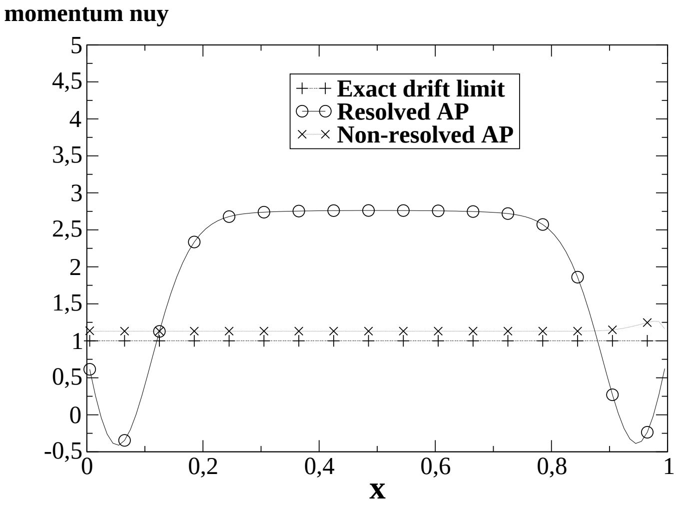
$\varepsilon = 1$, Resolved case $\Delta t \le \varepsilon$

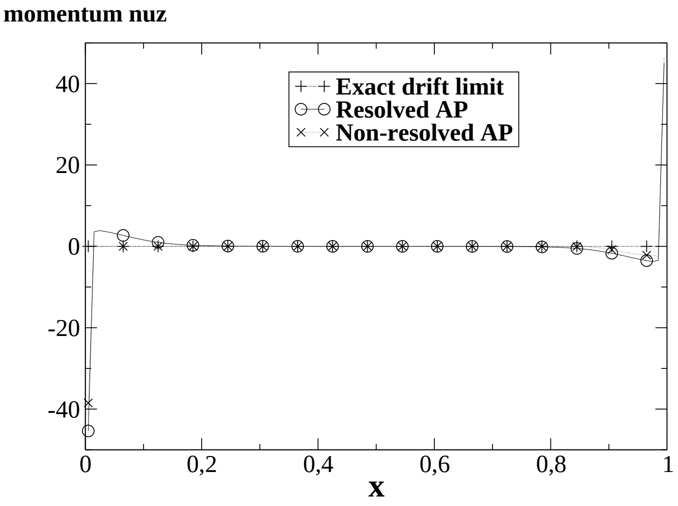


Non well prepared boundary conditions









4. Conclusion

Summary

- An AP scheme for the Euler-Lorentz system in the Drift-Fluid limit has been proposed
  - Drift-fluid $u_{||}$ is computable through an elliptic problem
  - Reformulation of the Euler-Lorentz system into a wave eq. for $u_{||}$
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- Numerical results show AP stability
  - Allows computer savings of 3 to 4 orders of magnitude compared with standard scheme
- Requires well-prepared Boundary Conditions
  - Boundary layer correctors for unprepared BC

Work in progress

- More complex geometry :
  - non uniform (and non constant) B
  - Arbitrary mesh wrt B-field geometry
- Full Euler eqs. (i.e. with energy eq.)
- Coupling with electrons via
  - Quasineutrality
  - or Poisson eq.

Work in progress (II)

- Kinetic model (Vlasov eq.)
  - Gyrokinetic model
  - cf. Vienna talk ...