Gyro-Kinetics Lectures. **From ITER Time-scales to Gyro-Kinetics.**

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My lectures are intended to introduce gyro-kinetics to a student who has some plasma physics knowledge and some mathematical ability. It would take at least 5 lectures to develop the theory in all its detail, but hopefully the key ideas can be communicated in 1 hour. I hope my choice of subject matter provides a basis for the student to understand the more advanced lectures by my colleagues.

Fusion plasmas are invariably turbulent. Since it is the turbulence which sets the confinement time (the time for heat to escape the confining magnetic field) understanding its dynamics is esential. A simple (naive) random walk estimate of the confinement time, τ_E, is:

$$\tau_E \sim \frac{\tau_c L^2}{\Delta^2}$$

where Δ is the typical turbulent cross field eddy size (the step length), τ_c is a typical turbulent correlation time and L is the size (radius) of the plasma. The key to confining a fusion plasma with a magnetic field is to reduce the cross field size of the turbulent eddies (Δ) to micro-scopic sizes. Then the turbulent random walk of heat and particles across the confining field is slowed to acceptable levels. The goal of gyro-kinetics is to calculate the turbulence and transport accurately and reliably. It may also enable us to find ways to reduce the transport. Further improvement in the performance of fusion devices is certainly desirable.

Calculating the turbulence in tokamaks directly from Newton's laws, for the particles, and Maxwell's equations, for the fields, would be impossible even on todays computers. Fortunately one can separate the length and time scales involved to reduce the problem to a computable system – we call this gyro-kinetics. In this lecture I will lay out the separation of scales that is most appropriate for ITER. This involves making some assumptions about the turbulence scales and amplitude. These assumptions are based on experimental measurements of the turbulence (not on ITER of course) and theoretical calculations of the expected instabilities. For simplicity we will not consider the edge of the plasma – specifically the pedestal where the temperature and density gradients are very steep. We will also focus on "Ion Scales" since these dominate the transport – you will hear about electron scale turbulence (*ETG usually*) this is only important when the ion scale turbulence is suppressed. It is possible (perhaps unlikely) that there are hidden components to the turbulence at scales we do not consider.

To set the numbers in context we use core ITER parameters ranging from the top of the pedestal to the middle of the plasma. These are presented in Table. 1 below. The major radius is taken as $R = 6.2m$ and the minor radius $a = 2m$. ITER is expected to achieve a confinement time of $\tau_E \sim 4s$.

Table 1: Parmeter Values for Typical ITER Scenario.

	Top of Pedestal	Center of Plasma
Ion Temperature	4 keV	18 keV
Magnetic Field	5.3 T	5.3 T
Particle Density	$10^{20}m^{-3}$	$1.2 \times 10^{20}m^{-3}$
Safety Factor	4	1.5
Deuterium Larmor Radius ρ_i	$1.6{\times}10^{-3}m$	$3.5{\times}10^{-3}m$
Deuterium Collision Rate ν_i	$200s^{-1}$	$25s^{-1}$
Deuterium Transit Rate $v_{thi}/(qR)$	$1.8 \times 10^4 s^{-1}$	$10^5 s^{-1}$

1 Gyro-kinetic and the ITER Ordering

Length Scales. There are two basic length scales:

- Macroscopic length L – might be size of plasma, or the density gradient length $(n/|\nabla n|)$ *etc.*. We will not distinguish between a and R in our ordering discussions.
- Microscopic length, the ion larmor radius ρ_i – we assume that the turbulent correlation length across the field, Δ, has this scale.

For example in ITER these lengths are approximately: $L \sim a \sim n/|\nabla n| \sim 2m$ and $\rho_i \sim 2 \times 10^{-3} m$. We use these length scales to define the fundamental small parameter of the theory:

$$\epsilon = \frac{\rho}{L} \ll 1 \tag{1}$$

In ITER $\epsilon \sim 10^{-3}$ – a good expansion parameter.

Ion Time Scales. There are *four* important ion frequency scales:

- The *fast* cyclotron frequency – Ω_{ci}. On ITER $\Omega_{ci} \sim 2.5 \times 10^8 rad/s$.
- The *medium* frequency – $\omega = v_{thi}/L \sim \epsilon\Omega_{ci}$. This is roughly the frequency of the turbulent fluctuations and the rate at which particles sense the inhomogeneity – thus $\tau_c \sim 1/\omega$. On ITER $\omega \sim 10^4 - 10^5 rad/s$
- The *slow* collision rate – $\nu_i \sim 25 - 200 s^{-1}$. This is rate at which the local ion Maxwellian is established. It is convenient to order the collision rate in ϵ as $\nu_i \sim \mathcal{O}(\epsilon^{1/2})\omega \sim \mathcal{O}(\epsilon^{3/2})\Omega_{ci}$. In the very collisionless center it might be more appropriate to use $\nu_i \sim \mathcal{O}(\epsilon)\omega$ but we will ignore this here.
- The *very slow* transport rate. Using the random walk estimate we obtain $1/\tau_E \sim (1/\tau_c)(\Delta/L)^2 \sim (v_{thi}/L)\epsilon^2 \sim \epsilon^3\Omega \sim 0.25 s^{-1}$. This is the evolution time for the equilibrium density and temperature.

It is convenient to order the mass ratio $m_e/m_i \sim \mathcal{O}(\epsilon)$ so that the electron time-scales can be treated within the ϵ expansion.

Electron Time Scales. There are *four* important electron frequency scales:

- The *very fast* cyclotron frequency – $\Omega_{ce} \sim \epsilon^{-1}\Omega_{ci}$.
- The *medium fast* transit frequency – $v_{the}/L \sim \epsilon^{1/2}\Omega_{ci}$.
- The *medium* collision rate – $\nu_e \sim \nu_i\epsilon^{-1/2} \sim \omega \sim \epsilon\Omega_{ci}$. This is rate at which the local electron Maxwellian is established.
- The *very slow* transport rate. Using the random walk estimate we obtain $1/\tau_E \sim (1/\tau_c)(\Delta/L)^2 \sim (v_{thi}/L)\epsilon^2 \sim \epsilon^3\Omega_{ci} \sim 0.25 s^{-1}$.
- The *ulta slow* resistive diffusion rate. Using the neoclassical resistivity (η) we get $1/\tau_\eta \sim (\eta/(4\pi))(c/L)^2 \sim \epsilon^4\Omega_{ci} \sim 0.25 \times 10^{-3} s^{-1}$.

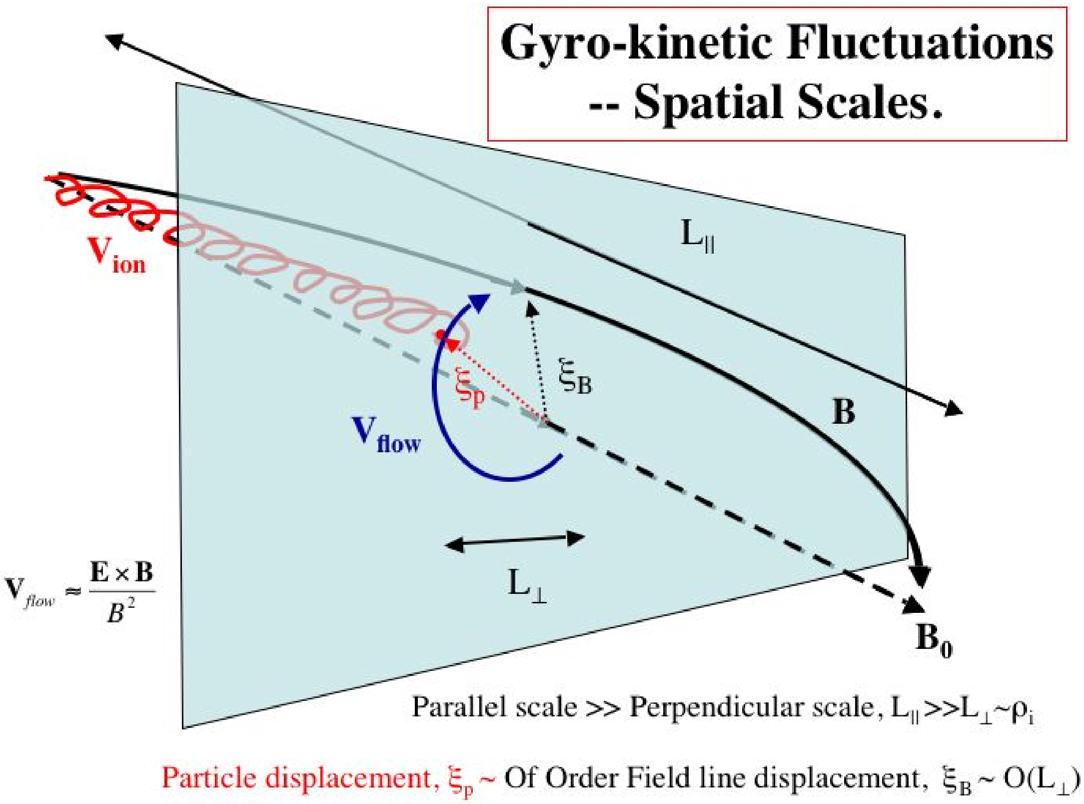

Figure 1: Gyro-kinetic fluctuations, space-scales. Typical fluctuation makes cigar shaped potential surface with $L \gg \lambda_\perp \sim \rho_i$. Particle drift off field line gives a step of order the larmor radius, $\xi_p \sim \rho$. Field displacement is also of order the larmor radius, $\xi_B \sim \rho$.

The fluctuating density and electric field in current fusion devices is small – $\delta n/n_0 < 0.01$. Therefore we split the distribution functions and fields into slowly varying (in time and space) equilibrium parts and medium time-scale fluctuating parts that vary fast in space. I will suppress any species label and deal for simplicity until we need to discuss electrons and ions separately. We define for the distribution functions:

$$f(\mathbf{r},\mathbf{v},t) = F_0(\mathbf{r},\mathbf{v},t) + \delta f_1(\mathbf{r},\mathbf{v},t) + \delta f_{3/2}(\mathbf{r},\mathbf{v},t) + \delta f_2(\mathbf{r},\mathbf{v},t) \;\; .... \tag{2}$$

and for the fields

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t) + \delta\mathbf{B}(\mathbf{r},t), \quad \mathbf{E}(\mathbf{r},t) = \delta\mathbf{E}(\mathbf{r},t) \tag{3}$$

Now we outline the ordering of all the quantities and their variations in time and space.

Small Fluctuations The fluctuations are order ϵ in the gyro-kinetic expansion *i.e.*

$$\frac{\delta f_1}{F_0} \sim \mathcal{O}(\epsilon), \quad \frac{\delta f_{3/2}}{F_0} \sim \mathcal{O}(\epsilon^{3/2}), \quad \frac{\delta f_2}{F_0} \sim \mathcal{O}(\epsilon^2) \quad ..... \; etc. \tag{4}$$

$$\frac{|\delta\mathbf{B}|}{|\mathbf{B}_0|} \sim \mathcal{O}(\epsilon), \quad \frac{|\delta\mathbf{E}|}{|v_{th}\mathbf{B}_0|} \sim \mathcal{O}(\epsilon). \tag{5}$$

Slowly varying Equlilibrium The equilibrium varies in space on the macroscopic length scale and in time on the transport time τ, *i.e.*

$$\nabla F_0 \sim \mathcal{O}(\frac{F_0}{L}), \qquad \nabla \mathbf{B_0} \sim \mathcal{O}(\frac{\mathbf{B_0}}{L}), \tag{6}$$

$$\frac{\partial F_0}{\partial t} \sim \mathcal{O}(\frac{F_0}{\tau}) \sim \mathcal{O}(\frac{v_{th}}{L}\epsilon^2 F_0), \qquad \frac{\partial \mathbf{B_0}}{\partial t} \sim \mathcal{O}(\frac{\mathbf{B_0}}{\tau}) \sim \mathcal{O}(\frac{v_{th}}{L}\epsilon^2 \mathbf{B_0}), \tag{7}$$

Fast Spatial Variation of Fluctuations across $\mathbf{B_0}$. The variation of the fluctuating quantities across the magnetic field is on the microscopic length scale, *i.e.*

$$|\mathbf{b_0} \times \nabla \delta f| \sim \mathcal{O}(\frac{\delta f}{\rho}), \quad |\mathbf{b_0} \times \nabla|\delta \mathbf{B} \sim \mathcal{O}(\frac{\delta \mathbf{B}}{\rho}), \qquad |\mathbf{b_0} \times \nabla|\delta \mathbf{E} \sim \mathcal{O}(\frac{\delta \mathbf{E}}{\rho}), \tag{8}$$

where $\mathbf{b_0} \sim \frac{\mathbf{B_0}}{B_0}$ is the unit vector along $\mathbf{B_0}$. We will often loosely write $k_\perp$ to mean the approximate inverse perpendicular scale, thus $k_\perp \rho \sim 1$.

Slow Spatial Variation Along $\mathbf{B_0}$ The variation of the fluctuating quantities along the magnetic field is on the macroscopic length scale, *i.e.*

$$\mathbf{b_0} \cdot \nabla \delta f \sim \mathcal{O}(\frac{\delta f}{L}), \qquad \mathbf{b_0} \cdot \nabla \delta \mathbf{B} \sim \mathcal{O}(\frac{\delta \mathbf{B}}{L}), \qquad \mathbf{b_0} \cdot \nabla \delta \mathbf{E} \sim \mathcal{O}(\frac{\delta \mathbf{E}}{\rho}), \tag{9}$$

Medium Time Scale Variation of Fluctuations. The fluctuating quantities vary on the mediium time scale, *i.e.*

$$\frac{\partial \delta f}{\partial t} \sim \mathcal{O}(\frac{v_{th}\delta f}{L}), \quad \frac{\partial \delta \mathbf{E}}{\partial t} \sim \mathcal{O}(\frac{v_{th}\delta \mathbf{E}}{L}), \qquad \frac{\partial \delta \mathbf{B}}{\partial t} \sim \mathcal{O}(\frac{v_{th}\delta \mathbf{B}}{L}) \tag{10}$$

Ion fluctuations develop small scales in velocity. The small collision rate allows the fluctuating distribution to develop two scales in velocity: the long scale v_{thi} and small scales $\Delta v_i \sim \nu^{1/2} v_{thi} \sim \epsilon^{1/4} v_{thi}$. Thus for the small scales

$$\frac{\partial \delta f_i}{\partial \tilde{\mathbf{v}}} \sim \mathcal{O}(\epsilon^{-1/4}\frac{\delta f_i}{v_{thi}}), \tag{11}$$

This ordering is chosen so that collisions smooth the small scale velocity variations on the medium time-scale. We will label the small velocity scales with tildes ($\tilde{\mathbf{v}}$).

These orderings have the simple consequences for the fluctuations illustrated in Figure 1.. Specifically: the typical perpendicular flow velocity, roughly the $\mathbf{E} \times \mathbf{B}$ velocity, is of order ϵv_{th}; the typical fluid displacement is roughly $\xi_p \sim \rho$ and the typical field line displacement is roughly $\xi_B \sim \rho$. Note also that $\nabla \delta f \sim \mathcal{O}(\nabla f)$ – *i.e.* the perturbed gradients are comparable with the equilibrium gradients. Thus the fluctuations can locally flatten the gradients driving the turbulence.

2 Potentials and Field Equations.

The orderings given in the previous section have some simple consequences for the fields and Maxwell's equations. Consider Faraday's law,

$$\frac{\partial \delta \mathbf{B}}{\partial t} = -\nabla \times \delta \mathbf{E}. \tag{12}$$

However from our ordering:

$$\nabla \times \delta \mathbf{E} \sim \mathcal{O}(\epsilon \Omega \mathbf{B_0}) \quad \text{and} \tag{13}$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} \sim \mathcal{O}(\epsilon^2 \Omega \mathbf{B_0}). \tag{14}$$

Thus the dominant order electric field must satisfy $\nabla \times \delta \mathbf{E} = 0$, which (of course) means that we can write:

$$\delta \mathbf{E} \sim -\nabla \phi \tag{15}$$

Thus the largest part of the electric field is electrostatic. However we shall need the inductive part of the electric field to get the dynamics right (as we see below). It is therefore convenient to write the fields in terms of the scalar and vector potentials, *i.e.*:

$$\delta \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \delta \mathbf{B} = \nabla \times \mathbf{A}. \tag{16}$$

For definiteness we use the coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0. \tag{17}$$

Which (using Eq. (8)) yields $\nabla_\perp \cdot \mathbf{A}_\perp = 0$ to the order we keep. Note $\perp$ and $\|$ refer to the perpendicular and parallel to the equilibrium field line $\mathbf{b}_0$. Thus,

$$\mathbf{A} = A_\| \mathbf{b}_0 + \nabla \xi \times \mathbf{b}_0. \tag{18}$$

where ξ is a scalar. To the order we keep:

$$\delta \mathbf{B}_\perp = \nabla A_\| \times \mathbf{b}_0, \qquad \delta \mathbf{B} \cdot \mathbf{b}_0 = \delta B_\| = -\nabla^2 \xi, \qquad \delta B_\| \sim \mathcal{O}(|\delta \mathbf{B}_\perp|) \tag{19}$$

and

$$\delta \mathbf{E}_\perp = -\nabla \phi - \nabla \frac{\partial \xi}{\partial t} \times \mathbf{b}_0, \quad \delta \mathbf{E} \cdot \mathbf{b}_0 = \delta E_\| = -\nabla_\| \phi - \tfrac{\partial A_\|}{\partial t} \quad \delta E_\| \sim \mathcal{O}(\epsilon |\delta \mathbf{E}_\perp|). \tag{20}$$

We note that the inductive part of the parallel electric field is comparable to the electrostatic part. The inductive part of the perpendicular electric field is, however, small compared to the electrostatic part. It must be kept because it yields a compressive part to the $\mathbf{E} \times \mathbf{B}$ velocity and a net acceleration as a particle goes around a gyro-orbit. This sounds a bit cryptic, but it will be more apparent in the derivation of the gyro-kinetic equations.

The key reason for introducing the potentials is that we reduce the field quantities from six scalers, the components of $\delta \mathbf{E}$ and $\delta \mathbf{B}$, to three, ϕ, $A_\|$ and ξ (or equivalently $\delta B_\|$). We need to extract from Maxwell's equations three equations for these three unknown fields in terms of current density and charge density. It is trivial to show that the displacement current is small in this ordering. The appropriate equations are then, Poisson's equation and two components of Ampere's law.

Poisson's Equation. Since to dominant order $\nabla \cdot \delta \mathbf{E} = -\nabla^2 \phi$,

$$\nabla^2 \phi = -\frac{1}{\epsilon_0}(q n_i - e n_e) \tag{21}$$

where n_i and n_e are the ion and electron densities. When $k_\perp^{-1}$ is long compared to the Debye length one can drop the left hand side of Poisson's equation and obtain *quasi-neutrality* – *i.e.* $q n_i = e n_e$.

Parallel Ampere's Law. Using $\nabla \times \delta \mathbf{B} = -\nabla^2 \mathbf{A}$ we obtain,

$$\nabla^2 A_\| = -\mu_0 J_\| = -\mu_0 (q n_i \mathbf{V}_i - e n_e \mathbf{V}_e) \cdot \mathbf{b}_0 \tag{22}$$

where $\mathbf{V}_i$ and $\mathbf{V}_e$ are the mean ion and electron flow velocities.

Perpendicular Ampere's Law:

$$\nabla_\perp B_0 + \nabla_\perp \delta B_\parallel = \mu_0 \mathbf{b}_0 \times \mathbf{J} = \mu_0 \mathbf{b}_0 \times (q n_i \mathbf{V}_i - e n_e \mathbf{V}_e). \tag{23}$$

Clearly we must solve for the distribution functions of ions and electrons to obtain the charge. In the slab equilibrium we have equilibrium currents in the y direction and these should be balance the variation of B_0. This just gives the equilibrium relation $p(x) + B_0^2(x)/(2\mu_o) =$ *constant* and it will not be needed here. I have dropped the equilibrium *Electric Field* $\mathbf{E}_0$. since because the variation of B_0 is slow in time does not enter the equations at the order we want to keep.

3 Gyro-Kinetic Particle Motion

Before we plough through the derivation of the gyro-kinetic equation and sweat over the algebra we can gain a little physical insight by looking at the single particle motion in the gyro-kinetic ordering. I will start this in a general slowly varying field and then give the non-uniform slab result. First we define the gyro-center position by a vector version of the simple uniform field (slab) result.

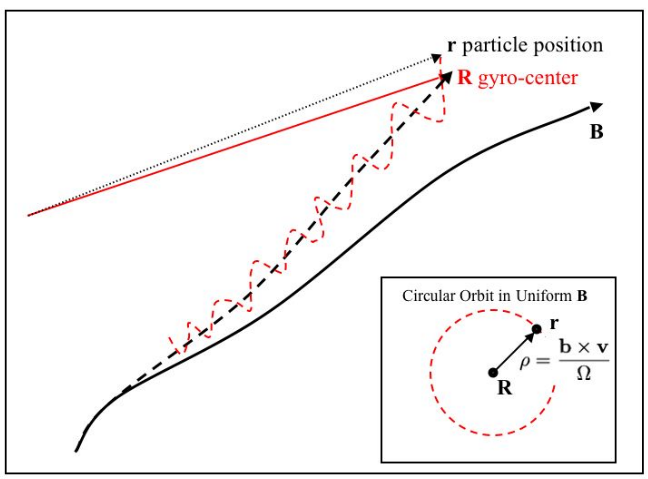

Figure 2: Defining the Gyro-center.

The exact gyro-center position is not actually a well defined quantity. However to lowest and first order our ordering shows that the particle orbit looks locally to be like the orbit in a uniform field. Thus we define (see Figure 2.) the gyro-center position $\mathbf{R}$ in terms of the particle position $\mathbf{r}$ and particle velocity $\mathbf{v}$:

$$\mathbf{R} = \mathbf{r} + \frac{\mathbf{v} \times \mathbf{b}_0}{\Omega_0} \tag{24}$$

where (as before) $\mathbf{b}_0 = \mathbf{b}_0(\mathbf{r}) = \mathbf{B}_0/B_0$ is the unit vector along the local equilibrium field and $\Omega_0 = \Omega_0(\mathbf{r}) = qB_0/m$ is the local equilibrium gyro-frequency. The transformation to gyro-center position is sometimes called the *Catto Transformation* after its inventor. We define the perpendicular, $v_\perp$, and parallel, $v_\parallel$, and gyro-angle, θ with respect to the equilibrium field from the expression:

$$\mathbf{v} = v_\parallel \mathbf{b}_0 + v_\perp (\cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2). \tag{25}$$

The unit vectors $\mathbf{b}_0$, $\mathbf{e}_1$ and $\mathbf{e}_2$ form a local right handed coordinate basis *i.e.* $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}_0$, and they vary on the macroscopic, L, spacial scale and the slow, τ, time scale. In the straight field (electrostatic) case so that $\mathbf{b}_0 = \mathbf{z}$, $\mathbf{e}_1 = \mathbf{x}$ and $\mathbf{e}_2 = \mathbf{y}$. The fastest motion is the gyro-motion and indeed;

$$\frac{d\theta}{dt} = -\Omega_0 + \mathcal{O}(\epsilon\Omega). \tag{26}$$

We will show shortly that both $v_\perp$ and $v_\parallel$ vary on the medium time scale (with a small part varying on the fast cyclotron time scale) and therefore can be considered constant on the fast (Ω) time scale. Now consider the evolution of $\mathbf{R}$. We differentiate Eq. (24) with respect to time:

$$\frac{d\mathbf{R}}{dt} = \mathbf{v} + \frac{d\mathbf{v}}{dt} \times \frac{\mathbf{b}_0}{\Omega_0} + \mathbf{v} \times \frac{d}{dt}(\frac{\mathbf{b}_0}{\Omega_0}) \tag{27}$$

now using the equation of motion:

$$m\frac{d\mathbf{v}}{dt} = q(\delta\mathbf{E} + \mathbf{v} \times \mathbf{B}_0 + \mathbf{v} \times \delta\mathbf{B}_0), \tag{28}$$

we obtain to order ϵ,

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b}_0 + \delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0} + v_\parallel \frac{\delta\mathbf{B}_\perp}{B_0} + \mathbf{v}_\perp \frac{\delta B_\parallel}{B_0} + \mathbf{v} \times \left(\mathbf{v} \cdot \nabla(\frac{\mathbf{b}_0}{\Omega_0})\right). \tag{29}$$

Note that the dominant motion is along the field lines and the cross field motion comes from the perturbed fields and the inhomogeneity of the equilibrium fields.

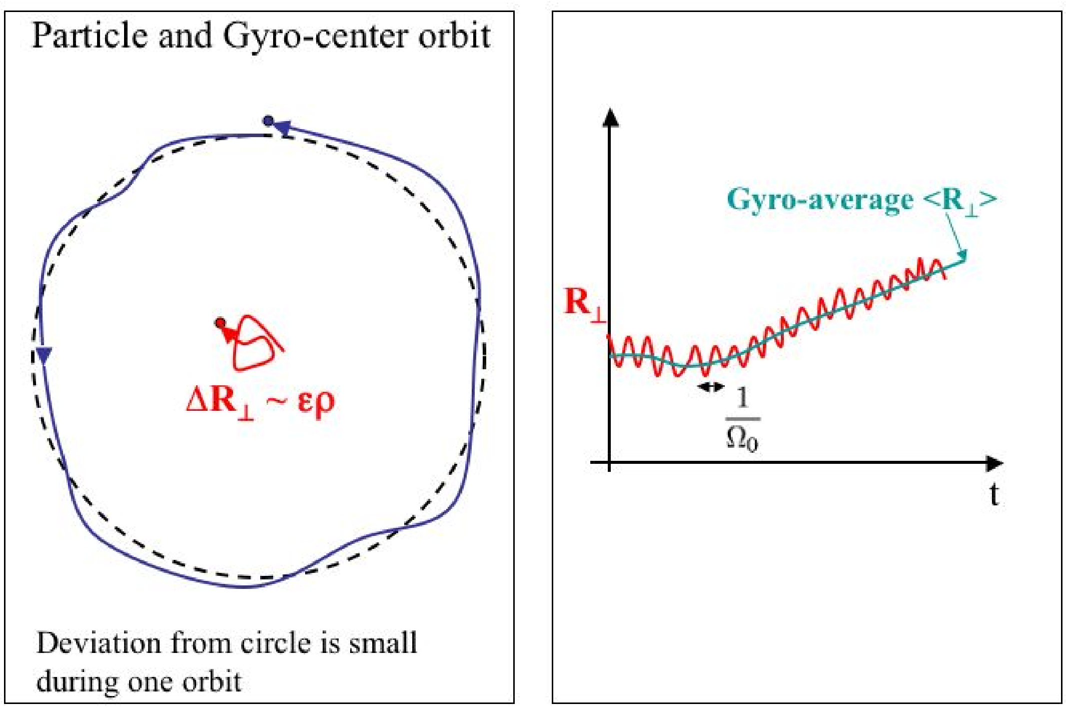

Figure 3: Motion of Gyro-center $\mathbf{R}$ and its Average. Note that in one orbit the gyro-center moves a small distance of order $\epsilon\rho$. However over ϵ^{-1} gyro-orbits the gyro-center "drifts" a distance $\mathcal{O}(\rho)$.

We wish to know the motion of the gyro-center, $\mathbf{R}$, over the medium time scale (times of order L/v_th). The right hand side of Eq. (54) oscillates on the fast time scale Ω^{-1} but when we integrate the perpendicular motion this averages out *i.e.*:

$$\delta\mathbf{R}_\perp = \int_0^t \left[\delta\mathbf{E}\times\frac{\mathbf{b}_0}{B_0} + v_\parallel\frac{\delta\mathbf{B}_\perp}{B_0} + \mathbf{v}_\perp\frac{\delta B_\parallel}{B_0} + \mathbf{v}\times\left(\mathbf{v}\cdot\nabla(\frac{\mathbf{b}_0}{\Omega_0})\right)\right]dt$$

$$= \int_0^t \left[< \delta\mathbf{E}\times\frac{\mathbf{b}_0}{B_0} >_{\mathbf{R}} + < v_\parallel\frac{\delta\mathbf{B}_\perp}{B_0} >_{\mathbf{R}} + < \mathbf{v}_\perp\frac{\delta B_\parallel}{B_0} >_{\mathbf{R}} + < \mathbf{v}\times\left(\mathbf{v}\cdot\nabla(\frac{\mathbf{b}_0}{\Omega_0})\right) >_{\mathbf{R}}\right]dt + \mathcal{O}(\epsilon\rho). \quad (30)$$

where the gyro-average (ring average) at fixed $\mathbf{R}$ is defined by:

$$< a(\mathbf{r},\mathbf{v},t) >_{\mathbf{R}} = \frac{1}{2\pi}\int_0^{2\pi} a(\mathbf{R}-\frac{\mathbf{v}\times\mathbf{b}_0}{\Omega_0},\mathbf{v},t)d\theta. \quad (31)$$

In Eq. (31) the θ integration is done keepin $\mathbf{R}$, $v_\perp$ and $v_\parallel$ fixed. Thus this gyro-average is an **average over a ring centered about R of radius** $\mathbf{v}_\perp/\boldsymbol{\Omega}_\mathbf{0}$. Thus we think of the gyro-center motion as the motion of this ring obeying the equation:

$$< \frac{d\mathbf{R}}{dt} > = < v_\parallel\mathbf{b}_0 >_{\mathbf{R}} + < \delta\mathbf{E}\times\frac{\mathbf{b}_0}{B_0} >_{\mathbf{R}} + < v_\parallel\frac{\delta\mathbf{B}_\perp}{B_0} >_{\mathbf{R}} + < \mathbf{v}_\perp\frac{\delta B_\parallel}{B_0} >_{\mathbf{R}} + < \mathbf{v}\times\left(\mathbf{v}\cdot\nabla(\frac{\mathbf{b}_0}{\Omega_0})\right) >_{\mathbf{R}} \quad (32)$$

After some straightforward algebra we obtain:

$$< \frac{d\mathbf{R}}{dt} > = v_\parallel\mathbf{b}_0 \quad - \quad \frac{\partial < \chi >_{\mathbf{R}}}{\partial\mathbf{R}}\times(\frac{\mathbf{b}_0}{B_0}) + v_\parallel^2(\frac{\mathbf{b}_0}{\Omega_0})\times\mathbf{b}_0\cdot\nabla\mathbf{b}_0 + \frac{v_\perp^2}{2B_0}(\frac{\mathbf{b}_0}{\Omega_0})\times\nabla B_0. \quad (33)$$

where:

$$\chi = \phi - \mathbf{v}\cdot\mathbf{A} \quad (34)$$

and we have dropped the $\mathcal{O}(\epsilon v_{th})$ corrections to the parallel motion as they are small compared to the $v_\parallel\mathbf{b}_0$ term and they are not needed. The expression, Eq. (32) is obviously not very familiar so let us expand out the terms in χ and look at the physical meaning of each term by referring to the drift kinetic limit. Thus:

- $v_\parallel\mathbf{b}_0$ is the motion along the equilibrium field.
- $-\frac{\partial<\phi>_{\mathbf{R}}}{\partial\mathbf{R}}\times(\frac{\mathbf{b}_0}{B_0})$ is the ring averaged $\mathbf{E}\times\mathbf{B}$ drift.
- $\frac{\partial<v_\parallel A_\parallel>_{\mathbf{R}}}{\partial\mathbf{R}}\times(\frac{\mathbf{b}_0}{B_0})$ is the correction to the parallel motion due to motion along the ring averaged tilted perturbed field line.
- $\frac{\partial<\mathbf{v}_\perp\cdot\mathbf{A}_\perp>_{\mathbf{R}}}{\partial\mathbf{R}}\times(\frac{\mathbf{b}_0}{B_0})$ is the ring averaged perturbed ∇B drift.
- $v_\parallel^2(\frac{\mathbf{b}_0}{\Omega_0})\times\mathbf{b}_0\cdot\nabla\mathbf{b}_0$ is the curvature drift in the equilibrium fields. This is zero in the straight field case.
- $\frac{v_\perp^2}{2B_0}(\frac{\mathbf{b}_0}{\Omega_0})\times\nabla B_0$ is the ∇B drift in the equilibrium fields. In the slab case this is $\frac{v_\perp^2}{2B_0}\frac{dB_0}{dx}(\frac{\mathbf{y}}{\Omega_0})$.

Note that the perpendicular drifts, both equilibrium and perturbed, are $\mathcal{O}(\epsilon v_{th})$ in the gyro-kinetic ordering. This is because although the perturbed fields are small they vary on the micro-scale. Because perpendicular structures are small scale the small drifts can move the gyro-center across the turbulent structure (eddy) on the turbulent time-scale.

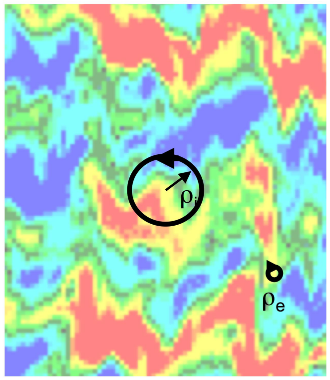

Figure 4: Gyro-average of the fluctuations over a ring of radius the larmor radius $\rho = v_\perp/\Omega$. For small radii the average is almost the same as the value at the center, for large radii the average tends to cancel and is almost zero. Electrons have smaller rings by the factor $\sqrt{m_e/m_i}$
.

To complete our derivation of the particle motion we need the equations for the variation of $v_\perp$ and $v_\parallel$. The variation of energy, $\mathcal{E} = \frac{1}{2}mv^2 + q\phi(\mathbf{r}, t)$, follows in a similar manner to the derivation above, specifically:

$$< \frac{d\mathcal{E}}{dt} >= q\frac{\partial < \chi >_{\mathbf{R}}}{\partial t} \tag{35}$$

note that $\mathcal{E}$ varies on the medium time scale whereas the kinetic energy has an $\mathcal{O}(\epsilon)$ variation on the fast time scale due to the variation of ϕ over the gyro-orbit. The net work done on a particle over the medium time scale and longer comes from integrating the right hand side of Eq. (55) over time. Note the same ring averaged perturbed quantity $< \chi >_{\mathbf{R}}$ enters the energy and gyro-center evolution – you might suspect that this is due to some underlying property of the equations, indeed it is related to the Hamiltonian properties of the collisionless motion. I will not elaborate on this here as it does not illuminate the physical picture. We have kept energy variations up to $\mathcal{O}(\epsilon)$ (they are needed), but we shall only need the $\mathcal{O}(1)$ part of the magnetic moment variation. Thus:

$$\mu = \frac{mv_\perp^2}{2B_0(\mathbf{R}, t)} = constant. \tag{36}$$

To the order that is required Eqs (26), (33), (55) and (36) provide a set of equations to find the particle orbits and energy variation. The perturbed distribution function is formally represented in the gyro-center variables as:

$$\delta f_i = \delta f_i(\mathbf{R}, \mathcal{E}, \tilde{\mathcal{E}}, \mu, \tilde{\mu}, \theta, \tilde{\theta}, \sigma, t). \tag{37}$$

where we have introduced the variable $\sigma = \pm 1$ to signify the sign of $v_\parallel$. The fast velocity scales are explicitly denoted by the tildes.

We see in the next lecture how this enters the kinetic equations.

4 Ordered Ion Fokker-Planck Equation.

From now on we will specialize to the case where the field is straight and the equilibrium depends only on x – *i.e.* $\mathbf{B} = B_0(x)\mathbf{z}$ and $F_0(x, \mathbf{v}, t)$. The system is assumed periodic over y and we identify the points (x, y, z_0) and $(x, y + y_0, -z_0)$. Thus following the magnetic field through the system covers the surface $x =$ constant – the "flux surface". We will also assume the perturbation is entirely electrostatic ($\delta\mathbf{B} = 0$). The full electromagnetic toroidal case is not much harder – but certainly longer. Here we expand the Ion Fokker Planck equation – to make things simple I will drop the subscript i. For convenience we write (see last lecture) $\delta f = \delta f_1 + \delta f_{3/2} + ...etc.$. We use the orderings stated in the last sections. The FP equation with order (relative to $v_{th}F_0/L$) under each term is:

$$\underbrace{\frac{\partial F_0}{\partial t}}_{\epsilon^2} + \underbrace{\frac{\partial \delta f}{\partial t}}_{\epsilon} + \underbrace{\mathbf{v}\cdot\nabla F_0}_{1} + \underbrace{\mathbf{v}_\perp\cdot\nabla\delta f}_{1} + \underbrace{v_\parallel \mathbf{z}\cdot\nabla\delta f}_{\epsilon} + \frac{q}{m}\left(\underbrace{-\nabla\phi}_{1} + \underbrace{\mathbf{v}\times\mathbf{B}}_{\epsilon^{-1}}\right)\cdot\frac{\partial F_0}{\partial \mathbf{v}} + \frac{q}{m}\left(\underbrace{-\nabla\phi}_{\epsilon^{3/4}} + \underbrace{\mathbf{v}\times\mathbf{B}}_{\epsilon^{-1/4}}\right)\cdot\frac{\partial \delta f}{\partial \tilde{\mathbf{v}}}$$
$$+\frac{q}{m}\left(\underbrace{-\nabla\phi}_{\epsilon} + \underbrace{\mathbf{v}\times\mathbf{B}}_{1}\right)\cdot\frac{\partial \delta f}{\partial \mathbf{v}} = \underbrace{C(F_0, F_0)}_{\epsilon^{1/2}} + \underbrace{\tilde{C}(\delta f, F_0)}_{\epsilon} + \underbrace{\tilde{C}(F_0, \delta f)}_{\epsilon} + ... + \underbrace{\tilde{C}(\delta f), \delta f))}_{\epsilon^2}. \tag{38}$$

Note that since δf has an expansion in $\epsilon^{1/2}$ the indications are the highest order of each term. The collisional terms are denoted by the bilinear integro-differential operator $C(f, f)$ – this operator has two velocity space derivatives which are enhanced when acting on the small ($\tilde{\mathbf{v}}$) velocity scale (see Eq. (11)). Thus $\tilde{C}(f, f)$ indicates the enhanced collision operator.

We expect that a solution is most easily obtained in terms of the gyro-center variables. Thus we transform the FP equation into these variables: Substituting the form Eq. (37) into Eq. (38) and dropping some terms $\mathcal{O}(\epsilon^2)$ and higher we obtain:

$$\underbrace{\frac{\partial \delta f}{\partial t}}_{\epsilon} + \underbrace{\frac{d\mathbf{R}}{dt}\cdot\frac{\partial \delta f}{\partial \mathbf{R}}}_{\epsilon} + \underbrace{\frac{d\mathcal{E}}{dt}\cdot\frac{\partial \delta f}{\partial \mathcal{E}}}_{\epsilon^{7/4}:\epsilon^2} + \underbrace{\frac{d\mu}{dt}\cdot\frac{\partial \delta f}{\partial \mu}}_{\epsilon^{7/4}:\epsilon^2} - \underbrace{\tilde{C}(\delta f, F_0)}_{\epsilon} - \underbrace{\tilde{C}(F_0, \delta f)}_{\epsilon} + \underbrace{\frac{d\theta}{dt}\left(\frac{\partial \delta f}{\partial \theta}\right)_{\mathbf{R}}}_{\epsilon^{-1/4}:1}$$
$$= -\underbrace{\frac{d\theta}{dt}\left(\frac{\partial F_0}{\partial \theta}\right)_{\mathbf{R}}}_{\epsilon^{-1}} - \underbrace{\frac{d\mathbf{R}}{dt}\cdot\frac{\partial F_0}{\partial \mathbf{R}}}_{1} - \underbrace{\frac{d\mathcal{E}}{dt}\cdot\frac{\partial F_0}{\partial \mathcal{E}}}_{\epsilon} - \underbrace{\frac{d\mu}{dt}\cdot\frac{\partial F_0}{\partial \mu}}_{\epsilon} + \underbrace{C(F_0, F_0)}_{\epsilon^{1/2}} \tag{39}$$

where (see previous sections) we have

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b}_0 + \delta\mathbf{E}\times\frac{\mathbf{b}_0}{B_0} - (\mathbf{v}\times\frac{\mathbf{b}_0}{\Omega_0})\frac{\mathbf{v}\cdot\nabla B_0}{B_0}. \tag{40}$$

and

$$\frac{d\mathcal{E}}{dt} = q\frac{\partial(\phi)}{\partial t}. \tag{41}$$

The process of simplifying the equations involves equating orders and solving the resulting equations. In principle we need to go to $\mathcal{O}(\epsilon^2)$ to get the long transport time evolution of F_0 – in fact we will only go to $\mathcal{O}(\epsilon)$ and then use the moment equations to get the slow evolution of F_0. Those of you who know the Chapman -Enskog expansion or neoclassical transport theory will recognize this approach. We now expand Eq. (39) order by order.

4.1 $\mathcal{O}(\epsilon^{-1})$:

At this order from Eq. (39) we have simply:

$$-\Omega_0(x)\left(\frac{\partial F_0}{\partial \theta}\right)_{\mathbf{R},\mathcal{E},\mu} = 0 \tag{42}$$

from which we deduce that F_0 is independent of gyro-angle θ (any initial dependance would be wiped out by the fast gyration) so that:

$$F_0 = F_0(\mathbf{R},\mathcal{E},\mu,t) \tag{43}$$

and recall that F_0 depends only on the long transport time scale. Now we proceed to $\mathcal{O}(\epsilon^{-1/4})$:

4.2 $\mathcal{O}(\epsilon^{-1/4})$:

The only contribution at this order is the $\tilde{\theta}$ variation of δf.

$$-\Omega_0(X)\left(\frac{\partial \delta f_1}{\partial \tilde{\theta}}\right)_{\mathbf{R},\mathcal{E},\mu} = 0 \tag{44}$$

thus $\delta f_i = \delta f_i(\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\theta,\sigma,t)$.

4.3 $\mathcal{O}(1)$:

From Eq. (39) we obtain:

$$\frac{d\mathbf{R}}{dt}\cdot\nabla F_0 = v_\parallel \mathbf{b}\cdot\nabla F_0 = \Omega_0\left(\frac{\partial \delta f_1}{\partial \theta}\right)_{\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\theta,\sigma,t} \tag{45}$$

where we have dropped the $\mathcal{O}(\epsilon)$ parts of $\frac{d\mathbf{R}}{dt}$ they will be included in next order. We have already defined the ring average in Eq. (31) but in our new variables we think of the average at fixed $\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\sigma,t$. *i.e.*

$$< a >_{\mathbf{R}} (\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\sigma,t) = \frac{1}{2\pi}\int_0^{2\pi} a(\mathbf{R}-\frac{\mathbf{v}\times\mathbf{b}_0}{\Omega_0},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\theta,\sigma,t)d\theta. \tag{46}$$

This average annihilates the right hand side of Eq. (45) and yields the condition for solution:

$$\mathbf{b}\cdot\nabla F_0 \tag{47}$$

i.e. the equilibrium must be constant along the field lines at constant $\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\sigma,t$. Thus since a field line covers the flux surface we make F_0 a function of $X = \mathbf{x}\cdot\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\sigma$ and t. Note that in fact F_0 contains some of the perturbation since it includes ϕ in $\mathcal{E}$. From Eqs. (45) and (47) we have

$$\delta f_1 = h(\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\sigma,t) \tag{48}$$

In $\mathcal{O}(\epsilon)$ we derive an equation – *the gyro-kinetic equation* – that determines the evolution of h.

4.4 $\mathcal{O}(\epsilon^{1/2})$:

In this order we obtain

$$C(F_0, F_0) = \Omega_0 \left(\frac{\partial \delta f_{3/2}}{\partial \theta} \right)_{\mathbf{R},\mathcal{E},\tilde{\mathcal{E}},\mu,\tilde{\mu},\theta,\sigma,t} \tag{49}$$

from Eq. (39). As before we ring average this equation to annihilate the right hand side. We obtain

$$C(F_0, F_0) = \mathcal{O}(\epsilon^{3/2}). \tag{50}$$

From Bolzmann's H theorem and the constraint on the form of F_0, Eq. (43), it is easy to show that F_0 is the "almost" Maxwellian:

$$F_0(\mathbf{r}, \mathbf{v}, t) = n(t, X) \left(\frac{m}{2\pi T(t, X)} \right)^{3/2} \exp[-(\frac{\mathcal{E}}{T(t, X)})] \tag{51}$$

Where as before $\mathcal{E} = (1/2)mv^2 + q\phi(\mathbf{r}, t)$ and $X = x + v_y/\Omega_0$. While this is the form of the distribution function we still need to derive equations for $h(\mathbf{R}, v, v_\perp, t)$, $n(t, X)$ and $T(t, X)$. This form of F_0 is essentially a local equilibrium response. It is common to expand F_0 as:

$$F_0(\mathbf{r}, \mathbf{v}, t) = n(t, x) \left(\frac{m}{2\pi T(t, x)} \right)^{3/2} \exp[-(\frac{(1/2)mv^2}{T(t, x)})] \left(1 - \frac{q\phi}{T} - \boldsymbol{\rho} \cdot \nabla \ln F_0 \right) \tag{52}$$

Where $\boldsymbol{\rho} = \frac{\mathbf{z}\times\mathbf{v}}{\Omega_0}$ is the larmor radius. The ϕ term is confusingly called the adiabatic response. We will use Eq. (51) and keep all the Boltzmann response to simplify equations at higher order. The term $C(F_0, F_0) = \mathcal{O}(\epsilon^{3/2})$ is dropped to higher order (eventually it gives rise to the classical transport). Now we proceed to $\mathcal{O}(\epsilon)$ where we obtain the **gyro-kinetic equation** as a solubility constraint for δf_2.

4.5 $\mathcal{O}(\epsilon)$:

Substituting the form Eq. (51) into Eq. (38) and dropping terms $\mathcal{O}(\epsilon^2)$ and higher we obtain:

$$\frac{\partial h}{\partial t} + \frac{d\mathbf{R}}{dt} \cdot \frac{\partial h}{\partial \mathbf{R}} - \tilde{C}(h, F_0) - \tilde{C}(F_0, h) = \Omega_0 \left(\frac{\partial \delta f_2}{\partial \theta} \right)_{\mathbf{R}} + \frac{d\mathbf{R}}{dt} \cdot \frac{\partial F_0}{\partial \mathbf{R}} + \frac{d\mathcal{E}}{dt} \cdot \frac{\partial F_0}{\partial \mathcal{E}} \tag{53}$$

where (see Lecture # 1) we have

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b}_0 + \delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0} - (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0}) \frac{\mathbf{v} \cdot \nabla B_0}{B_0}. \tag{54}$$

and

$$\frac{d\mathcal{E}}{dt} = q \frac{\partial \langle \phi \rangle}{\partial t}. \tag{55}$$

To obtain an equation for h we must annihilate δf_2 from Eq. (54) – to do this we ring average. The ring distribution $h(\mathbf{R}, \mu, \mathcal{E}, \sigma, t)$ satisfies the **Gyro-kinetic equation**:

$$\frac{\partial h}{\partial t} + v_\parallel \frac{\partial h}{\partial Z} + \mathbf{v_D} \cdot \frac{\partial h}{\partial \mathbf{R}} - \frac{\partial < \phi >_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial h}{\partial \mathbf{R}} - \left\langle \tilde{C}(h) \right\rangle_{\mathbf{R}} = q \frac{F_0}{T_0} \frac{\partial \langle \phi \rangle_{\mathbf{R}}}{\partial t} - \frac{\partial < \phi >_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial F_0}{\partial \mathbf{R}} \tag{56}$$

and

$$\mathbf{v_D} = -\frac{v_\perp^2}{2\Omega_0} \frac{\nabla B_0}{B_0} \times \mathbf{b}_0 = \frac{v_\perp^2}{2B_0} (\frac{1}{\Omega_0}) \frac{dB_0}{dx} \mathbf{y}$$

is the equilibrium grad B drift. In some loose sense the **Gyro-kinetic equation** is the kinetic equation for rings of charge centered at $\mathbf{R}(t)$ of radius $v_\perp/\Omega$. It is important to note that ϕ and h have zero spatial average over the box. The physical interpretation of the terms in Eq. (56) is straight forward, for example:

- $\mathbf{v_D} \cdot \frac{\partial h}{\partial \mathbf{R}}$ is the convection of the perturbed ring distribution by the equilibrium grad B drift.
- $-\frac{\partial <\phi>_\mathbf{R}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial h}{\partial \mathbf{R}}$ is the convection of the perturbed distribution by the ring averaged E cross B drift. This is the only nonlinear term.
- $q\frac{F_0}{T_0}\frac{\partial \langle\phi\rangle_\mathbf{R}}{\partial t}$ is the work done on the particles by the field.
- $-\frac{\partial <\phi>_\mathbf{R}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial F_0}{\partial \mathbf{R}}$ is the convection of the equilibrium distribution by the ring averaged E cross B drift.

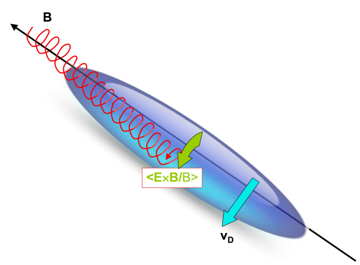

Figure 5: Perpendicular motion of the guiding center is the E cross B drift plus the equilibrium grad B drift.

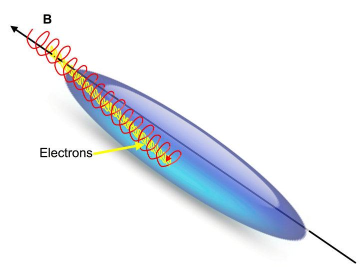

Figure 6: Perpendicular motion of the guiding center is the E cross B drift plus the equilibrium grad B drift.

5 Ordered Electron Fokker-Planck Equation.

The expansion of the electron equation is similar but differs because electrons move faster than ions ($v_{the} \sim \epsilon^{-1/2} v_{thi}$), have a smaller larmor radius ($\rho_e \sim \epsilon^{1/2}\rho_i$) and collide more than ions ($\nu_e \sim \epsilon^{-1/2}\nu_i$). Thus after some algebra one learns that:

$$F_{0e}(\mathbf{r}, \mathbf{v}, t) = n_e(t, x)\left(\frac{m}{2\pi T_e(t, x)}\right)^{3/2} \exp\left[-(\frac{\mathcal{E}_e}{T_e(t, x)})\right] \tag{57}$$

Where $\mathcal{E}_e = (1/2)mv^2 - e\phi(\mathbf{r}, t)$. We also learn that $\delta f_1 = h_e$ and $\delta f_{3/2}$ are independent of θ.

5.1 $\mathcal{O}(\epsilon^{1/2})$:

At this order we learn that:

$$\mathbf{b} \cdot \nabla h_e = 0 \tag{58}$$

The solution of this has $h_e = h_e(X, \mathcal{E}, \mu, \sigma, t)$ – *i.e.* constant on flux surface. We determine h_e in next order.

5.2 $\mathcal{O}(\epsilon)$:

Ring averaging the equation at this order we obtain the electron gyro-kinetic equation:

$$\frac{\partial h_e}{\partial t} + v_\parallel \frac{\partial \delta f_{3/2}}{\partial Z} + \mathbf{v_D} \cdot \frac{\partial h_e}{\partial \mathbf{R}} - \frac{\partial <\phi>_\mathbf{R}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial h_e}{\partial \mathbf{R}} - \langle C(h_e) \rangle_\mathbf{R} = -e \frac{F_0}{T_0} \frac{\partial \langle \phi \rangle_\mathbf{R}}{\partial t} - \frac{\partial <\phi>_\mathbf{R}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial F_0}{\partial \mathbf{R}} \tag{59}$$

We can remove $\delta f_{3/2}$ by averaging over Y and Z. Because the electron larmor radius is small we can ignore the difference between $\mathbf{r}$ and $\mathbf{R}$. Thus:

$$\frac{\partial h_e}{\partial t} - C(h_e) = -e \frac{F_0}{T_0} \frac{\partial \bar{\phi}}{\partial t} \tag{60}$$

where $\bar{\phi}$ is the flux surface $(y,\ z)$ averaged ϕ. This equation has the solution:

$$h_e = \frac{-e F_0 \bar{\phi}}{T_0} \tag{61}$$

6 Quasi-Neutrality.

We define a second ring average at fixed $\mathbf{r}$ as:

$$\langle a(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_\mathbf{r} = \frac{1}{2\pi} \oint d\theta a(\mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathcal{E}, \mu, \sigma, \theta, t),$$

This average arises in Maxwell's equations where for example the charge at $\mathbf{r}$ is due to particles with gyro-centers on a circle of radius $v_\perp/\Omega$ about $\mathbf{r}$. [1]Thus quasi neutrality can be written:

$$-\frac{n_i q^2 \phi}{T_i} + 2\pi q \sum_\sigma \int \int v_\perp dv_\perp dv_\parallel \langle h_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, t) \rangle_\mathbf{r} = \frac{n_e e^2 (\phi - \bar{\phi})}{T_e} \tag{64}$$

where we have expanded $\mathcal{E}$ in F_0 to get the Boltzmann terms.

[1]**Gyro-averages and Bessel Functions**. Strictly speaking the Eq. (56) and Eq. (64) are an integro-differential system in space since they involve the gyro-averages. It is common to use a fourier basis in both $\mathbf{r}$ and $\mathbf{R}$ since this "diagonalizes" the gyro-average. Specifically

$$< \exp i\mathbf{k} \cdot \mathbf{r} >_\mathbf{R} = J_0(\frac{k_\perp v_\perp}{\Omega}) \exp i\mathbf{k} \cdot \mathbf{R} \tag{62}$$

$$< \exp i\mathbf{k} \cdot \mathbf{R} >_\mathbf{r} = J_0(\frac{k_\perp v_\perp}{\Omega}) \exp i\mathbf{k} \cdot \mathbf{r} \tag{63}$$

Where $J_0(x)$ is the zeroth order Bessel function of the first kind. Thus in the fourier space gyro-averaging just becomes multiplication by a Bessel function (that depends on $v_\perp$).

The two equations, Eq. (56) and Eq. (64) are essentially an autonomous set on the turbulent time-scale. They are the **Electrostatic Gyro-kinetic system**. Of course F_0 must also be known, this requires calculating evolution on the long transport time-scale. However on the turbulent time-scale F_0 must be kept fixed and we can simply take it as known.

7 Equilibrium Transport Evolution.

To calculate the evolution of F_0 can be computed by considering the moment equations for n_0 and T_0. Thus the ion density evolution comes from,

$$\frac{\partial n_i}{\partial t} + \nabla \cdot \mathbf{\Gamma_i} = 0. \tag{65}$$

Where the flux is $\mathbf{\Gamma_i} = \int \mathbf{v} f_i d^3\mathbf{v}$. To isolate the mean density evolution we must average the fluctuating parts.

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