# Gyrokinetics on Transport Time Scales

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# Overview

- Global or full f GKs needs to evolve $\Phi$ as well as n & T and flows on transport time scales
- Must avoid introducing an extraneous $\Phi$
- Desire to retain neoclassical ion heat and momentum transport effects on evolution
- Want to avoid doing GKs to very high order
- Today: Transport time scale gyrokinetics using hybrid gyrokinetic - fluid description

# Hybrid gyrokinetics-fluid

- Retains **all** neoclassical effects as well as turbulence
- Simplification: electrostatic
- Simplification: drift kinetic electrons (ITG & TEM)
- Evolve n, $T_i$, $T_e$, $\Phi$, $\vec{V}$ and $\vec{J}$ with conservation equations
- Strategy: f used only for closure (heat flows and viscosities) - f **not** used to evaluate n, T, $\vec{V}$ and $\vec{J}$ directly
- Use higher order GK variables to retain ion viscosity
- Valid for PIC or continuum GKs
- No need to solve GK equation in a conservative form: conservation properties built into fluid description

# Gyrokinetic limitations

- Numerically implemented GKs typically valid thru $O(\rho/L)$
  - Evolves n & T without neoclassical transport effects
  - Often does not satisfy intrinsic ambipolarity
  - Can't evolve the axisymmetric, long wavelength $\Phi$
  - Moments of the gyrokinetic equation contain less information than moments of the full Fokker-Planck equation
  - Need to extend GKs
  - Desire to avoid having to solve GKE to higher order
  - Need GK variables to $O(\delta^2)$ but not GKE

  [since $f = f_M + \delta f$ for neoclassical effects, only need $\delta f$ to $O(\delta)$]

# Gyrokinetic validity reminder

- GKE normally derived using $\vec{R} = \vec{r} + \Omega^{-1}\vec{v}\times\vec{n}$ for which

  $\langle d\vec{R}/dt\rangle_\varphi = \vec{v}_d + u\vec{n}$ and $d\vec{R}/dt - \langle d\vec{R}/dt\rangle_\varphi \sim \delta v_i \sim \vec{v}_d \sim v_p$

- Therefore

$$df/dt - \langle df/dt\rangle_\varphi = -\Omega\partial\tilde{f}/\partial\varphi + (\dot{\vec{R}} - \langle\dot{\vec{R}}\rangle_\varphi)\cdot\nabla f + \ldots$$

  gives

$$\tilde{f} \sim \Omega^{-1}\int d\varphi(\dot{\vec{R}} - \langle\dot{\vec{R}}\rangle_\varphi)\cdot\nabla f_M + \ldots \sim \delta^2 f_M$$

- GKE normally gives $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t) + O(\delta^2)$ error even though GKs good for arbitrary $k_\perp\rho$: only good to $O(\delta)$
- Desire GK variables to $O(\delta^2)$ at $k_\perp$L~1 with leading collisional gyrophase dependence [it can be evaluated to $O(\delta^2)$] : then can evaluate $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t) + O(\delta^3)$

# Gyrokinetic equation reminder

- Variables G $\Rightarrow \vec{R}$, E = $v^2/2 + Ze\Phi/M$, $\mu$, $\varphi$
- Changing variables, Fokker-Planck equation becomes:

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \nabla_R f + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{\mu}\frac{\partial f}{\partial \mu} + \frac{Ze}{M}\frac{\partial \Phi}{\partial t}\frac{\partial f}{\partial E} = C\{f\}$$

- Variables G constructed so $dG_j/dt = \langle dG_j/dt \rangle_\varphi$ + small.
- Leading $\varphi$ dependence from $-\Omega \partial \tilde{f}/\partial\varphi = C\{f\} - \langle C\{f\}\rangle_\varphi$
- Gyroaveraging at fixed $\vec{R}$,E,$\mu$ (recall $\langle d\mu/dt\rangle_\varphi = 0$) gives

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \nabla_R f + \frac{Ze}{M}\frac{\partial \langle \Phi \rangle_\varphi}{\partial t}\frac{\partial f}{\partial E} = \langle C\{f\}\rangle_\varphi$$

to O($\delta$) when we ignore O($\delta^2$) from f & variable change

Here $\dot{\vec{R}} = u\vec{n}(\vec{R}) + \vec{v}_d$ with u parallel velocity & $\vec{v}_d$ drift velocity

# Hybrid features

- Hybrid gyrokinetic - fluid description is a way forward
  - Can solve any consistent gyrokinetic equation to order $\delta = \rho/L$
  - Conservation of number, charge, momentum & energy
  - Insures intrinsic ambipolarity
  - Evolve n, T, $\Phi$, $\vec{V}$ and $\vec{J}$
  - Use next order gyrokinetic variables only in ion viscosity
  - Use moments of full Fokker-Planck equation to gain an order

# Moment approach in a tokamak

- In a strongly magnetized ($B \to \infty$) plasma easier to evaluate certain moments of f indirectly
- Direct evaluation of $n\vec{V} \cdot \nabla\psi = \int d^3 v f \, \vec{v} \cdot \nabla\psi$ using a stationary Maxwellian gives a vanishing radial particle flux
- Taking the $(Mc/Ze)R^2\nabla\zeta\cdot\vec{v}$ moment of the full Fokker-Planck equation

$$\partial f / \partial t + \vec{v} \cdot \nabla f + (Ze/M)(-\nabla\Phi + c^{-1}\vec{v} \times \vec{B}) \cdot \nabla_v f = C$$

using $\vec{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi = B\vec{n}$ and $R^2\vec{B} \times \nabla\zeta = \nabla\psi$ then inserting a Maxwellian gives order $\delta \equiv \rho/L$ corrections

$n\vec{V} \cdot \nabla\psi = cn\partial\Phi/\partial\zeta + \nabla \cdot [(cR^2 nT/Ze)\nabla\zeta]$ & $\langle n\vec{V} \cdot \nabla\psi \rangle_\theta = c\langle n\partial\Phi/\partial\zeta \rangle_\theta$

with $\langle ... \rangle_\theta \equiv (1/V')\oint d\zeta d\theta (...) / \vec{B} \cdot \nabla\theta$

# Moment evaluation of ion heat flux

- A direct evaluation of the classical radial collisional heat flux $\vec{q}_\perp = \int d^3v\, f\, \vec{v}_\perp (Mv^2/2 - 5T/2)$ requires f to order O(νρ/ΩL)
- The diamagnetic flow only requires f to O(ρ/L) and can be evaluated by using $\tilde{f} = \Omega^{-1} f_M (Mv^2/2 - 5T/2)\vec{v}\cdot\vec{n}\times\nabla \ell nT$ to obtain $\vec{q}_{\perp d} = (5cnT/2ZeB)\vec{n}\times\nabla \ell nT$
- To avoid calculating f to higher order (we only need gyrophase dependent terms), form $\vec{v}(Mv^2/2 - 5T/2)$ moment of the full Fokker=Planck equation to find the classical term $\vec{q}_{\perp\nu} = -(1/2\Omega)\vec{n}\times\int d^3v\,\vec{v}(Mv^2 - 5T)C\{\tilde{f}\} = -(2\nu nT/M\Omega^2)\nabla_\perp T$
- Notice this evaluation only required the leading order gyrophase dependent correction to the Maxwellian!

# Neoclassical vs. classical in a tokamak

- The $(Mc/2Ze)R^2v^2\nabla\zeta\cdot\vec{v}$ moment of the full FP equation gives the flux surface averaged collisional radial heat flux

$$\langle \vec{q}_\nu \cdot \nabla\psi \rangle_\theta = -(M^2c/2Ze)\langle R^2 \int d^3v v^2 \vec{v}\cdot\nabla\zeta C\{f - f_M\}\rangle_\theta$$

- The leading gyrophase dependent correction to the Maxwellian gives classical radial ion heat transport
- The leading gyrophase independent correction to the Maxwellian gives the neoclassical radial ion heat transport
- Only need the leading corrections to the Maxwellian to evaluate the radial tranport of heat
- Can use same procedure with turbulence

# Hybrid overview

- Conservation of number, charge and energies
- Moment evaluation of heat flux
- Gyrokinetic reminders
- Conservation of total and electron momentum
- Moment evaluation of viscosity (not for the faint hearted)
- Retains all turbulent, neoclassical & classical effects to evolve profiles including the axisymmetric radial electric field!

# Number, charge & energy

- Number: $\partial n/\partial t + \nabla\cdot(n\vec{V}) = S_n$
- Charge: $\nabla \cdot \vec{J} = 0$ with $\vec{J} = en(\vec{V}_i - \vec{V}_e)$
- Ion energy:

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla\cdot\left(\vec{q}_i + \frac{5}{2}p_i\vec{V}\right) = -en\vec{V}\cdot\nabla\Phi + W + S_{pi}$$

- Electron energy:

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla\cdot\left(\vec{q}_e + \frac{5}{2}p_e\vec{V}_e\right) = en\vec{V}\cdot\nabla\Phi - W + S_{pe}$$

energy exchange $= W = 3mn\nu_e(T_e - T_i)/M + (mn\nu_e\vec{V} - \vec{F})\cdot\vec{V}$

momentum exchange $= \vec{F} = mn\nu_e\vec{V} - 2\gamma_e mn\int d^3v f_e \vec{v}/v^3$

# Ion heat flow

- Start with $\vec{q}_i = \int d^3v f_i \vec{v}(Mv^2 - 5T_i)/2$
- A direct evaluation of $\langle \vec{q}_i \cdot \nabla\psi \rangle_\theta$ using the lowest order gyrokinetic f is independent of ν in the axisymmetric limit so misses collisional radial heat flux
- To pick up an order form $\vec{v}v^2/2$ moment of full FP

$$\Omega_i \vec{n} \times \vec{q}_i + \partial \vec{q}_i / \partial t + \nabla \cdot [\int d^3v f_i (Mv^2 - 5T_i)\vec{v}\vec{v}/2] + \frac{5p_i}{2M}\nabla T_i +$$

$$\vec{\pi}_i \cdot (\frac{e}{M}\nabla\Phi + \frac{5}{2M}\nabla T_i)\} = (1/2)\int d^3v (Mv^2 - 5T_i)\vec{v} C_i\{f_i\}$$

- To lowest order find (Q known)

$$\langle \vec{q}_i \cdot \nabla\psi \rangle_\theta \approx (5c/2e)\langle p_i \frac{\partial T_i}{\partial \zeta} \rangle_\theta - \frac{\nu_i B T_i}{\Omega_i}\langle R^2 \int d^3v f_i Q \vec{v} \cdot \nabla\zeta \rangle_\theta$$

anomalous collisional

# Full ion heat flow expression

- Putting everything together and neglecting the time derivative term

$$\vec{q}_i = \frac{1}{\Omega_i}\vec{n}\times\{\nabla\cdot[\int d^3v f_i(Mv^2 - 5T_i)\vec{v}\vec{v}/2]$$

$$+\frac{5p_i}{2M}\nabla T_i + \vec{\pi}_i\cdot[(e/M)\nabla\Phi + (5/2M)\nabla T_i]\}$$

$$+(\nu_i/\Omega_i)T_i\int d^3v f_i Q(x_i)\vec{v}\times\vec{n} + \vec{n}\int d^3v f_i v_{||}(Mv^2 - 5T_i)/2$$

- The time derivative term is ignored since it gives an $O(\delta)$ correction to the fluctuating heat flux that when averaged over a turbulent saturation time results in a $O(\delta^2)$ correction to the background evolution

# Neoclassical and classical limit

- The neoclassical and classical terms are in
  $\langle \vec{q}_i \cdot \nabla\psi \rangle_\theta \Rightarrow -\langle (\nu_i/\Omega_i) T_i B R^2 \int d^3v f_i Q(x_i) \vec{v} \cdot \nabla\zeta \rangle_\theta$
- Only $\nabla T_i$ terms should contribute since
  $f = f_M + (Mc/Ze) R \nabla\zeta \cdot \vec{v} \partial f_M/\partial\psi + \overline{h}_1$ and
  $v_{\|} \vec{n} \cdot \nabla \overline{h}_1 = C_1\{\overline{h}_1 - (I v_{\|}/\Omega) f_M (Mv^2/2T - 5/2) \partial \ell n T/\partial\psi\}$
- Q has the property $\int d^3v\, \vec{v}\vec{v} Q f_M \equiv 0$ so only $\vec{v}_\perp v^2 \partial T/\partial\psi$ term from $\partial f_M/\partial\psi$ will enter and it gives classical transport
- $h_1$ & $v_{\|} v^2 \partial T/\partial\psi$ give neoclassical (only depends on $\partial T/\partial\psi$)
- Really have to do all this gyrokinetically

# Need a gyrokinetic f for ions

- Use your favorite GK variables & a PIC or Eulerian code
- For example, $\vec{R}$ , K = v²/2 + Ze(Φ– $\langle\Phi\rangle_\varphi$)/M, μ, φ

$$\frac{\partial f}{\partial t} + (u\vec{n} + \vec{v}_d)\cdot[\nabla_R f - \frac{Ze}{M}\nabla_R(\Phi - \langle\Phi\rangle_\varphi)\frac{\partial f}{\partial K}] = \langle C\{f\}\rangle_\varphi$$

$$\vec{v}_d = \vec{v}_M - (c/B)\nabla_R\langle\Phi\rangle_\varphi \times \vec{n} \qquad \langle\Phi\rangle_\varphi = (2\pi)^{-1}\oint d\varphi\,\Phi(\vec{r},t)$$

with the gyroaverage performed at fixed $\vec{R}$, K, μ and
f = f($\vec{R}$,K,μ,t) in velocity integrals performed at fixed $\vec{r}$

- For heat flux we only need the leading gyrokinetic variables

$\vec{R} = \vec{r} + \Omega^{-1}\vec{v}\times\vec{n}$ , K = v²/2 + Ze(Φ– $\langle\Phi\rangle_\varphi$)/M & $\mu_0 = v_\perp^2/2B$

# Usual gyrokinetic orderings apply

- Small parameters: $\delta = \dfrac{\rho}{L} \sim \dfrac{\omega_*}{\Omega} \sim \dfrac{\nu}{\Omega} << 1$
- $f$ and $\Phi$ have $k_\perp \rho \sim 1$ but $k_{||} L \sim 1$
- For $k_\perp L \sim 1$, $e\Phi/T \sim 1$ and $f \approx f_M \equiv$ Maxwellian
- For $k_\perp \rho \sim 1$, $e\Phi_k/T \sim f_k / f_M \sim \delta$
- For general $k_\perp$: $\dfrac{e\phi_k}{T} \sim \dfrac{f_k}{f_M} \sim \dfrac{1}{k_\perp L}$
  - Note $\nabla\Phi \sim T/eL \sim k_\perp \Phi_k$ and $\nabla f_k \sim \nabla f_M$
  - Drift ordering: $V_{ExB} \sim \delta v_i << v_i$

# Global of full f gyrokinetic evolution

- Evolution of n and T only contains what is in the density and energy moments of the full f gyrokinetic equation
- Drift term in GKE contains ExB turbulent transport but no neoclassical & classical collisional transport in GKE
- Profile evolution only due turbulence
- Can't properly evolve the long wavelength axisymmetric potential profile since Reynolds stress incomplete and collisional terms missing

# Momentum conservation

- Electrons (neglect inertia & gyro+perp viscosity):

$$en(-\nabla\Phi + c^{-1}\vec{V}_e \times \vec{B}) + \nabla\cdot(p_e\vec{I} + \overleftrightarrow{\pi}_e) = \vec{F} + \vec{S}_{me}$$

- Ions + electrons:

$$\frac{\partial(Mn\vec{V})}{\partial t} + \nabla\cdot[(p_i + p_e)\vec{I} + \overleftrightarrow{\pi}_i + \overleftrightarrow{\pi}_e] = \frac{1}{c}\vec{J}\times\vec{B} + \vec{S}_{mi} + \vec{S}_{me}$$

- Solve electron momentum for $n\vec{V}_{e\perp}$
  - To lowest order radial particle flux

$$e\langle n\vec{V}_e\cdot\nabla\psi\rangle_\theta = c\langle en\partial\Phi/\partial\zeta + R^2\vec{F}\cdot\nabla\zeta\rangle_\theta$$

- Intrinsically ambipolar in axisymmetric limit since we use $\nabla\cdot\vec{J} = 0$ requiring $\langle\vec{J}\cdot\nabla\psi\rangle_\theta = 0$

# Ion viscosity

- Start with $\overleftrightarrow{\pi}_i = M\int d^3v f_i(\vec{v}\vec{v} - \overleftrightarrow{I}v^2/3) = \overleftrightarrow{\pi}_{i\|} + \overleftrightarrow{\pi}_{ig} + \overleftrightarrow{\pi}_{i\perp}$
- Parallel anisotropy: $\overleftrightarrow{\pi}_{i\|} = (p_{i\|} - p_{i\perp})(\vec{n}\vec{n} - \overleftrightarrow{I}/3)$ with

$$p_{i\|} - p_{i\perp} = M\int d^3v f_i(v_{\|}^2 - \mu B)$$

- Gyroviscosity & perpendicular viscosity evaluated using $\vec{v}\vec{v}$ moment of full FP equation to find form

$$\overleftrightarrow{\pi}_{ig,\perp} = (4\Omega)^{-1}[\vec{n}\times\overleftrightarrow{K}_{ig,\perp}\cdot(\overleftrightarrow{I} + 3\vec{n}\vec{n}) - (\overleftrightarrow{I} + 3\vec{n}\vec{n})\cdot\overleftrightarrow{K}_{ig,\perp}\times\vec{n}]$$

- Perpendicular viscosity (using self-adjointness of $C_{ii}^{\ell}$):
  $\overleftrightarrow{K}_{i\perp} = -M\int d^3v\vec{v}\vec{v}[\nu_i F(x_i)f_i + C_{ii}^{n\ell}\{f_i - f_{iM}, f_i - f_{iM}\}]$
  F known, neglect $\partial/\partial t$ & need variables to $O(\delta^2)$ in first term

# Ion viscosity & comments

- Ion gyroviscosity

$$\overleftrightarrow{K}_{ig} = \nabla\cdot\left(M\int d^3v\,\vec{v}\vec{v}\vec{v}f_i\right) + (en\nabla\Phi + \vec{n}F_{\parallel})\vec{V} + \vec{V}(en\nabla\Phi + \vec{n}F_{\parallel})$$

  - Reynolds stress part of gyroviscosity
  - Need f to O($\delta^2$) in first term for classical (need leading order collisional correction to f from $-\Omega\partial\tilde{f}/\partial\varphi = C\{f\} - \langle C\{f\}\rangle_{\varphi}$ )

Viscosity comments

- Good news:
  - Long wavelengths can be done analytically [Simakov & Catto PPCF]
  - Can assume $B_p/B << 1$ to retain O($B\delta^2/B_p$) poloidal gyroradius neoclassical corrections and ignore O($\delta^2$) classical transport
- Bad news:
  - GKs gives $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{R}, E, \mu, t)$ but need to evaluate integrals at fixed $\mathbf{r}$ and the expressions are complicated

# Radial momentum transport

- Conservation of toroidal angular momentum determines $\partial\Phi/\partial\psi$ and it enters both Reynolds stress & collisional viscosity as

$$\left\langle \frac{\tilde{n}}{\bar{n}} \frac{\partial}{\partial\zeta}\left(\frac{e\tilde{\Phi}}{T}\right)\right\rangle_{\theta} \text{ vs } \frac{q^2 R}{L_\perp}\frac{\nu}{\Omega}$$

- Both ~ $10^{-5}$ for ITER: B=5.3 T, T = 8 keV, n = $10^{19}$ m$^{-3}$, R = 6 m, and $L_\perp \sim q^2\rho_p$ with 0.1 de-phasing of $\tilde{n}$ & $\tilde{\Phi}$

- Expect relaxation to steady state to be anomalous since balancing ion inertia & Reynolds stress $Mn\vec{V}\vec{V} \lesssim O(\delta^2 p)$ gives time to establish zonal flow $\partial(e\Phi/T)/\partial t \sim \delta^2\Omega >> \delta^2\nu$

- But for times $>> 1/\delta^2\Omega \sim 1/\delta\omega_*$ should retain collisional viscosity and Reynolds stress to establish global $\partial\Phi/\partial\psi$

# Vorticity “replaces” quasineutrality

- Vorticity is used along with quasineutrality:
  - The plasma is still quasineutral
  - Vorticity must retain all physics in quasineutrality
  - Must at least satisfy intrinsic ambipolarity to $O(\delta^2)$ [not determine long wavelength axisymmetric $\Phi$ to $O(\delta^2)$]
  - Must retain $\Phi$ evolution including neoclassical effects for global or full f descriptions
  - Could evolve axisymmetric and non-axisymmetric pieces separately

# Vorticity requirements

- Vorticity = charge conservation must be evaluated carefully:
  - Need full $\vec{J}_\perp$ from momentum conservation in $\nabla \cdot \vec{J} = 0$
  - Ion inertial term gives time derivative of vorticity
  - Must retain gyroviscosity/Reynolds stress and perpendicular viscosity to get neoclassical effects
- Vorticity requirements differ for $\delta$f and full f
  - Desire vorticity for a $\delta$f code to not determine the long wavelength axisymmetric radial electric field
  - Vorticity for a full f or global code needs to keep more physics to determine the long wavelength axisymmetric radial electric field

# Final Comments

- Global gyrokinetics must satisfy intrinsic ambipolarity

- Hybrid gyrokinetic-fluid description needed to properly evolve turbulence with neoclassical retained
  - Density, temperatures, potential, ion flow, current evolved by conservation equations
  - Gyrokinetic f only used for closure and (almost) anyones will do!